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M^2-Type sharp estimates and boundedness on a Morrey space for Toeplitz-type operators associated to singular integral operators satisfying a variant of Hörmander's condition

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Abstract

In this paper, we prove the M^2-type sharp maximal function estimates for the Toeplitz-type operators associated to certain singular integral operators satisfying a variant of Hörmander's condition. As an application, we obtain the weighted boundedness of the operators on the Lebesgue and Morrey spaces.

MSC: 42B20; 42B25

Keywords: Toeplitz-type operator; singular integral operator; sharp maximal function; *BMO*; Morrey space

1 Introduction

As the development of singular integral operators (see [1, 2]), their commutators have been well studied. In [3, 4], the authors prove that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [5]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [6, 7], some Toeplitz-type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by *BMO* and Lipschitz functions is obtained. In [6], some singular integral operators satisfying a variant of Hörmander's condition are introduced, and the boundedness for the operators is obtained (see [6], [20]). In this paper, we prove the sharp maximal function inequalities for the Toeplitz-type operator related to some singular integral operators satisfying a variant of Hörmander's condition. As an application, we obtain the weighted boundedness of the Toeplitz-type operator on Lebesgue and Morrey spaces.

2 Preliminaries

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left| f(y) - f_Q \right| dy,$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)\,dx$. We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and define $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$. It has been known that (see [2])

$$\|f - f_{2^kQ}\|_{BMO} \le Ck\|f\|_{BMO}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

For $\eta > 0$, let $M_\eta(f) = M(|f|^\eta)^{1/\eta}$. For $k \in N$, we denote by M^k the operator M iterated k times, *i.e.*, $M^1(f) = M(f)$ and

$$M^k(f) = M\big(M^{k-1}(f)\big) \quad \text{when } k \ge 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ. We denote the Φ-average by, for a function f,

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \le 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x\in Q}\|f\|_{\Phi,Q}.$$

The Young functions to be used in this paper are $\Phi(t) = t(1 + \log t)$ and $\tilde{\Phi}(t) = \exp(t)$, the corresponding average and maximal functions denoted by $\|\cdot\|_{L(\log L),Q}$, $M_{L(\log L)}$ and $\|\cdot\|_{\exp L,Q}$, $M_{\exp L}$. Following [2], we know that the generalized Hölder inequality and the following inequalities hold:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|\,dy \le \|f\|_{\Phi,Q}\|g\|_{\tilde{\Phi},Q},$$

$$\|f\|_{L(\log L),Q} \le M_{L(\log L)}(f) \le CM^2(f),$$

$$\|f - f_Q\|_{\exp L,Q} \le C\|f\|_{BMO}$$

and

$$\|f - f_Q\|_{\exp L,2^kQ} \le Ck\|f\|_{BMO}.$$

The A_p weight is defined by (see [1])

$$A_p = \left\{w \in L^1_{\mathrm{loc}}(R^n) : \sup_Q\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1} < \infty\right\},$$

$$1 < p < \infty,$$

and

$$A_1 = \{w \in L^p_{\mathrm{loc}}(R^n) : M(w)(x) \le Cw(x), \text{a.e.}\}.$$

Given a weight function w, for $1 \le p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty.$$

Definition 1 Let $\Phi = \{\phi_1, \dots, \phi_l\}$ be a finite family of bounded functions in R^n. For any locally integrable function f, the Φ sharp maximal function of f is defined by

$$M^{\#}_{\Phi}(f)(x) = \sup_{Q \ni x} \inf_{\{c_1,\dots,c_l\}} \frac{1}{|Q|} \int_Q \left| f(y) - \sum_{i=1}^{l} c_i \phi_i(x_Q - y)\right| dy,$$

where the infimum is taken over all m-tuples $\{c_1, \dots, c_l\}$ of complex numbers and x_Q is the center of Q. For $\eta > 0$, let

$$M^{\#}_{\Phi,\eta}(f)(x) = \sup_{Q \ni x} \inf_{\{c_1,\dots,c_l\}} \left(\frac{1}{|Q|} \int_Q \left| f(y) - \sum_{i=1}^{l} c_j \phi_i(x_Q - y)\right|^{\eta} dy\right)^{1/\eta}.$$

Remark We note that $M^{\#}_{\Phi} \approx f^{\#}$ if $l = 1$ and $\phi_1 = 1$.

Definition 2 Given a positive and locally integrable function f in R^n, we say that f satisfies the reverse Hölder condition (write this as $f \in RH_\infty(R^n)$) if for any cube Q centered at the origin, we have

$$0 < \sup_{x \in Q} f(x) \le C \frac{1}{|Q|} \int_Q f(y)\,dy.$$

Definition 3 Let φ be a positive, increasing function on R^+, and there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \quad \text{for } t \ge 0.$$

Let w be a weight function and f be a locally integrable function on R^n. Set, for $1 \le p < \infty$,

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{x \in R^n,\, d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p w(y)\,dy\right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x - y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n, w) = \{f \in L^1_{\mathrm{loc}}(R^n) : \|f\|_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\eta$, $\eta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\eta}(R^n, w)$, which is the classical weighted Morrey spaces (see [8, 9]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, which is the weighted Lebesgue spaces (see [6]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [5, 8–11]).

In this paper, we study some singular integral operators as follows (see [12]).

Definition 4 Let $K \in L^2(R^n)$ and satisfy

$$\|K\|_{L^\infty} \le C,$$
$$\big|K(x)\big| \le C|x|^{-n},$$

there exist functions $B_1, \dots, B_l \in L^1_{\mathrm{loc}}(R^n - \{0\})$ and $\Phi = \{\phi_1, \dots, \phi_l\} \subset L^\infty(R^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$, and for a fixed $\delta > 0$ and any $|x| > 2|y| > 0$,

$$\left|K(x-y) - \sum_{i=1}^{l} B_i(x)\phi_i(y)\right| \le C\frac{|y|^\delta}{|x-y|^{n+\delta}}.$$

For $f \in C_0^\infty$, we define the singular integral operator related to the kernel K by

$$T(f)(x) = \int_{R^n} K(x-y)f(y)\,dy.$$

Moreover, let b be a locally integrable function on R^n. The Toeplitz-type operator related to T is defined by

$$T^b = \sum_{j=1}^{m} T^{j,1} M_b T^{j,2},$$

where $T^{j,1}$ are T or $\pm I$ (the identity operator), $T^{j,2}$ are the bounded linear operators on $L^p(w)$ for $1 < p < \infty$ and $w \in A_1$, $j = 1, \dots, m$, $M_b(f) = bf$.

Remark Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 4 (see [13], [19]). Also note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz-type operators T^b. The Toeplitz-type operators T^b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [12, 14]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz-type operator T^b. As the application, we obtain the weighted L^p-norm inequality and Morrey space boundedness for the Toeplitz-type operators T^b.

3 Theorems and lemmas

We shall prove the following theorems.

Theorem 1 *Let T be the singular integral operator as Definition* 4, $0 < r < 1$ *and* $b \in BMO(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^{\#}_{\Phi,r}\big(T^b(f)\big)(\tilde{x}) \le C\|b\|_{BMO} \sum_{j=1}^{m} M^2\big(T^{j,2}(f)\big)(\tilde{x}).$$

Theorem 2 *Let T be the singular integral operator as Definition* 4, $1 < p < \infty$, $w \in A_1$ *and* $b \in BMO(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$, *then* T^b *is bounded on* $L^p(w)$.

Theorem 3 *Let T be the singular integral operator as Definition* 4, $0 < D < 2^n$, $1 < p < \infty$, $w \in A_1$ *and* $b \in BMO(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$, *then* T^b *is bounded on* $L^{p,\varphi}(R^n, w)$.

To prove the theorems, we need the following lemmas.

Lemma 1 ([1, p.485]) *Let* $0 < p < q < \infty$ *and for any function* $f \geq 0$. *We define that for* $1/r = 1/p - 1/q$,

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda \left|\{x \in R^n : f(x) > \lambda\}\right|^{1/q},$$

$$N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p} / \|\chi_E\|_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. *Then*

$$\|f\|_{WL^q} \leq N_{p,q}(f) \leq \left(q/(q-p)\right)^{1/p} \|f\|_{WL^q}.$$

Lemma 2 (see [2]) *We have*

$$\frac{1}{|Q|} \int_Q \left|f(x)g(x)\right| dx \leq \|f\|_{\exp L,Q} \|g\|_{L(\log L),Q}.$$

Lemma 3 (see [15]) *Let T be the singular integral operator as Definition* 4. *Then T is bounded on* $L^p(w)$ *for* $1 < p < \infty$, $w \in A_1$ *and weak* (L^1, L^1) *bounded*.

Lemma 4 (see [12]) *Let* $1 < p < \infty$, $0 < \eta < \infty$, $w \in A_\infty$ *and* $\Phi = \{\phi_1, \dots, \phi_l\} \subset L^\infty(R^n)$ *such that* $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$. *Then, for any smooth function f for which the left-hand side is finite*,

$$\int_{R^n} M_\eta(f)(x)^p w(x)\,dx \leq C \int_{R^n} M^{\#}_{\Phi,\eta}(f)(x)^p w(x)\,dx.$$

Lemma 5 (see [5, 11]) *Let* $1 < p < \infty$, $w \in A_1$ *and* $0 < D < 2^n$. *Then, for any smooth function f for which the left-hand side is finite*,

$$\left\|M(f)\right\|_{L^{p,\varphi}(w)} \leq C\|f\|_{L^{p,\varphi}(w)}.$$

Lemma 6 *Let* $1 < p < \infty$, $0 < \eta < \infty$, $w \in A_1$, $0 < D < 2^n$ *and* $\Phi = \{\phi_1, \dots, \phi_l\} \subset L^\infty(R^n)$ *such that* $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$. *Then, for any smooth function f for which the left-hand side is finite*,

$$\left\|M_\eta(f)\right\|_{L^{p,\varphi}(w)} \leq C\left\|M^{\#}_{\Phi,\eta}(f)\right\|_{L^{p,\varphi}(w)}.$$

Proof For any cube $Q = Q(x_0, d)$ in R^n, we know $M(w\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [3]. By Lemma 4, we have, for $f \in L^{p,\varphi}(R^n, w)$,

$$\begin{aligned}
&\int_Q \left|M_\eta(f)(y)\right|^p w(y)\,dy \\
&\quad = \int_{R^n} \left|M_\eta(f)(y)\right|^p w(y)\chi_Q(y)\,dy \\
&\quad \le \int_{R^n} \left|M_\eta(f)(y)\right|^p M(w\chi_Q)(y)\,dy \\
&\quad \le C\int_{R^n} \left|M^\#_{\Phi,\eta}(f)(y)\right|^p M(w\chi_Q)(y)\,dy \\
&\quad = C\left(\int_Q \left|M^\#_{\Phi,\eta}(f)(y)\right|^p M(w\chi_Q)(y)\,dy + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} \left|M^\#_{\Phi,\eta}(f)(y)\right|^p M(w\chi_Q)(y)\,dy\right) \\
&\quad \le C\left(\int_Q \left|M^\#_{\Phi,\eta}(f)(y)\right|^p w(y)\,dy + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} \left|M^\#_{\Phi,\eta}(f)(y)\right|^p \frac{w(Q)}{|2^{k+1}Q|}\,dy\right) \\
&\quad \le C\left(\int_Q \left|M^\#_{\Phi,\eta}(f)(y)\right|^p w(y)\,dy + \sum_{k=0}^\infty \int_{2^{k+1}Q} \left|M^\#_{\Phi,\eta}(f)(y)\right|^p \frac{M(w)(y)}{2^{n(k+1)}}\,dy\right) \\
&\quad \le C\left(\int_Q \left|M^\#_{\Phi,\eta}(f)(y)\right|^p w(y)\,dy + \sum_{k=0}^\infty \int_{2^{k+1}Q} \left|M^\#_{\Phi,\eta}(f)(y)\right|^p \frac{w(y)}{2^{nk}}\,dy\right) \\
&\quad \le C\left\|M^\#_{\Phi,\eta}(f)\right\|^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty 2^{-nk}\varphi\left(2^{k+1}d\right) \\
&\quad \le C\left\|M^\#_{\Phi,\eta}(f)\right\|^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty \left(2^{-n}D\right)^k\varphi(d) \\
&\quad \le C\left\|M^\#_{\Phi,\eta}(f)\right\|^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M_\eta(f)(x)^p w(x)\,dx\right)^{1/p} \le C\left(\frac{1}{\varphi(d)}\int_Q M^\#_{\Phi,\eta}(f)(x)^p w(x)\,dx\right)^{1/p}$$

and

$$\left\|M_\eta(f)\right\|_{L^{p,\varphi}(w)} \le C\left\|M^\#_{\Phi,\eta}(f)\right\|_{L^{p,\varphi}(w)}.$$

This finishes the proof. □

Lemma 7 *Let T be the singular integral operator as Definition* 3 *or the bounded linear operator on $L^r(w)$ for any $1 < r < \infty$ and $w \in A_1$, $1 < p < \infty$, $w \in A_1$ and $0 < D < 2^n$. Then*

$$\left\|T(f)\right\|_{L^{p,\varphi}(w)} \le C\|f\|_{L^{p,\varphi}(w)}.$$

The proof of the lemma is similar to that of Lemma 6 by Lemma 3, we omit the details.

4 Proofs of theorems

Proof of Theorem 1 It suffices to prove that for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^r dx\right)^{1/r} \le C\|b\|_{BMO}\sum_{j=1}^m M^2\left(T^{j,2}(f)\right)(\tilde{x}),$$

where Q is any cube centered at x_0, $C_0 = \sum_{j=1}^m \sum_{i=1}^l g_j^i \phi_i(x_0 - x)$ and $g_j^i = \int_{R^n} B_i(x_0 - y) M_{(b-b_Q)\chi_{(2Q)^c}} T^{j,2}(f)(y)\,dy$. Without loss of generality, we may assume $T^{j,1}$ are T ($j = 1, \ldots, m$). Let $\tilde{x} \in Q$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write

$$T^b(f)(x) = T^{b-b_{2Q}}(f)(x) = T^{(b-b_{2Q})\chi_{2Q}}(f)(x) + T^{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^r dx\right)^{1/r} \le C\left(\frac{1}{|Q|}\int_Q \left|f_1(x)\right|^r dx\right)^{1/r} + C\left(\frac{1}{|Q|}\int_Q \left|f_2(x) - C_0\right|^r dx\right)^{1/r} = I + II.$$

For I, by Lemmas 1, 2 and 3, we obtain

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|T^{j,1}M_{(b-b_{2Q})\chi_{2Q}}T^{j,2}(f)(x)\right|^r dx\right)^{1/r} \\
&\quad\le |Q|^{-1}\frac{\|T^{j,1}M_{(b-b_{2Q})\chi_{2Q}}T^{j,2}(f)\chi_Q\|_{L^r}}{|Q|^{1/r-1}} \\
&\quad\le C|Q|^{-1}\left\|T^{j,1}M_{(b-b_{2Q})\chi_{2Q}}T^{j,2}(f)\right\|_{WL^1} \\
&\quad\le C|Q|^{-1}\left\|M_{(b-b_{2Q})\chi_{2Q}}T^{j,2}(f)\right\|_{L^1} \\
&\quad\le C|Q|^{-1}\int_{2Q}\left|b(x) - b_{2Q}\right|\left|T^{j,2}(f)(x)\right| dx \\
&\quad\le C\|b - b_{2Q}\|_{\exp L,2Q}\left\|T^{j,2}(f)\right\|_{L(\log L),2Q} \\
&\quad\le C\|b\|_{BMO}M^2\left(T^{j,2}(f)\right)(\tilde{x}),
\end{aligned}$$

thus

$$I \le C\sum_{j=1}^m\left(\frac{1}{|Q|}\int_Q \left|T^{j,1}M_{(b-b_{2Q})\chi_{2Q}}T^{j,2}(f)(x)\right|^r dx\right)^{1/r} \le C\|b\|_{BMO}\sum_{j=1}^m M^2\left(T^{j,2}(f)\right)(\tilde{x}).$$

For II, we get, for $x \in Q$,

$$\begin{aligned}
&\left|T^{j,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{j,2}(f)(x) - \sum_{i=1}^l g_j^i\phi_i(x_0 - x)\right| \\
&\quad\le \left|\int_{R^n}\left(K(x-y) - \sum_{i=1}^l B_i(x_0 - y)\phi_i(x_0 - x)\right)\left(b(y) - b_{2Q}\right)\chi_{(2Q)^c}(y)T^{j,2}(f)(y)\,dy\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \left| K(x-y) - \sum_{i=1}^{l} B_i(x_0-y)\phi_i(x_0-x) \right| \left| b(y) - b_{2Q} \right| \left| T^{j,2}(f)(y) \right| dy \\
&\le C \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \frac{|x-x_0|^{\delta}}{|y-x_0|^{n+\delta}} \left| b(y) - b_{2Q} \right| \left| T^{j,2}(f)(y) \right| dy \\
&\le C \sum_{k=1}^{\infty} \frac{d^{\delta}}{(2^k d)^{n+\delta}} \left(2^k d\right)^n \|b - b_{2Q}\|_{\exp L, 2^{k+1} Q} \left\| T^{j,2}(f) \right\|_{L(\log L), 2^{k+1} Q} \\
&\le C \|b\|_{BMO} M^2\left(T^{j,2}(f)\right)(\tilde{x}) \sum_{k=1}^{\infty} k 2^{-k\delta} \\
&\le C \|b\|_{BMO} M^2\left(T^{j,2}(f)\right)(\tilde{x}),
\end{aligned}
$$

thus

$$
II \le \frac{1}{|Q|} \int_Q \sum_{j=1}^{m} \left| T^{j,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{j,2}(f)(x) - C_0 \right| dx \le C \|b\|_{BMO} \sum_{j=1}^{m} M^2\left(T^{j,2}(f)\right)(\tilde{x}).
$$

This completes the proof of Theorem 1. □

Proof of Theorem 2 By Theorem 1 and Lemmas 3-4, we have

$$
\begin{aligned}
\left\| T^b(f) \right\|_{L^p(w)} &\le \left\| M_r(\left(T^b(f)\right) \right\|_{L^p(w)} \le C \left\| M^{\#}_{\Phi,r}\left(T^b(f)\right) \right\|_{L^p(w)} \\
&\le C \|b\|_{BMO} \sum_{j=1}^{m} \left\| M^2\left(T^{j,2}(f)\right) \right\|_{L^p(w)} \le C \|b\|_{BMO} \sum_{j=1}^{m} \left\| T^{j,2}(f) \right\|_{L^p(w)} \\
&\le C \|b\|_{BMO} \|f\|_{L^p(w)}.
\end{aligned}
$$

This completes the proof. □

Proof of Theorem 3 By Theorem 1 and Lemmas 5-7, we have

$$
\begin{aligned}
\left\| T^b(f) \right\|_{L^{p,\varphi}(w)} &\le \left\| M_r\left(T^b(f)\right) \right\|_{L^{p,\varphi}(w)} \le C \left\| M^{\#}_{\Phi,r}\left(T^b(f)\right) \right\|_{L^{p,\varphi}(w)} \\
&\le C \|b\|_{BMO} \sum_{j=1}^{m} \left\| M^2\left(T^{j,2}(f)\right) \right\|_{L^{p,\varphi}(w)} \le C \|b\|_{BMO} \sum_{j=1}^{m} \left\| T^{j,2}(f) \right\|_{L^{p,\varphi}(w)} \\
&\le C \|b\|_{BMO} \|f\|_{L^{p,\varphi}(w)}.
\end{aligned}
$$

This completes the proof. □

Competing interests

The author declares that they have no competing interests.

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