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A Grüss type inequality for vector-valued functions in Hilbert C^*-modules

Amir Ghasem Ghazanfari*

*Correspondence:
ghazanfari.amir@gmail.com
Department of Mathematics,
Lorestan University, P.O. Box 465,
Khoramabad, Iran

Abstract
In this paper we prove a version of Grüss' integral inequality for mappings with values in Hilbert C^*-modules. Some applications for such functions are also given.
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1 Introduction

In 1934, Grüss [1] showed that for two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{R}$,

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \le \frac{1}{4}(M-m)(N-n),$$

provided m, M, n, N are real numbers with the property $-\infty < m \le f \le M < \infty$ and $-\infty < n \le g \le N < \infty$ a.e. on $[a,b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The following inequality of Grüss type in real or complex inner product spaces is well known [2].

Theorem 1 *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $e \in H$, $\|e\| = 1$. If $\alpha, \beta, \lambda, \mu \in \mathbb{K}$ and $x, y \in H$ are such that the conditions*

$$\operatorname{Re}\langle \alpha e - x, x - \beta e\rangle \ge 0, \qquad \operatorname{Re}\langle \lambda e - y, y - \mu e\rangle \ge 0$$

hold, or, equivalently, if

$$\left\|x - \frac{\alpha+\beta}{2}e\right\| \le \frac{1}{2}|\alpha-\beta|, \qquad \left\|y - \frac{\lambda+\mu}{2}e\right\| \le \frac{1}{2}|\lambda-\mu|$$

hold, then the following inequality holds:

$$\left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right| \le \frac{1}{4}|\alpha-\beta||\lambda-\mu|. \tag{1.1}$$

The constant $\frac{1}{4}$ is the best possible in equation (1.1).

Let $\langle H; \langle\cdot,\cdot\rangle\rangle$ be a real or complex Hilbert space, $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set and $\rho : \Omega \to [0,\infty)$ a Lebesgue measurable function with $\int_\Omega \rho(s)\,ds = 1$. We denote by $L_{2,\rho}(\Omega, H)$ the set of all strongly measurable functions f on Ω such that $\|f\|_{2,\rho}^2 := \int_\Omega \rho(s)\|f(s)\|^2\,ds < \infty$.

A further extension of the Grüss-type inequality for Bochner integrals of vector-valued functions in real or complex Hilbert spaces is given in [3].

Theorem 2 *Let $\langle H; \langle\cdot,\cdot\rangle\rangle$ be a real or complex Hilbert space, $\Omega \subset \mathbb{R}^n$ a Lebesgue measurable set and $\rho : \Omega \to [0,\infty)$ a Lebesgue measurable function with $\int_\Omega \rho(s)\,ds = 1$. If f, g belong to $L_{2,\rho}(\Omega, H)$ and there exist vectors $x, X, y, Y \in H$ such that*

$$\begin{aligned} &\int_\Omega \rho(t)\operatorname{Re}\bigl\langle X - f(t), f(t) - x\bigr\rangle\,dt \geq 0, \\ &\int_\Omega \rho(t)\operatorname{Re}\bigl\langle Y - g(t), g(t) - y\bigr\rangle\,dt \geq 0, \end{aligned} \tag{1.2}$$

or, equivalently,

$$\begin{aligned} &\int_\Omega \rho(t)\left\|f(t) - \frac{X+x}{2}\right\|^2 dt \leq \frac{1}{4}\|X - x\|^2, \\ &\int_\Omega \rho(t)\left\|g(t) - \frac{Y+y}{2}\right\|^2 dt \leq \frac{1}{4}\|Y - y\|^2, \end{aligned} \tag{1.3}$$

then the following inequalities hold:

$$\begin{aligned} &\left|\int_\Omega \rho(t)\bigl\langle f(t), g(t)\bigr\rangle\,dt - \left\langle \int_\Omega \rho(t)f(t)\,dt, \int_\Omega \rho(t)g(t)\,dt \right\rangle\right| \\ &\quad\leq \frac{1}{4}\|X - x\|\|Y - y\| - \left[\int_\Omega \rho(t)\operatorname{Re}\bigl\langle X - f(t), f(t) - x\bigr\rangle\,dt \right. \\ &\qquad \left. \times \int_\Omega \rho(t)\operatorname{Re}\bigl\langle Y - g(t), g(t) - y\bigr\rangle\,dt\right]^{\frac{1}{2}} \\ &\quad\leq \frac{1}{4}\|X - x\|\|Y - y\|. \end{aligned} \tag{1.4}$$

The constant $\frac{1}{4}$ is sharp in the sense mentioned above.

The Grüss inequality has been investigated in inner product modules over H^*-algebras and C^*-algebras [4, 5], completely bounded maps [6], n-positive linear maps [7] and semi-inner product C^*-modules [8].

Also Jocić *et al.* in [9] presented the following Grüss-type inequality:

$$\left|\left|\left|\int_\Omega \mathcal{A}_t X \mathcal{B}_t\,d\mu(t) - \int_\Omega \mathcal{A}_t\,d\mu(t) X \int_\Omega \mathcal{B}_t\,d\mu(t)\right|\right|\right| \leq \frac{\|D - C\| \cdot \|F - E\|}{4} |||X|||$$

for all bounded self-adjoint fields satisfying $C \leq \mathcal{A}_t \leq D$ and $E \leq \mathcal{B}_t \leq F$ for all $t \in \Omega$ and some bounded self-adjoint operators C, D, E, and F, and for all $X \in \mathcal{C}_{|||\cdot|||}(H)$.

The main aim of this paper is to obtain a generalization of Theorem 2 for vector-valued functions in Hilbert C^*-modules. Some applications for such functions are also given.

2 Preliminaries

Hilbert C^*-modules are used as the framework for Kasparov's bivariant K-theory and form the technical underpinning for the C^*-algebraic approach to quantum groups. Hilbert C^*-modules are very useful in the following research areas: operator K-theory, index theory for operator-valued conditional expectations, group representation theory, the theory of AW^*-algebras, noncommutative geometry, and others. Hilbert C^*-modules form a category in between Banach spaces and Hilbert spaces and obey the same axioms as a Hilbert space except that the inner product takes values in a general C^*-algebra rather than in the complex numbers $\mathbb{C}$. This simple generalization gives a lot of trouble. Fundamental and familiar Hilbert space properties like Pythagoras' equality, self-duality and decomposition into orthogonal complements must be given up. Moreover, a bounded module map between Hilbert C^*-modules does not need to have an adjoint; not every adjoinable operator needs to have a polar decomposition. Hence to get its applications, we have to use it with great care.

Let $\mathcal{A}$ be a C^*-algebra. A semi-inner product module over $\mathcal{A}$ is a right module X over $\mathcal{A}$ together with a generalized semi-inner product, that is, with a mapping $\langle\cdot,\cdot\rangle$ on $X \times X$, which is $\mathcal{A}$-valued and has the following properties:

(i) $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$ for all $x,y,z \in X$,

(ii) $\langle x, ya\rangle = \langle x,y\rangle a$ for $x,y \in X$, $a \in \mathcal{A}$,

(iii) $\langle x,y\rangle^* = \langle y,x\rangle$ for all $x,y \in X$,

(iv) $\langle x,x\rangle \geq 0$ for $x \in X$.

We will say that X is a semi-inner product C^*-module. The absolute value of $x \in X$ is defined as the square root of $\langle x,x\rangle$, and it is denoted by $|x|$. If, in addition,

(v) $\langle x,x\rangle = 0$ implies $x = 0$,

then $\langle\cdot,\cdot\rangle$ is called a generalized inner product and X is called an inner product module over $\mathcal{A}$ or an inner product C^*-module. An inner product C^*-module which is complete with respect to the norm $\|x\| := \|\langle x,x\rangle\|^{\frac{1}{2}}$ $(x \in X)$ is called a Hilbert C^*-module.

As we can see, an inner product module obeys the same axioms as an ordinary inner product space, except that the inner product takes values in a more general structure than in the field of complex numbers.

If $\mathcal{A}$ is a C^*-algebra and X is a semi-inner product $\mathcal{A}$-module, then the following Schwarz inequality holds:

$$\langle x,y\rangle\langle y,x\rangle \leq \big\|\langle x,x\rangle\big\|\langle y,y\rangle \quad (x,y \in X) \tag{2.1}$$

(*e.g.* [10, Proposition 1.1]).

It follows from the Schwarz inequality equation (2.1) that $\|x\|$ is a semi-norm on X.

Now let $\mathcal{A}$ be a $*$-algebra, φ a positive linear functional on $\mathcal{A}$, and let X be a semi-inner $\mathcal{A}$-module. We can define a sesquilinear form on $X \times X$ by $\sigma(x,y) = \varphi(\langle x,y\rangle)$; the Schwarz inequality for σ implies that

$$\big|\varphi\langle x,y\rangle\big|^2 \leq \varphi\langle x,x\rangle\varphi\langle y,y\rangle. \tag{2.2}$$

In [11, Proposition 1, Remark 1] the authors present two other forms of the Schwarz inequality in semi-inner $\mathcal{A}$-module X, one for a positive linear functional φ on $\mathcal{A}$:

$$\varphi\big(\langle x,y\rangle\langle y,x\rangle\big) \leq \varphi\langle x,x\rangle r\langle y,y\rangle, \tag{2.3}$$

where r is the spectral radius, and another one for a C^*-seminorm γ on $\mathcal{A}$:

$$\big(\gamma\langle x,y\rangle\big)^2 \le \gamma\langle x,x\rangle\gamma\langle y,y\rangle. \tag{2.4}$$

3 The main results

Let $\mathcal{A}$ be a C^*-algebra; first we state some basic properties of integrals of $\mathcal{A}$-value functions with respect to a positive measure for Bochner integrability of functions which we need to use in our discussion. For basic properties of the integrals of vector-valued functions with respect to scalar measures and the integrals of scalar-valued functions with respect to vector measures, see Chapter II in [12].

Lemma 1 *Let $\mathcal{A}$ be a C^*-algebra, X a Hilbert C^*-module and $(\Omega,\mathfrak{M},\mu)$ be a measure space. If $f:\Omega\to X$ is Bochner integrable and $a\in\mathcal{A}$ then*

(a) *fa is Bochner integrable, where $fa(t)=f(t)a$ $(t\in\Omega)$,*

(b) *the function $f^*:\Omega\to\mathcal{A}$ defined by $f^*(t)=(f(t))^*$ is Bochner integrable and $\int_\Omega f^*\,d\mu=(\int_\Omega f\,d\mu)^*$.*

Furthermore,

(c) *if $\mu(\Omega)<\infty$ and f is positive, i.e., $f(t)\ge 0$ for all $t\in\Omega$, then $\int_\Omega f(t)\,d\mu(t)\ge 0$.*

Proof (a) Suppose $\varphi:\Omega\to X$ is a simple function with finite support *i.e.*, $\varphi=\sum_{i=1}^n \chi_{E_i}x_i$ with $\mu(E_i)<\infty$ for each non-zero $x_i\in X$ $(i=1,2,3,\dots,n)$, then for every $a\in\mathcal{A}$ the function $\varphi a:\Omega\to X$ defined by $\varphi a(t)=\varphi(t)a$ $(t\in\Omega)$ is a simple function and $\int_\Omega \varphi a\,d\mu=(\int_\Omega\varphi\,d\mu)a$. Since $f:\Omega\to X$ is Bochner integrable and $\|f(t)a\|\le\|f(t)\|\|a\|$, fa is strongly measurable and $\int_\Omega fa\,d\mu=(\int_\Omega f\,d\mu)a$, therefore fa is Bochner integrable.

(b) For every simple function $\varphi=\sum_{i=1}^n\chi_{E_i}a_i$ we have $\varphi^*=\sum_{i=1}^n\chi_{E_i}a_i^*$ and consequently $\int_\Omega\varphi^*\,d\mu=(\int_\Omega\varphi\,d\mu)^*$. The result therefore follows.

(c) Suppose that $\mu(\Omega)<\infty$ and f is a Bochner integrable function and positive, *i.e.*, $f(t)\ge 0$ for all $t\in\Omega$. Since f is Bochner integrable, $\int_\Omega\|f(t)\|\,d\mu(t)<\infty$ by Theorem 2 in [12, Chapter II, Section 2]. Using the Holder inequality for Lebesgue integrable functions we get

$$\int_\Omega \big\|f^{\frac12}(t)\big\|\,d\mu(t)=\int_\Omega \big\|f(t)\big\|^{\frac12}\,d\mu(t)\le\left(\int_\Omega\big\|f(t)\big\|\,d\mu(t)\right)^{\frac12}\mu(\Omega)^{\frac12}<\infty.$$

So $f^{\frac12}$ is Bochner integrable; thus there is a sequence of simple functions φ_n such that $\varphi_n(t)\to f^{\frac12}(t)$ for almost all $t\in\Omega$ and $\int_\Omega\varphi_n(t)\,d\mu(t)\to\int_\Omega f^{\frac12}(t)\,d\mu(t)$ in the norm topology in $\mathcal{A}$.

This implies that $\varphi_n(t)^*\varphi_n(t)\to f(t)$, *i.e.*, for every positive Bochner integrable function f there is a sequence of positive simple functions ψ_n such that $\psi_n(t)\to f(t)$ for almost all $t\in\Omega$ and $\int_\Omega\psi_n(t)\,d\mu(t)\to\int_\Omega f(t)\,d\mu(t)$ in the norm topology in $\mathcal{A}$. By proposition (1.6.1) in [13] the set of positive elements in a C^*-algebra is a closed convex cone, therefore $\int_\Omega f(t)\,d\mu(t)\ge 0$ since $\int_\Omega\psi_n(t)\,d\mu(t)\ge 0$. □

If μ is a probability measure on Ω, we denote by $L_2(\Omega,X)$ the set of all strongly measurable functions f on Ω such that $\|f\|_2^2:=\int_\Omega\|f(s)\|^2\,d\mu(s)<\infty$.

For every $a\in X$, we define the constant function $e_a\in L_2(\Omega,X)$ by $e_a(t)=a$ $(t\in\Omega)$. In the following lemma we show that a special kind of invariant property holds, which we will use in the sequel.

Lemma 2 *If $f, g \in L_2(\Omega, X)$, $a, b \in X$, and e_a, e_b are measurable then*

$$\begin{aligned}&\int_\Omega \langle f(t) - e_a(t), g(t) - e_b(t)\rangle \, d\mu(t) - \left\langle \int_\Omega \big(f(t) - e_a(t)\big)\, d\mu(t), \int_\Omega \big(g(t) - e_b(t)\big)\, d\mu(t)\right\rangle \\ &\quad = \int_\Omega \langle f(t), g(t)\rangle \, d\mu(t) - \left\langle \int_\Omega f(t)\, d\mu(t), \int_\Omega g(t)\, d\mu(t)\right\rangle. \end{aligned} \tag{3.1}$$

In particular,

$$\int_\Omega |f(t)|^2 \, d\mu(t) - \left|\int_\Omega f(t)\, d\mu(t)\right|^2 \le \int_\Omega |f(t) - e_a(t)|^2 \, d\mu(t). \tag{3.2}$$

Proof We must state that the functions under the integrals of equation (3.1) are Bochner integrable on Ω, since they are strongly measurable and we can state the following obvious results.

For every $\Lambda \in X^*$ we have $\Lambda(\int_\Omega f(t)\, d\mu(t)) = \int_\Omega \Lambda(f(t))\, d\mu(t)$. Therefore

$$\begin{aligned}&\int_\Omega \langle f(t), b\rangle \, d\mu(t) = \left\langle \int_\Omega f(t)\, d\mu(t), b\right\rangle, \\ &\int_\Omega \langle a, g(t)\rangle \, d\mu(t) = \left\langle a, \int_\Omega g(t)\, d\mu(t)\right\rangle. \end{aligned}$$

Also for almost all $t \in \Omega$ we have

$$\begin{aligned}&\int_\Omega \|f(t)\| \, d\mu(t) \le \big(\mu(\Omega)\big)^{\frac{1}{2}} \left(\int_\Omega \|f(t)\|^2 \, d\mu(t)\right)^{\frac{1}{2}} = \|f\|_2, \\ &\int_\Omega \|g(t)\| \, d\mu(t) \le \big(\mu(\Omega)\big)^{\frac{1}{2}} \left(\int_\Omega \|g(t)\|^2 \, d\mu(t)\right)^{\frac{1}{2}} = \|g\|_2 \end{aligned}$$

and

$$\int_\Omega \big\|\langle f(t), g(t)\rangle\big\| \, d\mu(t) \le \|f\|_2 \|g\|_2.$$

A simple calculation shows that

$$\begin{aligned}&\int_\Omega \langle f(t) - e_a(t), g(t) - e_b(t)\rangle \, d\mu(t) - \left\langle \int_\Omega \big(f(t) - e_a(t)\big)\, d\mu(t), \int_\Omega \big(g(t) - e_b(t)\big)\, d\mu(t)\right\rangle \\ &\quad = \int_\Omega \big(\langle f(t), g(t)\rangle - \langle f(t), b\rangle - \langle a, g(t)\rangle + \langle a, b\rangle\big)\, d\mu(t) \\ &\qquad - \left\langle \int_\Omega f(t)\, d\mu(t) - a, \int_\Omega g(t)\, d\mu(t) - b\right\rangle \\ &\quad = \int_\Omega \langle f(t), g(t)\rangle \, d\mu(t) - \left\langle \int_\Omega f(t)\, d\mu(t), \int_\Omega g(t)\, d\mu(t)\right\rangle, \end{aligned}$$

and for $f = g$ and $a = b$ we deduce equation (3.2). □

The following result concerning a generalized semi-inner product on $L_2(\Omega, X)$ may be stated.

Lemma 3 *If $f,g \in L_2(\Omega,X)$, $x,x',y,y' \in X$, then*

(i) *the following inequalities, equations* (3.3) *and* (3.4), *are equivalent*:

$$\begin{aligned}&\int_\Omega \operatorname{Re}\langle x'-f(t),f(t)-x\rangle\,d\mu(t)\ge 0,\\&\int_\Omega \operatorname{Re}\langle y'-g(t),g(t)-y\rangle\,d\mu(t)\ge 0,\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\int_\Omega \left|f(t)-\frac{x'+x}{2}\right|^2 d\mu(t)\le\frac{1}{4}\left|x'-x\right|^2,\\&\int_\Omega \left|g(t)-\frac{y'+y}{2}\right|^2 d\mu(t)\le\frac{1}{4}\left|y'-y\right|^2;\end{aligned}\tag{3.4}$$

(ii) *the map $[f,g]: L_2(\Omega,X)\times L_2(\Omega,X)\to\mathcal{A}$,*

$$[f,g]:=\int_\Omega\langle f(t),g(t)\rangle\,d\mu(t)-\left\langle\int_\Omega f(t)\,d\mu(t),\int_\Omega g(t)\,d\mu(t)\right\rangle,\tag{3.5}$$

is a generalized semi-inner product on $L_2(\Omega,X)$.

Proof (i) If $f\in L_2(\Omega,X)$, since for any $y,x,x'\in X$

$$\left|y-\frac{x'+x}{2}\right|^2-\frac{1}{4}\left|x'-x\right|^2=\operatorname{Re}\langle y-x',y-x\rangle,$$

we have

$$\begin{aligned}&\int_\Omega \operatorname{Re}\langle x'-f(t),f(t)-x\rangle\,d\mu(t)\\&\quad=\int_\Omega\left[\frac{1}{4}\left|x'-r\right|^2-\left|f(t)-\frac{x'+x}{2}\right|^2\right]d\mu(t)\\&\quad=\frac{1}{4}\left|x'-x\right|^2-\int_\Omega\left|f(t)-\frac{x'+x}{2}\right|^2 d\mu(t),\end{aligned}\tag{3.6}$$

showing that, indeed, the inequalities of equations (3.3) and (3.4) are equivalent.

(ii) We note that the first integral in equation (3.5) belongs to $\mathcal{A}$ and the later integrals are in X, and the following Korkine-type identity for Bochner integrals holds:

$$\begin{aligned}&\int_\Omega\langle f(t),g(t)\rangle\,d\mu(t)-\left\langle\int_\Omega f(t)\,d\mu(t),\int_\Omega g(t)\,d\mu(t)\right\rangle\\&\quad=\frac{1}{2}\int_\Omega\int_\Omega\langle f(t)-f(s),g(t)-g(s)\rangle\,d\mu(t)\,d\mu(s).\end{aligned}\tag{3.7}$$

By an application of the identity equation (3.7),

$$\int_\Omega\left|f(t)\right|^2 d\mu(t)-\left|\int_\Omega f(t)\,d\mu(t)\right|^2=\frac{1}{2}\int_\Omega\int_\Omega\left|f(t)-f(s)\right|^2 d\mu(t)\,d\mu(s)\ge 0.\tag{3.8}$$

It is easy to show that $[\cdot,\cdot]$ is a generalized semi-inner product on $L_2(\Omega,X)$. □

The following theorem is a generalization of Theorem 2 for Hilbert C^*-modules.

Theorem 3 *Let X be a Hilbert C^*-module, μ a probability measure on Ω. If f, g belong to $L_2(\Omega, X)$ and there exist vectors $x, x', y, y' \in X$ such that*

$$\begin{aligned} &\int_\Omega \operatorname{Re}\langle x' - f(t), f(t) - x\rangle \, d\mu(t) \geq 0, \\ &\int_\Omega \operatorname{Re}\langle y' - g(t), g(t) - y\rangle \, d\mu(t) \geq 0, \end{aligned} \tag{3.9}$$

or, equivalently,

$$\begin{aligned} &\int_\Omega \left| f(t) - \frac{x' + x}{2} \right|^2 d\mu(t) \leq \frac{1}{4} \left| x' - x \right|^2, \\ &\int_\Omega \left| g(t) - \frac{y' + y}{2} \right|^2 d\mu(t) \leq \frac{1}{4} \left| y' - y \right|^2. \end{aligned} \tag{3.10}$$

Then the following inequalities hold:

$$\begin{aligned} &\left\| \int_\Omega \langle f(t), g(t)\rangle \, d\mu(t) - \left\langle \int_\Omega f(t)\, d\mu(t), \int_\Omega g(t)\, d\mu(t) \right\rangle \right\| \\ &\quad \leq \left\| \int_\Omega \left| f(t) \right|^2 d\mu(t) - \left| \int_\Omega f(t)\, d\mu(t) \right|^2 \right\|^{\frac{1}{2}} \left\| \int_\Omega \left| g(t) \right|^2 d\mu(t) - \left| \int_\Omega g(t)\, d\mu(t) \right|^2 \right\|^{\frac{1}{2}} \\ &\quad \leq \left\| \frac{1}{4} \left| x' - x \right|^2 - \int_\Omega \operatorname{Re}\langle x' - f(t), f(t) - x\rangle \, d\mu(t) \right\|^{\frac{1}{2}} \\ &\qquad \times \left\| \frac{1}{4} \left| y' - y \right|^2 - \int_\Omega \operatorname{Re}\langle y' - g(t), g(t) - y\rangle \, d\mu(t) \right\|^{\frac{1}{2}} \\ &\quad \leq \frac{1}{4} \left\| x' - x \right\| \left\| y' - y \right\|. \end{aligned} \tag{3.11}$$

The coefficient 1 *in the second inequality and the constant $\frac{1}{4}$ in the last inequality are sharp in the sense that they cannot be replaced by a smaller quantity.*

Proof Since equation (3.5) is a generalized semi-inner product on $L_2(\Omega, X)$, the Schwarz inequality holds, *i.e.*,

$$\left\| [f, g] \right\|^2 \leq \left\| [f, f] \right\| \left\| [g, g] \right\|. \tag{3.12}$$

Using equation (3.2) with $a = \frac{x + x'}{2}$ and equation (3.6) we get

$$\begin{aligned} [f, f] &= \int_\Omega \left| f(t) \right|^2 d\mu(t) - \left| \int_\Omega f(t)\, d\mu(t) \right|^2 \\ &\leq \int_\Omega \left| f(t) - \frac{x' + x}{2} \right|^2 d\mu(t) \\ &= \frac{1}{4} \left| x' - x \right|^2 - \int_\Omega \operatorname{Re}\langle x' - f(t), f(t) - x\rangle \, d\mu(t) \\ &\leq \frac{1}{4} \left| x' - x \right|^2. \end{aligned} \tag{3.13}$$

Similarly,

$$\begin{aligned}[g,g] &= \int_\Omega \left|g(t)\right|^2 d\mu(t) - \left|\int_\Omega g(t)\,d\mu(t)\right|^2 \\ &\leq \int_\Omega \left|g(t) - \frac{y'+y}{2}\right|^2 d\mu(t) \\ &= \frac{1}{4}\left|y'-y\right|^2 - \int_\Omega \operatorname{Re}\langle y' - g(t), g(t) - y\rangle\,d\mu(t) \\ &\leq \frac{1}{4}\left|y'-y\right|^2. \end{aligned} \tag{3.14}$$

By the Schwarz inequality (3.12) and the inequalities (3.13) and (3.14) we deduce equation (3.11).

Now, suppose that equation (3.11) holds with the constants $C, D > 0$ in the third and fourth inequalities. That is,

$$\begin{aligned}&\left\|\int_\Omega \langle f(t), g(t)\rangle\,d\mu(t) - \left\langle \int_\Omega f(t)\,d\mu(t), \int_\Omega g(t)\,d\mu(t)\right\rangle\right\| \\ &\quad\leq C\left\|\frac{1}{4}\left|x'-x\right|^2 - \int_\Omega \operatorname{Re}\langle x' - f(t), f(t) - x\rangle\,d\mu(t)\right\|^{\frac{1}{2}} \\ &\qquad\times \left\|\frac{1}{4}\left|y'-y\right|^2 - \int_\Omega \operatorname{Re}\langle y' - g(t), g(t) - y\rangle\,d\mu(t)\right\|^{\frac{1}{2}} \\ &\quad\leq D\left\|x'-x\right\|\left\|y'-y\right\|. \end{aligned} \tag{3.15}$$

Every Hilbert space H can be regarded as a Hilbert $\mathbb{C}$-module. If we choose $\Omega = [0,1] \subseteq \mathbb{R}$, $X = \mathbb{C}$, $f, g : [0,1] \to \mathbb{R} \subseteq X$,

$$f(t) = g(t) = \begin{cases} -1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \tag{3.16}$$

then for $x' = y' = 1$, $x = y = -1$ and μ a Lebesgue measure on Ω, the conditions (3.10) hold. By equation (3.15) we deduce

$$1 \leq C \leq 4D,$$

giving $C \geq 1$ and $D \geq \frac{1}{4}$, and the theorem is proved. □

4 Applications

1. Let X be a Hilbert C^*-module and $\mathcal{B}(X)$ the set of all adjoinable operators on X. We recall that if $A \in \mathcal{B}(X)$ then its operator norm is defined by

$$\|A\| = \sup\left\{\|Ax\| : x \in X, \|x\| \leq 1\right\},$$

with this norm $\mathcal{B}(X)$ is a C^*-algebra.

Let $\Omega = [0,1]$ and $f(t) = e^{tA}$ for $t \in \Omega$, where A is an invertible element in $\mathcal{B}(X)$. Since for each $t \in [0,1]$ one has

$$\|e^{tA}\| \le e^{t\|A\|} \le e^{\|A\|},$$

an application of the first inequality in equation (3.13) for $x' = 2e^A$, $x = -e^A$ gives

$$0 \le \int_0^1 |e^{tA}|^2\,dt - \left|\int_0^1 e^{tA}\,dt\right|^2 \le \frac{9}{4}|e^A|^2.$$

This implies that

$$\int_0^1 |e^{tA}|^2\,dt \le \frac{9}{4}|e^A|^2 + |A^{-1}(e^A - I)|^2.$$

2. For square integrable functions f and g on $[0,1]$ and

$$D(f,g) = \int_0^1 f(t)g(t)\,dt - \int_0^1 f(t)\,dt \int_0^1 g(t)\,dt$$

Landau proved [14]

$$|D(f,g)| \le \sqrt{D(f,f)}\sqrt{D(g,g)}.$$

Jocić *et al.* in [9] have proved for a probability measure μ and for square integrable fields $(\mathcal{A}_t)$ and $(\mathcal{B}_t)$ $(t \in \Omega)$ of commuting normal operators that the following Landau-type inequality holds:

$$\begin{aligned}
&\left|\!\left|\!\left| \int_\Omega \mathcal{A}_t X \mathcal{B}_t\,d\mu(t) - \int_\Omega \mathcal{A}_t\,d\mu(t) X \int_\Omega \mathcal{B}_t\,d\mu(t) \right|\!\right|\!\right| \\
&\le \left|\!\left|\!\left| \sqrt{\int_\Omega |\mathcal{A}_t|^2\,d\mu(t) - \left|\int_\Omega \mathcal{A}_t\,d\mu(t)\right|^2} X \sqrt{\int_\Omega |\mathcal{B}_t|^2\,d\mu(t) - \left|\int_\Omega \mathcal{B}_t\,d\mu(t)\right|^2} \right|\!\right|\!\right|
\end{aligned}$$

for all $X \in B(H)$ and for all unitarily invariant norms $|||\cdot|||$.

Every C^*-algebra can be regarded as a Hilbert C^*-module over itself with the inner product defined by $\langle a, b\rangle = a^*b$. If we apply the first inequality in equation (3.11) of Theorem 3, we obtain the following result.

Corollary 1 *Let $\mathcal{A}$ be a C^*-algebra, μ a probability measure on Ω. If f, g belong to $L_2(\Omega, \mathcal{A})$, then the following inequality holds*:

$$\begin{aligned}
&\left\| \int_\Omega f(t)g(t)\,d\mu(t) - \int_\Omega f(t)\,d\mu(t) \int_\Omega g(t)\,d\mu(t) \right\| \\
&\quad \le \left\| \int_\Omega |f(t)|^2\,d\mu(t) - \left|\int_\Omega f(t)\,d\mu(t)\right|^2 \right\|^{\frac{1}{2}} \left\| \int_\Omega |g(t)|^2\,d\mu(t) - \left|\int_\Omega g(t)\,d\mu(t)\right|^2 \right\|^{\frac{1}{2}}. \qquad (4.1)
\end{aligned}$$

Competing interests

The author declares that he has no competing interests.

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