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Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli

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Abstract

In this paper, the upper bound of the Hankel determinant $H_3(1)$ for a subclass of analytic functions associated with right half of the lemniscate of Bernoulli $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$ is investigated.

MSC: 30C45; 30C50

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1 Introduction and preliminaries

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$. A function f is said to be subordinate to a function g , written as $f \prec g$, if there exists a Schwartz function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if g is univalent in E , then $f(0) = g(0)$ and $f(E) \subset g(E)$.

Let P denote the class of analytic functions p normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (1.2)$$

such that $\operatorname{Re} p(z) > 0$. Let SL^* be the class of functions defined by

$$SL^* = \left\{ f \in A : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \right\}, \quad z \in E.$$

Thus a function $f \in SL^*$ is such that $\frac{zf'(z)}{f(z)}$ lies in the region bounded by the right half of the lemniscate of Bernoulli given by the relation $|w^2 - 1| < 1$. It can easily be seen that $f \in SL^*$ if it satisfies the condition

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \quad z \in E. \quad (1.3)$$

This class of functions was introduced by Sokół and Stankiewicz [1] and further investigated by some authors. For details, see [2, 3].

Noonan and Thomas [4] have studied the q th Hankel determinant defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{1.4}$$

where $n \geq 1$ and $q \geq 1$. The Hankel determinant plays an important role in the study of singularities; for instance, see [5, p.329] and Edrei [6]. This is also important in the study of power series with integral coefficients [5, p.323] and Cantor [7]. For the use of the Hankel determinant in the study of meromorphic functions, see [8], and various properties of these determinants can be found in [9, Chapter 4]. It is well known that the Fekete-Szegő functional $|a_3 - a_2^2| = H_2(1)$. This functional is further generalized as $|a_3 - \mu a_2^2|$ for some μ (real as well as complex). Fekete and Szegő gave sharp estimates of $|a_3 - \mu a_2^2|$ for μ real and $f \in S$, the class of univalent functions. It is a very great combination of the two coefficients which describes the area problems posted earlier by Gronwall in 1914-15. Moreover, we also know that the functional $|a_2 a_4 - a_3^2|$ is equivalent to $H_2(2)$. The q th Hankel determinant for some subclasses of analytic functions was recently studied by Arif *et al.* [10] and Arif *et al.* [11]. The functional $|a_2 a_4 - a_3^2|$ has been studied by many authors, see [12-14]. Babalola [15] studied the Hankel determinant $H_3(1)$ for some subclasses of analytic functions. In the present investigation, we determine the upper bounds of the Hankel determinant $H_3(1)$ for a subclass of analytic functions related with lemniscate of Bernoulli by using Toeplitz determinants.

We need the following lemmas which will be used in our main results.

Lemma 1.1 [16] *Let $p \in P$ and of the form (1.2). Then*

$$|p_2 - \nu p_1^2| \leq \begin{cases} -4\nu + 2, & \nu < 0, \\ 2, & 0 \leq \nu \leq 1, \\ 4\nu - 2, & \nu > 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if $p(z) = (\frac{1}{2} + \frac{\eta}{2})\frac{1+z}{1-z} + (\frac{1}{2} - \frac{\eta}{2})\frac{1-z}{1+z}$ ($0 \leq \eta \leq 1$) or one of its rotations. If $\nu = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$. Although the above upper bound is sharp, when $0 < \nu < 1$, it can be improved as follows:

$$|p_2 - \nu p_1^2| + \nu |p_1|^2 \leq 2 \quad (0 < \nu \leq 1/2)$$

and

$$|p_2 - \nu p_1^2| + (1 - \nu) |p_1|^2 \leq 2 \quad (1/2 < \nu \leq 1).$$

Lemma 1.2 [16] *If $p(z) = 1 + p_1z + p_2z^2 + \dots$ is a function with positive real part in E , then for v a complex number*

$$|p_2 - vp_1^2| \leq 2 \max(1, |2v - 1|).$$

This result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 1.3 [17] *Let $p \in P$ and of the form (1.2). Then*

$$2p_2 = p_1^2 + x(4 - p_1^2)$$

for some x , $|x| \leq 1$, and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some z , $|z| \leq 1$.

2 Main results

Although we have discussed the Hankel determinant problem in the paper, the first two problems are specifically related with the Fekete-Szegő functional, which is a special case of the Hankel determinant.

Theorem 2.1 *Let $f \in SL^*$ and of the form (1.1). Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{16}(1 - 4\mu), & \mu < -\frac{3}{4}, \\ \frac{1}{4}, & -\frac{3}{4} \leq \mu \leq \frac{5}{4}, \\ \frac{1}{16}(4\mu - 1), & \mu > \frac{5}{4}. \end{cases}$$

Furthermore, for $-\frac{3}{4} < \mu \leq \frac{1}{4}$,

$$|a_3 - \mu a_2^2| + \frac{1}{4}(4\mu + 3)|a_2|^2 \leq \frac{1}{4},$$

and for $\frac{1}{4} < \mu \leq \frac{5}{4}$,

$$|a_3 - \mu a_2^2| + \frac{1}{4}(5 - 4\mu)|a_2|^2 \leq \frac{1}{4}.$$

These results are sharp.

Proof If $f \in SL^*$, then it follows from (1.3) that

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \tag{2.1}$$

where $\phi(z) = \sqrt{1+z}$. Define a function

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1z + p_2z^2 + \dots$$

It is clear that $p \in P$. This implies that

$$w(z) = \frac{p(z)-1}{p(z)+1}.$$

From (2.1), we have

$$\frac{zf'(z)}{f(z)} = \phi(w(z)),$$

with

$$\phi(w(z)) = \left(\frac{2p(z)}{p(z)+1} \right)^{\frac{1}{2}}.$$

Now

$$\left(\frac{2p(z)}{p(z)+1} \right)^{\frac{1}{2}} = 1 + \frac{1}{4}p_1z + \left[\frac{1}{4}p_2 - \frac{5}{32}p_1^2 \right]z^2 + \left[\frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3 \right]z^3 + \dots$$

Similarly,

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + [2a_3 - a_2^2]z^2 + [3a_4 - 3a_2a_3 + a_2^3]z^3 + \dots$$

Therefore

$$a_2 = \frac{1}{4}p_1, \tag{2.2}$$

$$a_3 = \frac{1}{8}p_2 - \frac{3}{64}p_1^2, \tag{2.3}$$

$$a_4 = \frac{1}{12}p_3 - \frac{7}{96}p_1p_2 + \frac{13}{768}p_1^3. \tag{2.4}$$

This implies that

$$|a_3 - \mu a_2^2| = \frac{1}{8} \left| p_2 - \frac{1}{8}(4\mu + 3)p_1^2 \right|.$$

Now, using Lemma 1.1, we have the required result. □

The results are sharp for the functions $K_i(z)$, $i = 1, 2, 3, 4$, such that

$$\frac{zK_1'(z)}{K_1(z)} = \sqrt{1+z} \quad \text{if } \mu < -\frac{3}{4} \text{ or } \mu > \frac{5}{4},$$

$$\frac{zK_2'(z)}{K_2(z)} = \sqrt{1+z^2} \quad \text{if } -\frac{3}{4} < \mu < \frac{5}{4},$$

$$\frac{zK_3'(z)}{K_3(z)} = \sqrt{1 + \Phi(z)} \quad \text{if } \mu = -\frac{3}{4},$$

$$\frac{zK_4'(z)}{K_4(z)} = \sqrt{1 - \Phi(z)} \quad \text{if } \mu = \frac{5}{4},$$

where $\Phi(z) = \frac{z(z+\eta)}{1+\eta z}$ with $0 \leq \eta \leq 1$.

Theorem 2.2 *Let $f \in SL^*$ and of the form (1.1). Then for a complex number μ ,*

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} \max \left\{ 1; \left| \mu - \frac{1}{4} \right| \right\}.$$

Proof Since

$$|a_3 - \mu a_2^2| = \frac{1}{8} \left| p_2 - \frac{1}{8}(4\mu + 3)p_1^2 \right|,$$

therefore, using Lemma 1.2, we get the result. This result is sharp for the functions

$$\frac{zf'(z)}{f(z)} = \sqrt{1+z}$$

or

$$\frac{zf'(z)}{f(z)} = \sqrt{1+z^2}.$$

□

For $\mu = 1$, we have $H_2(1)$.

Corollary 2.3 *Let $f \in SL^*$ and of the form (1.1). Then*

$$|a_3 - a_2^2| \leq \frac{1}{4}.$$

Theorem 2.4 *Let $f \in SL^*$ and of the form (1.1). Then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{16}.$$

Proof From (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned} a_2 a_4 - a_3^2 &= \frac{1}{48} \left(p_1 p_3 - \frac{7}{8} p_1^2 p_2 + \frac{13}{64} p_1^4 \right) - \left(\frac{1}{8} p_2 - \frac{3}{64} p_1^2 \right)^2 \\ &= \frac{1}{48} p_1 p_3 - \frac{1}{64} p_2^2 - \frac{5}{768} p_1^2 p_2 + \frac{25}{12,288} p_1^4 \\ &= \frac{1}{12,288} (256 p_1 p_3 - 192 p_2^2 - 80 p_1^2 p_2 + 25 p_1^4). \end{aligned}$$

Putting the values of p_2 and p_3 from Lemma 1.3, we assume that $p > 0$, and taking $p_1 = p \in [0, 2]$, we get

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{1}{12,288} |64 p_1 \{ p_1^3 + 2(4 - p_1^2) p_1 x - (4 - p_1^2) p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2) z \} \\ &\quad - 48 \{ p_1^2 + x(4 - p_1^2) \}^2 - 40 p_1^2 \{ p_1^2 + x(4 - p_1^2) \} + 25 p_1^4|. \end{aligned}$$

After simple calculations, we get

$$|a_2a_4 - a_3^2| = \frac{1}{12,288} |41p^4 - 8(4 - p^2)p^2x - 128(4 - p^2)(1 - |x|^2)z + x^2(4 - p^2)(64p^2 + 48)(4 - p^2)|.$$

Now, applying the triangle inequality and replacing $|x|$ by ρ , we obtain

$$|a_2a_4 - a_3^2| \leq \frac{1}{12,288} [41p^4 + 128(4 - p^2) + 8(4 - p^2)p^2\rho + \rho^2(4 - p^2)(16p^2 + 64)] = F(p, \rho) \quad (\text{say}).$$

Differentiating with respect to ρ , we have

$$\frac{\partial F(p, \rho)}{\partial \rho} = \frac{1}{12,288} [8(4 - p^2)p^2 + 2\rho(4 - p^2)(16p^2 + 64)].$$

It is clear that $\frac{\partial F(p, \rho)}{\partial \rho} > 0$, which shows that $F(p, \rho)$ is an increasing function on the closed interval $[0, 1]$. This implies that maximum occurs at $\rho = 1$. Therefore $\max F(p, \rho) = F(p, 1) = G(p)$ (say). Now

$$G(p) = \frac{1}{12,288} [17p^4 - 96p^2 + 768].$$

Therefore

$$G'(p) = \frac{1}{12,288} [68p^3 - 192p]$$

and

$$G''(p) = \frac{1}{12,288} [204p^2 - 192] < 0$$

for $p = 0$. This shows that maximum of $G(p)$ occurs at $p = 0$. Hence, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{768}{12,288} = \frac{1}{16}.$$

This result is sharp for the functions

$$\frac{zf'(z)}{f(z)} = \sqrt{1+z}$$

or

$$\frac{zf'(z)}{f(z)} = \sqrt{1+z^2}.$$

□

Theorem 2.5 *Let $f \in SL^*$ and of the form (1.1). Then*

$$|a_2a_3 - a_4| \leq \frac{1}{6}.$$

Proof Since

$$\begin{aligned} a_2 &= \frac{1}{4}p_1, \\ a_3 &= \frac{1}{8}p_2 - \frac{3}{64}p_1^2, \\ a_4 &= \frac{1}{12}p_3 - \frac{7}{96}p_1p_2 + \frac{13}{768}p_1^2. \end{aligned}$$

Therefore, by using Lemma 1.3, we can obtain

$$|a_2a_3 - a_4| \leq \frac{1}{768} \{7p^3 + 8p\rho(4 - p^2) + 32(4 - p^2) + 16\rho^2(p - 2)(4 - p^2)\}.$$

Let

$$F_1(p, \rho) = \frac{1}{768} \{7p^3 + 8p\rho(4 - p^2) + 32(4 - p^2) + 16\rho^2(p - 2)(4 - p^2)\}. \tag{2.5}$$

We assume that the upper bound occurs at the interior point of the rectangle $[0, 2] \times [0, 1]$. Differentiating (2.5) with respect to ρ , we get

$$\frac{\partial F_1}{\partial \rho} = \frac{1}{768} \{8p(4 - p^2) + 32\rho(p - 2)(4 - p^2)\}.$$

For $0 < \rho < 1$ and fixed $p \in (0, 2)$, it can easily be seen that $\frac{\partial F_1}{\partial \rho} < 0$. This shows that $F_1(p, \rho)$ is a decreasing function of ρ , which contradicts our assumption; therefore, $\max F_1(p, \rho) = F_1(p, 0) = G_1(p)$. This implies that

$$G'_1(p) = \frac{1}{768} \{21p^2 - 64p\}$$

and

$$G''_1(p) = \frac{1}{768} \{42p - 64\} < 0$$

for $p = 0$. Therefore $p = 0$ is a point of maximum. Hence, we get the required result. \square

Lemma 2.6 *If the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ belongs to the class SL^* , then*

$$|a_2| \leq 1/2, \quad |a_3| \leq 1/4, \quad |a_4| \leq 1/6, \quad |a_5| \leq 1/8.$$

These estimations are sharp. The first three bounds were obtained by Sokół [3] and the bound for $|a_5|$ can be obtained in a similar way.

Theorem 2.7 *Let $f \in SL^*$ and of the form (1.1). Then*

$$|H_3(1)| \leq \frac{43}{576}.$$

Proof Since

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_1a_3 - a_2^2).$$

Now, using the triangle inequality, we obtain

$$|H_3(1)| \leq |a_3| |a_2a_4 - a_3^2| + |a_4| |a_2a_3 - a_4| + |a_5| |a_1a_3 - a_2^2|.$$

Using the fact that $a_1 = 1$ with the results of Corollary 2.3, Theorem 2.4, Theorem 2.5 and Lemma 2.6, we obtain

$$|H_3(1)| \leq \frac{43}{576}. \quad \square$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

MR and SNM jointly discussed and presented the ideas of this article. MR made the text file and all the communications regarding the manuscript. All authors read and approved the final manuscript.

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