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On a more accurate half-discrete mulholland's inequality and an extension

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Abstract

By using the way of weight functions and Jensen-Hadamard's inequality, a more accurate half-discrete Mulholland's inequality with a best constant factor is given. The extension with multi-parameters, the equivalent forms as well as the operator expressions are considered.

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1 Introduction

Assuming that $f, g \in L^2(R_+),\ \|f\| = \left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}} < 0,\ \|g\| < 0$' we have the following Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dxdy < \pi\,\|f\|\,\|g\|\,, \tag{1}$$

where the constant factor π is the best possible. Moreover, for $a = \{a_m\}_{m=1}^\infty \in l^2, b = \{b_n\}_{n=1}^\infty \in l^2, \|a\| = \left\{\sum_{m=1}^\infty a_m^2\right\}^{\frac{1}{2}} < 0, \|b\| > 0$' we still have the following discrete Hilbert's inequality

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \pi\,\|a\|\,\|b\|, \tag{2}$$

with the same best constant factor π. Inequalities (1) and (2) are important in analysis and its applications (cf. [2-4]) and they still represent the field of interest to numerous mathematicians. Also we have the following Mulholland's inequality with the same best constant factor (cf. [1,5]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \pi \left\{\sum_{m=2}^\infty m a_m^2 \sum_{n=2}^\infty n b_n^2\right\}^{\frac{1}{2}}. \tag{3}$$

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In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [6] gave an extension of (1). By generalizing the results from [6], Yang [7] gave some best extensions of (1) and (2) as follows: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda$ satisfying $k(\lambda_1) = \int_0^\infty k_\lambda(t, 1) t^{\lambda_1 - 1} dt \in R_+$, $\phi(x) = x^{p(1-\lambda_1)-1}, \psi(x) = x^{q(1-\lambda_2)-1}, f(\geq 0) \in L_{p,\phi}(R_+) = \left\{ f \mid \|f\|_{p,\phi} := \left\{ \int_0^\infty \phi(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\}, g(\geq 0) \in L_{q,\psi}(R_+), \|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$ then

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dxdy < k(\lambda_1) \|f\|_{p,\phi} \|g\|_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover if $k_\lambda(x, y)$ is finite and $k_\lambda(x, y) x^{\lambda_1 - 1} \left(k_\lambda(x, y) y^{\lambda_2 - 1}\right)$ is decreasing for $x > 0 (y > 0)$, then for $a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} := \left\{ a \mid \|a\|_{p,\phi} := \left\{ \sum_{n=1}^\infty \phi(n) |a_n|^p \right\}^{\frac{1}{p}} < \infty \right\}, b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}, \|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$ we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m, n) a_m b_n < k(\lambda_1) \|a\|_{p,\phi} \|b\|_{q,\psi}, \tag{5}$$

where, $k(\lambda_1)$ is still the best value. Clearly, for $p = q = 2, \lambda = 1, k_1(x, y) = \frac{1}{x + y}, \lambda_1 = \lambda_2 = \frac{1}{2}$, inequality (4) reduces to (1), while (5) reduces to (2). Some other results about Hilbert-type inequalities are provided by [8-16].

On half-discrete Hilbert-type inequalities with the general non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the the constant factors in the inequalities are the best possible. However Yang [17] gave a result with the kernel $\frac{1}{(1 + nx)^\lambda}$ by introducing an interval variable and proved that the constant factor is the best possible. Recently, Yang [18] gave the following half-discrete Hilbert's inequality with the best constant factor $B(\lambda_1, \lambda_2)(\lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda)$:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x + n)^\lambda} dx < B(\lambda_1, \lambda_2) \|f\|_{p,\phi} \|a\|_{q,\psi}. \tag{6}$$

In this article, by using the way of weight functions and Jensen-Hadamard's inequality, a more accurate half-discrete Mulholland's inequality with a best constant factor similar to (6) is given as follows:

$$\int_{\frac{3}{2}}^\infty f(x) \sum_{n=2}^\infty \frac{a_n}{\ln \frac{4}{9} xn} dx < \pi \left\{ \int_{\frac{3}{2}}^\infty x f^2(x) dx \sum_{n=2}^\infty n a_n^2 \right\}^{\frac{1}{2}}. \tag{7}$$

Moreover, a best extension of (7) with multi-parameters, some equivalent forms as well as the operator expressions are also considered.

2 Some lemmas

Lemma 1 *If* $\lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda, \alpha \geq \frac{4}{9}$, *setting weight functions* $\omega(n)$ *and* $\varpi(x)$ *as follows:*

$$\omega(n) := \left(\ln\sqrt{\alpha}n\right)^{\lambda_2} \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{\left(\ln\sqrt{\alpha}x\right)^{\lambda_1-1}}{x(\ln\alpha xn)^{\lambda}} dx, \quad n \in \mathbf{N}\backslash\{1\}, \tag{8}$$

$$\varpi(x) := \left(\ln\sqrt{\alpha}x\right)^{\lambda_1} \sum_{n=2}^{\infty} \frac{\left(\ln\sqrt{\alpha}n\right)^{\lambda_2-1}}{n(\ln\alpha xn)^{\lambda}}, \quad x \in \left(\frac{1}{\sqrt{\alpha}}, \infty\right), \tag{9}$$

then we have

$$\varpi(x) < \omega(n) = B(\lambda_1, \lambda_2). \tag{10}$$

Proof. Applying the substitution $t = \frac{\ln\sqrt{\alpha}x}{\ln\sqrt{\alpha}n}$ to (8), we obtain

$$\omega(n) = \int_0^{\infty} \frac{1}{(1+t)^{\lambda}} t^{\lambda_1-1} dt = B(\lambda_1, \lambda_2).$$

Since by the conditions and for fixed $x \geq \frac{1}{\sqrt{\alpha}}$,

$$h(x, y) := \frac{\left(\ln\sqrt{\alpha}y\right)^{\lambda_2-1}}{y\left(\ln\alpha xy\right)^{\lambda}} = \frac{1}{y\left(\ln\sqrt{\alpha}x + \ln\sqrt{\alpha}y\right)^{\lambda}\left(\ln\sqrt{\alpha}y\right)^{1-\lambda_2}}$$

is decreasing and strictly convex in $y \in \left(\frac{3}{2}, \infty\right)$, then by Jensen-Hadamard's inequality (cf. [1]), we find

$$\varpi(x) < \left(\sqrt{\alpha}\ln x\right)^{\lambda_1} \int_{\frac{3}{2}}^{\infty} \frac{1}{y\left(\ln\alpha xy\right)^{\lambda}} \left(\ln\sqrt{\alpha}y\right)^{\lambda_2-1} dy$$

$$\overset{t = (\ln\sqrt{\alpha}y)/(\ln\sqrt{\alpha}x)}{=} \int_{\frac{\ln\left(3\sqrt{\alpha}/2\right)}{\ln\sqrt{\alpha}x}}^{\infty} \frac{t^{\lambda_2-1} dt}{(1+t)^{\lambda}} \leq B(\lambda_2, \lambda_1) B(\lambda_1, \lambda_2),$$

namely, (10) follows. □ ▪

Lemma 2 *Let the assumptions of Lemma* 1 *be fulfilled and additionally,* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n \geq 0, \in \mathbf{N}\backslash\{1\}, f(x)$*is a non-negative measurable function in* $\left(\frac{1}{\sqrt{\alpha}}, \infty\right)$. *Then we have the following inequalities:*

$$J := \left\{ \sum_{n=2}^{\infty} \frac{\left(\ln \sqrt{\alpha} n\right)^{p\lambda_2 - 1}}{n} \left[\int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{f(x)}{(\ln \alpha x n)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} \le [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \varpi(x) x^{p-1} \left(\ln \sqrt{\alpha} x\right)^{p(1-\lambda_1)} f^p(x)\, dx \right\}^{\frac{1}{p}}, \tag{11}$$

$$L_1 := \left\{ \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{\left(\ln \sqrt{\alpha} x\right)^{q\lambda_1 - 1}}{x[\varpi(x)]^{q-1}} \left[\sum_{n=2}^{\infty} \frac{a_n}{(\ln \alpha x n)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} \le \left\{ B(\lambda_1, \lambda_2) \sum_{n=2}^{\infty} n^{q-1} \left(\ln \sqrt{\alpha} n\right)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}. \tag{12}$$

Proof. By Hälder's inequality cf. [1] and (10), it follows

$$\begin{aligned}
\left[\int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{f(x)\,dx}{(\ln \alpha x n)^{\lambda}} \right]^p &= \left\{ \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{1}{(\ln \alpha x n)^{\lambda}} \left[\frac{\left(\ln \sqrt{\alpha} x\right)^{(1-\lambda_1)/q} x^{1/q}}{\left(\ln \sqrt{\alpha} n\right)^{(1-\lambda_2)/p} n^{1/p}} f(x) \right] \right. \\
&\quad \left. \times \left[\frac{\left(\ln \sqrt{\alpha} n^{(1-\lambda_2)/p} n^{1/p}\right)}{\left(\ln \sqrt{\alpha} x^{(1-\lambda_1)/q} x^{1/q}\right)} \right] dx \right\}^p \le \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{x^{p-1}}{(\ln \alpha x n)^{\lambda}} \frac{\left(\ln \sqrt{\alpha} x\right)^{(1-\lambda_1)(p-1)}}{n\left(\ln \sqrt{\alpha} n\right)^{1-\lambda_2}} f^p(x)\, dx \\
&\quad \times \left\{ \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{1}{(\ln \alpha x n)^{\lambda}} \frac{n^{q-1} \left(\ln \sqrt{\alpha} n\right)^{(1-\lambda_2)(q-1)}}{x\left(\ln \sqrt{\alpha} x\right)^{1-\lambda_1}} dx \right\}^{p-1} \\
&= \left\{ \omega(n) \frac{\left(\ln \sqrt{\alpha} n\right)^{q(1-\lambda_2)-1}}{n^{1-q}} \right\}^{p-1} \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{x^{p-1} \left(\ln \sqrt{\alpha} x\right)^{(1-\lambda_1)(p-1)}}{n(\ln \alpha x n)^{\lambda} \left(\ln \sqrt{\alpha} n\right)^{1-\lambda_2}} f^p(x)\, dx \\
&= \frac{[B(\lambda_1, \lambda_2)]^{p-1} n}{\left(\ln \sqrt{\alpha} n\right)^{p\lambda_2 - 1}} \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{x^{p-1} \left(\ln \sqrt{\alpha} x\right)^{(1-\lambda_1)(p-1)}}{n\left(\ln \sqrt{\alpha} x n\right)^{\lambda} \left(\ln \sqrt{\alpha} n\right)^{1-\lambda_2}} f^p(x)\, dx.
\end{aligned}$$

Then by Beppo Levi's theorem (cf. [19]), we have

$$J \leq [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \sum_{n=2}^{\infty} \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{x^{p-1}\left(\ln\sqrt{\alpha}x\right)^{(1-\lambda_1)(p-1)}}{n(\ln\alpha xn)^{\lambda}\left(\ln\sqrt{\alpha}n\right)^{1-\lambda_2}} f^p(x)\,dx \right\}^{\frac{1}{p}}$$

$$= [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \sum_{n=2}^{\infty} \frac{x^{p-1}\left(\ln\sqrt{\alpha}x\right)^{(1-\lambda_1)(p-1)}}{n(\ln\alpha xn)^{\lambda}\left(\ln\sqrt{\alpha}n\right)^{1-\lambda_2}} f^p(x)\,dx \right\}^{\frac{1}{p}}$$

$$= [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \varpi(x)\, x^{p-1}\left(\ln\sqrt{\alpha}x\right)^{p(1-\lambda_1)-1} f^p(x)\,dx \right\}^{\frac{1}{p}},$$

that is, (11) follows. Still by Hölder's inequality, we have

$$\left[\sum_{n=2}^{\infty} \frac{a}{(\ln\alpha xn)^{\lambda}}\right]^q = \left\{\sum_{n=2}^{\infty} \frac{1}{(\ln\alpha xn)^{\lambda}} \left[\frac{\left(\ln\sqrt{\alpha}x\right)^{(1-\lambda_1)/q} x^{1/q}}{\left(\ln\sqrt{\alpha}n\right)^{(1-\lambda_2)/p} n^{1/p}}\right]\right.$$

$$\left.\times \left[\frac{\left(\ln\sqrt{\alpha}n\right)^{(1-\lambda_2)/p} n^{1/p}}{\left(\ln\sqrt{\alpha}x\right)^{(1-\lambda_1)/q} x^{1/p}} a_n\right]\right\}^q \leq \left\{\sum_{n=2}^{\infty} \frac{x^{p-1}}{(\ln\alpha xn)^{\lambda}} \frac{\left(\ln\sqrt{\alpha}x\right)^{(1-\lambda_1)(p-1)}}{n\left(\ln\sqrt{\alpha}n\right)^{1-\lambda_2}}\right\}^{q-1}$$

$$\times \sum_{n=2}^{\infty} \frac{1}{(\ln\alpha xn)^{\lambda}} \frac{n^{q-1}\left(\ln\sqrt{\alpha}n\right)^{(1-\lambda_2)(q-1)}}{x\left(\ln\sqrt{\alpha}x\right)^{1-\lambda_1}} a_n^q$$

$$= \frac{x[\varpi(x)]^{q-1}}{\left(\ln\sqrt{\alpha}x\right)^{q\lambda_1-1}} \sum_{n=2}^{\infty} \frac{\left(\ln\sqrt{\alpha}x\right)^{\lambda_1-1}}{x(\ln\alpha xn)^{\lambda}} n^{q-1}\left(\ln\sqrt{\alpha}n\right)^{(q-1)(1-\lambda_2)} a_n^q.$$

Then by Beppo Levi's theorem, we have

$$L_1 \leq \left\{ \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \sum_{n=2}^{\infty} \frac{\left(\ln\sqrt{\alpha}x\right)^{\lambda_1-1}}{x(\ln\alpha xn)^{\lambda}} n^{q-1}\left(\ln\sqrt{\alpha}n\right)^{(q-1)(1-\lambda_2)} a_n^q dx \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=2}^{\infty} \left[\left(\ln\sqrt{\alpha}n\right)^{\lambda_2} \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{\left(\ln\sqrt{\alpha}x\right)^{\lambda_1-1}}{x(\ln\alpha xn)^{\lambda}} dx \right] n^{q-1}\left(\ln\sqrt{\alpha}n\right)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=2}^{\infty} \omega(n)\, n^{q-1}\left(\ln\sqrt{\alpha}n\right)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}},$$

and then in view of (10), inequality (12) follows. □ ▪

3 Main results

We introduce two functions

$$\Phi(x): \quad = x^{p-1}\left(\ln\sqrt{\alpha}x\right)^{p(1-\lambda_1)-1}\left(x \in \left(\frac{1}{\sqrt{\alpha}}, \infty\right)\right), \text{ and}$$

$$\Psi(n): \quad = n^{q-1}\left(\ln\sqrt{\alpha}n\right)^{q(1-\lambda_2)-1}\ (n \in \mathbf{N}\setminus\{1\}),$$

wherefrom, $[\Phi(x)]^{1-q} = \frac{1}{x}\left(\ln\sqrt{\alpha}x\right)^{q\lambda_1-1}$, and $[\Psi(n)]^{1-p} = \frac{1}{n}\left(\ln\sqrt{\alpha}n\right)^{p\lambda_2-1}$.

Theorem 3 *If* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, \lambda_1 > 0, 0 < \lambda_2 \le 1, \lambda_1+\lambda_2 = \lambda, \alpha \ge \frac{4}{9}, f(x), a_n \ge 0, f \in L_{p,\Phi}\left(\frac{1}{\sqrt{\alpha}}, \infty\right), a = \{a_n\}_{n=2}^{\infty} \in l_{q,\Psi}, \|f\|_{p,\Phi} > 0,$ *then we have the following equivalent inequalities:*

$$I := \sum_{n=2}^{\infty}\int_{\frac{1}{\sqrt{\alpha}}}^{\infty}\frac{a_n f(x)\,dx}{(\ln\alpha xn)^{\lambda}} = \int_{\frac{1}{\sqrt{\alpha}}}^{\infty}\sum_{n=2}^{\infty}\frac{f(x)\,a_n dx}{(\ln\alpha xn)^{\lambda}} < B(\lambda_1, \lambda_2)\|f\|_{p,\Phi}\|a\|_{q,\Psi}, \tag{13}$$

$$J = \left\{\sum_{n=2}^{\infty}[\Psi(n)]^{1-p}\left[\int_{\frac{1}{\sqrt{\alpha}}}^{\infty}\frac{f(x)\,dx}{(\ln\alpha xn)^{\lambda}}\right]^{p}\right\}^{\frac{1}{p}} < B(\lambda_1, \lambda_2)\|f\|_{p,\Phi}, \tag{14}$$

$$L := \left\{\int_{\frac{1}{\sqrt{\alpha}}}^{\infty}[\Phi(x)]^{1-q}\left[\sum_{n=2}^{\infty}\frac{a_n}{(\ln\alpha xn)^{\lambda}}\right]^{q}dx\right\}^{\frac{1}{q}} < B(\lambda_1, \lambda_2)\|a\|_{q,\Psi}, \tag{15}$$

where the constant $B(\lambda_1, \lambda_2)$ *is the best possible in the above inequalities.*

Proof. By Beppo Levi's theorem (cf. [19]), there are two expressions for I in (13). In view of (11), for $\varpi(x) < B(\lambda_1, \lambda_2)$, we have (14). By Hälder's inequality, we have

$$I = \sum_{n=2}^{\infty}\left[\Psi^{\frac{-1}{q}}(n)\int_{\frac{1}{\sqrt{\alpha}}}^{\infty}\frac{1}{(\ln\alpha xn)^{\lambda}}f(x)\,dx\right]\left[\Psi^{\frac{1}{q}}(n)\,a^{n}\right] \le J\|a\|_{q,\Psi}. \tag{16}$$

Then by (14), we have (13). On the other-hand, assuming that (13) is valid, setting

$$a_n := [\Psi(n)]^{1-p}\left[\int_{\frac{1}{\sqrt{\alpha}}}^{\infty}\frac{1}{(\ln\alpha xn)^{\lambda}}f(x)\,dx\right]^{p-1}, n \in \mathbf{N}\setminus\{1\},$$

then $J^{p-1} = ||a||_{q, \Psi}$. By (11), we find $J < \infty$. If $J = 0$, then (14) is valid trivially; if $J > 0$, then by (13), we have

$$\|a\|_{q,\Psi}^{q} = J^{p} = I < B(\lambda_1, \lambda_2) \|f\|_{p,\Psi} \|a\|_{q,\Psi}, \text{ i.e.}$$
$$\|a\|_{q,\Psi}^{q-1} = J < B(\lambda_1, \lambda_2) \|f\|_{p,\Phi},$$

that is, (14) is equivalent to (13). By (12), since $[\varpi(x)]^{1-q} > [B(\lambda_1, \lambda_2)]^{1-}q$, we have (15). By Hälder's inequality, we find

$$I = \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \left[\Phi^{\frac{1}{p}}(x) f(x)\right]\left[\Phi^{\frac{-1}{p}}(x) \sum_{n=2}^{\infty} \frac{a_n}{(\ln \alpha x n)^{\lambda}}\right] dx \le \|f\|_{p,\Phi} L. \tag{17}$$

Then by (15), we have (13). On the other-hand, assuming that (13) is valid, setting

$$f(x) := [\Phi(x)]^{1-q}\left[\sum_{n=2}^{\infty} \frac{a_n}{(\ln \alpha x n)^{\lambda}}\right]^{q-1}, \quad x \in \left(\frac{1}{\sqrt{\alpha}}, \infty\right),$$

then $L^{q-1} = \|f\|_{p, \Phi}$.. By (12), we find $L < \infty$. If $L = 0$, then (15) is valid trivially; if $L > 0$, then by (13), we have

$$\|f\|_{p,\Phi}^{p} = L^{q} = I < B(\lambda_1, \lambda_2) \|f\|_{p,\Phi} \|a\|_{q,\Psi}, \text{ i.e.}$$
$$\|f\|_{p,\Phi}^{p-1} = L < B(\lambda_1, \lambda_2) \|a\|_{q,\Psi},$$

That is, (15) is equivalent to (13). Hence inequalities (13), (14) and (15) are equivalent.

For $0 < \varepsilon < p\lambda_1$, setting $\tilde{a}_n = \frac{1}{n}(\ln \sqrt{\alpha} n)^{\lambda_2 - \frac{\in}{q} - 1}$, $n \in \mathbf{N}\backslash\{1\}$, and

$$\tilde{f}(x) := \begin{cases} 0, x \in \left(\frac{1}{\sqrt{\alpha}}, \frac{e}{\sqrt{\alpha}}\right) \\ \frac{1}{x}(\ln \sqrt{\alpha} x)^{\lambda_1 - \frac{\in}{p} - 1}, \quad x \in \left[\frac{e}{\sqrt{\alpha}}, \infty\right) \end{cases},$$

if there exists a positive number $k(\le B(\lambda_1, \lambda_2))$, such that (13) is valid as we replace $B(\lambda_1, \lambda_2)$ with k, then in particular, it follows

$$\begin{aligned} \tilde{I} := & \sum_{n=2}^{\infty} \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{1}{(\ln \alpha x n)^{\lambda}} \tilde{a}_n \tilde{f}(x)\, dx < k \|\tilde{f}\|_{p,\Phi} \|\tilde{a}\|_{q,\Psi} \\ & = k \left\{\int_{\frac{e}{\sqrt{\alpha}}}^{\infty} \frac{dx}{x(\ln \sqrt{\alpha} x)^{\varepsilon+1}}\right\}^{\frac{1}{p}} \left\{\frac{1}{2(\ln 2\sqrt{\alpha})^{\varepsilon+1}} + \sum_{n=3}^{\infty} \frac{1}{n(\ln \sqrt{\alpha} n)^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\ & < k\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}} \left\{\frac{1}{2(\ln 2\sqrt{\alpha})^{\varepsilon+1}} + \int_{2}^{\infty} \frac{1}{x(\ln \sqrt{\alpha} x)^{\varepsilon+1}} dx\right\}^{\frac{1}{q}} \\ & = \frac{k}{\varepsilon}\left\{\frac{\varepsilon}{2(\ln 2\sqrt{\alpha})^{\varepsilon+1}} + \frac{1}{(\ln 2\sqrt{\alpha})^{\varepsilon}}\right\}^{\frac{1}{q}}, \end{aligned} \tag{18}$$

$$\tilde{I} = \sum_{n=2}^{\infty} \left(\ln \sqrt{\alpha} n\right)^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{1}{n} \int_{\frac{e}{\sqrt{\alpha}}}^{\infty} \frac{1}{x(\ln \alpha x n)^{\lambda}} \left(\ln \sqrt{\alpha} x\right)^{\lambda_1 - \frac{\varepsilon}{p} - 1} dx$$

$$t = \left(\ln \sqrt{\alpha} x\right) / \left(\ln \sqrt{\alpha} n\right) \underset{=}{} \sum_{n=2}^{\infty} \frac{1}{n\left(\ln \sqrt{\alpha} n\right)^{\varepsilon+1}} \int_{1/\ln \sqrt{\alpha} n}^{\infty} \frac{t^{\lambda_1 - \frac{\varepsilon}{p} - 1}}{(t+1)^{\lambda}} dt$$

$$= B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) \sum_{n=2}^{\infty} \frac{1}{n\left(\ln \sqrt{\alpha} n\right)^{\varepsilon+1}} - A(\varepsilon) \quad (19)$$

$$> B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) \int_{2}^{\infty} \frac{1}{y\left(\ln \sqrt{\alpha} y\right)^{\varepsilon+1}} dy - A(\varepsilon)$$

$$= \frac{1}{\varepsilon\left(\ln 2\sqrt{\alpha}\right)^{\varepsilon}} B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) - A(\varepsilon),$$

$$A(\varepsilon) := \sum_{n=2}^{\infty} \frac{1}{n\left(\ln \sqrt{\alpha} n\right)^{\varepsilon+1}} \int_{0}^{1/\ln \sqrt{\alpha} n} \frac{1}{(t+1)^{\lambda}} t^{\lambda_1 - \frac{\varepsilon}{p} - 1} dt.$$

W find

$$0 < A(\varepsilon) \le \sum_{n=2}^{\infty} \frac{1}{n\left(\ln \sqrt{\alpha} n\right)^{\varepsilon+1}} \int_{0}^{1/\ln \sqrt{\alpha} n} t^{\lambda_1 - \frac{\varepsilon}{p} - 1} dt$$

$$= \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} \sum_{n=2}^{\infty} \frac{1}{n\left(\ln \sqrt{\alpha} n\right)^{\lambda_1 + \frac{\varepsilon}{q} + 1}} < \infty,$$

that is, $A(\varepsilon) = O(1)$ $(\varepsilon \to 0^+)$. Hence by (18) and (19), it follows

$$\frac{B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right)}{\left(\ln 2\sqrt{\alpha}\right)^{\varepsilon}} - \varepsilon O(1) < k\left\{\frac{\varepsilon}{2\left(\ln 2\sqrt{\alpha}\right)^{\varepsilon+1}} + \frac{1}{\left(\ln 2\sqrt{\alpha}\right)^{\varepsilon}}\right\}^{\frac{1}{q}}, \quad (20)$$

and $B(\lambda_1, \lambda_2) \le k (\varepsilon \to 0^+)$. Hence, $k = B(\lambda_1, \lambda_2)$ is the best value of (13).

Due to the equivalence, the constant factor $B(\lambda_1, \lambda_2)$ in (14) and (15) is the best possible. Otherwise, we can imply a contradiction by (16) and (17) that the constant factor in (13) is not the best possible. □ ▪

Remark 1 *(i) Define the first type half-discrete Mulholland's operator* $T : L_{p,\Phi}\left(\frac{1}{\sqrt{\alpha}}, \infty\right) \to l_{p,\Psi^{1-p}}$ *as follows: for* $f \in L_{p,\Phi}\left(\frac{1}{\sqrt{\alpha}}, \infty\right)$, *we define* $Tf \in l_{p,\Psi^{1-p}}$ *as*

$$Tf(n) = \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \frac{1}{(\ln \alpha x n)^{\lambda}} f(x) dx, \quad n \in \mathbf{N} \backslash \{1\}.$$

Then by (14), *it follows* $\|Tf\|_{p,\Psi^{1-p}} \le B(\lambda_1, \lambda_2) \|f\|_{p,\Phi}$ *and then T is a bounded operator with* $\|T\| \le B(\lambda_1, \lambda_2)$. *Since by Theorem* 1, *the constant factor in* (14) *is the best possible, we have* $\|T\| = B(\lambda_1, \lambda_2)$.

(ii) Define the second type half-discrete Mulholland's operator $\tilde{T}: l_{q,\Psi} \to L_{q,\Phi^{1-q}}\left(\frac{1}{\sqrt{\alpha}}, \infty\right)$ *as follows: For* $a \in l_{q,\Psi}$, *define* $\tilde{T}a \in L_{q,\Phi^{1-q}}\left(\frac{1}{\sqrt{\alpha}}, \infty\right)$ *as*

$$\tilde{T}a(x) = \sum_{n=2}^{\infty} \frac{1}{(\ln \alpha x n)^{\lambda}} a_n, \quad x \in \left(\frac{1}{\sqrt{\alpha}}, \infty\right).$$

Then by (15), *it follows* $\left\|\tilde{T}a\right\|_{q,\Phi^{1-q}} \leq B(\lambda_1, \lambda_2)\|a\|_{q,\Psi}$ *and then* $\tilde{T}$ *is a bounded operator with* $\left\|\tilde{T}\right\| \leq B(\lambda_1, \lambda_2)$. *Since by Theorem* 1, *the constant factor in* (15) *is the best possible, we have* $\left\|\tilde{T}\right\| = B(\lambda_1, \lambda_2)$.

Remark 2 *We set* $p = q = 2, \lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{2}$ *in* (13), (14) *and* (15). *(i) if* $\alpha = \frac{4}{9}$, *then we deduce* (7) *and the following equivalent inequalities:*

$$\sum_{n=2}^{\infty} \frac{1}{n}\left[\int_{\frac{3}{2}}^{\infty} \frac{f(x)}{\ln \frac{4}{9} x n} dx\right]^2 < \pi^2 \int_{\frac{3}{2}}^{\infty} x f^2(x) dx, \tag{21}$$

$$\int_{\frac{3}{2}}^{\infty} \frac{1}{x}\left[\sum_{n=2}^{\infty} \frac{a_n}{\ln \frac{4}{9} x n}\right]^2 dx < \pi^2 \sum_{n=2}^{\infty} n a_n^2; \tag{22}$$

(ii) if $\alpha = 1$, *then we have the following half-discrete Mulholland's inequality and its equivalent forms:*

$$\int_{1}^{\infty} f(x) \sum_{n=2}^{\infty} \frac{a_n}{\ln x n} dx < \pi \left\{\int_{1}^{\infty} x f^2(x) dx \sum_{n=2}^{\infty} n a_n^2\right\}^{\frac{1}{2}}, \tag{23}$$

$$\sum_{n=2}^{\infty} \frac{1}{n}\left[\int_{1}^{\infty} \frac{f(x)}{\ln x n} dx\right]^2 < \pi^2 \int_{1}^{\infty} x f^2(x) dx, \tag{24}$$

$$\int_{1}^{\infty} \frac{1}{x}\left[\sum_{n=2}^{\infty} \frac{a_n}{\ln x n}\right]^2 dx < \pi^2 \sum_{n=2}^{\infty} n a_n^2. \tag{25}$$

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Authors' contributions
QC conceived of the study, and participated in its design and coordination. BY wrote and reformed the article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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References

1. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1934)
2. Mitrinović, DS, Pečarić, JE, Fink, AM: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Acaremic Publishers, Boston (1991)
3. Yang, B: Hilbert-Type Integral Inequalities. Bentham Science Publishers Ltd., dubai (2009)
4. Yang, B: Discrete Hilbert-Type Inequalities. Bentham Science Publishers Ltd., dubai (2011)
5. Yang, B: An extension of Mulholand's inequality. Jordan J Math Stat. **3**(3):151–157 (2010)
6. Yang, B: On Hilbert's integral inequality. J Math Anal Appl. **220**, 778–785 (1998)
7. Yang, B: The Norm of Operator and Hilbert-type Inequalities. Science Press, Beijin (2009)
8. Yang, B, Brnetić, I, Krnić, M, Pečarić, J: Generalization of Hilbert and Hardy-Hilbert integral inequalities. Math Inequal Appl. **8**(2):259–272 (2005)
9. Krnić, M, Pečarić, J: Hilbert's inequalities and their reverses. Publ Math Debrecen. **67**(3-4):315–331 (2005)
10. Jin, J, Debnath, L: On a Hilbert-type linear series operator and its applications. J Math Anal Appl. **371**, 691–704 (2010)
11. Azar, L: On some extensions of Hardy-Hilbert's inequality and applications. J Inequal Appl **2009**, 14 (2009). Article ID 546829
12. Yang, B, Rassias, T: On the way of weight coefficient and research for Hilbert-type inequalities. Math Inequal Appl. **6**(4):625–658 (2003)
13. Arpad, B, Choonghong, O: Best constant for certain multilinear integral operator. J Inequal Appl **2006**, 12 (2006). Article ID 28582
14. Kuang, J, Debnath, L: On Hilbert's type inequalities on the weighted Orlicz spaces. Pac J Appl Math. **1**(1):95–103 (2007)
15. Zhong, W: The Hilbert-type integral inequality with a homogeneous kernel of Lambda-degree. J Inequal Appl **2008**, 13 (2008). Article ID 917392
16. Li, Y, He, B: On inequalities of Hilbert's type. Bull Aust Math Soc. **76**(1):1–13 (2007)
17. Yang, B: A mixed Hilbert-type inequality with a best constant factor. Int J Pure Appl Math. **20**(3):319–328 (2005)
18. Yang, B: A half-discrete Hilbert's inequality. J Guangdong Univ Edu. **31**(3):1–7 (2011)
19. Donald, LC: Measure Theorey. Birkhäuser, Boston (1980)

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