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The Beckenbach-Dresher inequality in the Ψ-direct sums of spaces and related results

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Abstract

Let $\tilde{\psi} : [0, 1] \to \mathbf{R}$ be a concave function with $\tilde{\psi}(0) = \tilde{\psi}(1) = 1$. There is a corresponding map $\|.\|_{\tilde{\psi}}$ for which the inverse Minkowski inequality holds. Several properties of that map are obtained. Also, we consider the Beckenbach-Dresher type inequality connected with ψ-direct sums of Banach spaces and of ordered spaces. In the last section we investigate the properties of functions $\psi_{\omega,q}$ and $\|.\|_{\omega,q}$, ($0 < \omega < 1$, $q < 1$) related to the Lorentz sequence space. Other posibilities for parameters ω and q are considered, the inverse Holder inequalities and more variants of the Beckenbach-Dresher inequalities are obtained.

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1 Preliminaries

In the fifties of the previous century the following result was obtained:

Let $1 \leq p \leq 2$ and $x_i, y_i > 0$, $i = 1, ..., n$. Then

$$\frac{\sum_{i=1}^{n} (x_i + y_i)^p}{\sum_{i=1}^{n} (x_i + y_i)^{p-1}} \leq \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} x_i^{p-1}} + \frac{\sum_{i=1}^{n} y_i^p}{\sum_{i=1}^{n} y_i^{p-1}}. \tag{1}$$

The above-mentioned discrete inequality was given by Beckenbach [1], and the integral version is due to Dresher [2] (see also [3]). From that time, some generalizations of the Beckenbach-Dresher inequality (1) have appeared. Here, we are pointing out articles of Pečarić and Beesack [4], Petree and Persson [5], Persson [6] and Varošanec [7], where the reader can find related literature about this inequality Here we consider inequalities of Beckenbach-Dresher type in more general structures, namely in ψ-direct sums.

In this article we follow definitions and notations from the paper [8]. Let Ψ denote the family of all convex functions ψ on $[0, 1]$ with $\psi(0) = \psi(1) = 1$ satisfying

$$\max\{1 - t, t\} \leq \psi(t) \leq 1, \quad (0 \leq t \leq 1).$$

It is known (see [9]), that Ψ is in one-to-one correspondence with the set N_a of all absolute and normalized norms on $\mathbf{C}^2$, i.e., such that

$$\|(x, y)\| = \|(|x|, |y|)\| \text{ and } \|(1, 0)\| = \|(0, 1)\| = 1.$$

Namely, if $\|.\| \in N_a$ and $\psi(t) = \|(1 - t, t)\|$, then $\psi \in \Psi$. Conversely, if $\psi \in \Psi$, then

$$
\|(x, y)\|_{\psi} = \begin{cases} (|x| + |y|)\psi\left(\dfrac{|y|}{|x| + |y|}\right), & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}
$$

is a norm and $\|.\|_{\psi} \in N_a$.

Some examples of convex functions and the corresponding norms are the following:

Example 1. *The convex functions* $\psi_p(t) = [(1 - t)^p + t^p]^{\frac{1}{p}}$, $1 \leq p < \infty$, $0 \leq t \leq 1$, *correspond to* l_p-*norms of* $\mathbf{C}^2$. *For* $p = \infty$, *the function* $\psi_{\infty}(t) = \max\{t, 1 - t\}$, $0 \leq t \leq 1$, *corresponds to the norm* l_{∞}.

Example 2. *Let*

$$
\|(x, y)\|_{\omega, q} = (x^{*q} + \omega y^{*q})^{\frac{1}{q}},
$$

where $\{x^*, y^*\}$ *is the non-increasing rearrangement of* $\{|x|, |y|\}$. *If* $0 < \omega < 1$, $1 \leq q$, *then* $\|.\|_{\omega, q}$ *is a norm of the two-dimensional Lorentz sequence space* $d^{(2)}(\omega, q)$, *it belongs to* N_a *and its dual norm was computed recently in* [10]. *The corresponding function from* Ψ *is*

$$
\psi_{\omega, q}(t) = \begin{cases} ((1 - t)^q + \omega t^q)^{\frac{1}{q}}, & \text{if } 0 \leq t \leq \dfrac{1}{2}, \\ (t^q + \omega(1 - t)^q)^{\frac{1}{q}}, & \text{if } \dfrac{1}{2} \leq t \leq 1. \end{cases}
$$

If $\omega = 2^{\frac{q}{p} - 1}$, *then we get a classical Lorentz* $l_{p,q}$-*norm.*

Example 3. *For* α, $1/2 \leq \alpha \leq 1$, *let us define the following function*

$$
\psi_{\alpha}(t) = \begin{cases} \dfrac{\alpha - 1}{\alpha}t + 1, & 0 \leq t \leq \alpha, \\ t, & \alpha \leq t \leq 1. \end{cases}
$$

$\psi_{\alpha} \in \Psi$ *and the corresponding norm is*

$$
\|(z, w)\|_{\psi_{\alpha}} = \max\{\|z\| + (2 - 1/\alpha)\|w\|, \|w\|\}.
$$

Example 4. *Let* $1 \leq q < p \leq \infty$ *and* $2^{\frac{1}{p} - \frac{1}{q}} < \lambda < 1$. *Let* $\psi_{p, q, \lambda} = \max\{\psi_p, \lambda\psi_q\}$. *Then the corresponding norm is* $\|.\|_{p, q, \lambda} = \max\{\|.\|_p, \lambda\|.\|_q\}$.

Example 5. *Let* $1/2 \leq \beta \leq 1$ *and let* $\psi_{\beta}(t) = \max\{1 - t, t, \beta\}$ *(note that neither* $\psi_{\beta} \geq \psi_2$ *nor* $\psi_{\beta} \leq \psi_2$.*). The corresponding norm is*

$$
\|(x, y)\|_{\psi_{\beta}}(t) = \begin{cases} |y|, & \dfrac{|x|}{|y|} \leq \dfrac{1 - \beta}{\beta}, \\ \beta, & \dfrac{1 - \beta}{\beta} \leq \dfrac{|x|}{|y|} \leq \dfrac{\beta}{1 - \beta}, \\ |x|, & \dfrac{|x|}{|y|} \geq \dfrac{\beta}{1 - \beta}. \end{cases}
$$

For $\psi \in \Psi$ let $\|.\|_{\psi}^*$ denote the dual of the norm $\|.\|_{\psi}$. From [11] we have that $\|.\|_{\psi}^*$ is an absolute normalized norm and the corresponding convex function $\psi^* \in \Psi$ is

$$\psi * (t) = \sup_{0 \le s \le 1} \frac{(1-s)(1-t)+st}{\psi(s)}$$

for t with $0 \le t \le 1$. For the norms $\|.\|_\psi$ and $\|.\|_\psi^*$ we have the following Hölder-type inequality:

$$\left\| (x_1, y_1) \right\|_\psi \left\| (x_2, y_2) \right\|_{\psi*} \ge |x_1 x_2| + |y_1 y_2|, \tag{2}$$

where $x_1, x_2, y_1, y_2 \in \mathbf{C}$.

Since the proof of Beckenbach-Dresher inequality can be obtained as an application of the Minkowski inequality the Holder inequality and its inverse inequalities in different cases, we are going to see what kind of such inequalities we could prove using some ideas of ψ-direct sums. In the following section we consider a family of concave functions $\tilde{\psi}$. We prove some properties of concave functions and the inverse Minkowski inequality Using these results and combining with the known results about the family Ψ and normalized absolute norms we obtain a variant of the Beckenbach-Dresher inequality related to those norms. In the third section we are considering Ψ-direct sums of Banach and ordered spaces. Finally, the last section is devoted to the two-dimensional Lorentz sequence space and its variants. There we obtain several inequalites of the Hölder type.

2 A family $\tilde{\Psi}$ of concave functions and a generalization of the Beckenbach-Dresher inequality

Let $\tilde{\Psi}$ denotes the family of all concave functions $\tilde{\psi}$ on $[0, 1]$ with $\tilde{\psi}(0) = \tilde{\psi}(1) = 1$. Let us define the map $\|.\|_{\tilde{\psi}}$ on $\mathbf{C}^2$ by

$$\left\| (z, w) \right\|_{\tilde{\psi}} = \begin{cases} (|z| + |w|) \tilde{\psi} \left(\dfrac{|w|}{|z| + |w|} \right), & (z, w) \ne (0, 0); \\ 0 & (z, w) = (0, 0). \end{cases} \tag{3}$$

As it is proved in the next proposition for this map $\|.\|_{\tilde{\psi}}$ the inverse Minkowski inequality holds. For that reason we call it a pseudo-norm.

Proposition 6. *Let $\tilde{\psi}$ be a concave function on $[0, 1]$. Then the inverse Minkowski inequality holds, i.e. for $u, v, z, w \in \mathbf{C}$ the following is valid:*

$$\left\| (|u| + |z|, |v| + |w|) \right\|_{\tilde{\psi}} \ge \left\| (|u|, |v|) \right\|_{\tilde{\psi}} + \left\| (|z|, |w|) \right\|_{\tilde{\psi}}. \tag{4}$$

Proof. Using concavity of the function $\tilde{\psi}$ and the equality

$$\frac{|v| + |w|}{|u| + |z| + |v| + |w|} = \frac{|u| + |v|}{|u| + |z| + |v| + |w|} \frac{|v|}{|u| + |v|} + \frac{|z| + |w|}{|u| + |z| + |v| + |w|} \frac{|w|}{|z| + |w|}$$

we get that

$$\begin{aligned} \left\| (|u| + |z|, |v| + |w|) \right\|_{\tilde{\psi}} &= (|u| + |z| + |v| + |w|) \tilde{\psi} \left(\frac{|v| + |w|}{|u| + |z| + |v| + |w|} \right) \\ &\ge (|u| + |v|) \tilde{\psi} \left(\frac{|v|}{|u| + |v|} \right) + (|z| + |w|) \tilde{\psi} \left(\frac{|w|}{|z| + |w|} \right) \\ &= \left\| (|u|, |v|) \right\|_{\tilde{\psi}} + \left\| (|z|, |w|) \right\|_{\tilde{\psi}}. \end{aligned}$$

The proof is complete.

Our next result reads:

Proposition 7. *Let $\tilde{\psi}$ be a concave function on* [0, 1], $\tilde{\psi}(t) \geq 0$. *Then* $\dfrac{\tilde{\psi}(t)}{t}$ *is non-increasing on* (0, 1] *and* $\dfrac{\tilde{\psi}(t)}{1-t}$ *is non-decreasing on* [0, 1). *If* $\tilde{\psi}(t) > 0$ *then the words non-increasing and non-decreasing are replaced by decreasing and increasing, correspondingly.*

Moreover, if $0 \leq p \leq r$, $0 \leq q \leq s$, *we have that*

$$\left\| (p,q) \right\|_{\tilde{\psi}} \leq \left\| r,s \right\|_{\tilde{\psi}}.$$

Proof. Let $0 < s < t < 1$.

(1) Put $p = t/s$ and write $s = \dfrac{1}{p}t + \dfrac{1}{p'}0$ where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Then, by concavity of $\tilde{\psi}$, we get that

$$\tilde{\psi}(s) \geq \frac{1}{p}\tilde{\psi}(t) + \frac{1}{p'}\tilde{\psi}(0) \geq \frac{1}{p}\psi(t).$$

Hence $\tilde{\psi}(s) \geq \dfrac{s}{t}\tilde{\psi}(t)$, i.e. $\dfrac{\tilde{\psi}(s)}{s} \geq \dfrac{\tilde{\psi}(t)}{t}$. If $\tilde{\psi}(0) > 0$, then the inequality is strict.

(2) Let now $p = \dfrac{1-s}{1-t}$. Then $t = \dfrac{p-1}{p} + \dfrac{s}{p} = \dfrac{1}{p'} \cdot 1 + \dfrac{1}{p} \cdot s$. By concavity of $\tilde{\psi}$ we find that

$$\tilde{\psi}(t) \geq \frac{1}{p}\tilde{\psi}(s) + \frac{1}{p'}\tilde{\psi}(1) \geq \frac{1}{p}\tilde{\psi}(s).$$

Hence $\tilde{\psi}(t) \geq \dfrac{1-t}{1-s}\tilde{\psi}(s)$ i.e. $\dfrac{\tilde{\psi}(t)}{1-t} \geq \dfrac{\tilde{\psi}(s)}{1-s}$. If $\tilde{\psi}(1) > 0$, then the inequality is strict.

To prove the monotonity property we proceed like in [8]. Firstly if $0 \leq p \leq r$, $0 \leq q \leq s$, then

$$\frac{q}{p+q} \geq \frac{q}{r+q} \quad \text{and} \quad \frac{q}{r+q} \leq \frac{s}{r+s}.$$

Using the fact that $\dfrac{\tilde{\psi}(t)}{t}$ is non-increasing and $\dfrac{\tilde{\psi}(t)}{1-t}$ is non-decreasing we obtain that

$$\left\| (p,q) \right\|_{\tilde{\psi}} = (p+q)\tilde{\psi}\left(\frac{q}{p+q}\right) = q \cdot \frac{\tilde{\psi}\left(\dfrac{q}{p+q}\right)}{\dfrac{q}{p+q}} \leq q \cdot \frac{\tilde{\psi}\left(\dfrac{q}{r+q}\right)}{\dfrac{q}{r+q}} = (r+q)\tilde{\psi}\left(\frac{q}{r+q}\right)$$

$$= r \cdot \frac{\tilde{\psi}\left(\dfrac{q}{r+q}\right)}{\dfrac{r}{r+q}} \leq r \cdot \frac{\tilde{\psi}\left(\dfrac{s}{s+r}\right)}{\dfrac{r}{s+r}} = (r+s)\tilde{\psi}\left(\frac{s}{r+s}\right) = \left\| (r,s) \right\|_{\tilde{\psi}}.$$

The proof is complete.

Let $\psi \in \tilde{\Psi}$. Denote

$$\psi_*(t) = \inf_{0 \leq s \leq 1} \frac{(1-s)(1-t)+st}{\psi(s)}$$

for $0 \le t \le 1$. The corresponding map $\|.\|_{\psi_*}$ is defined by (3). Using similar arguments as in [10] we can prove the following:

Proposition 8. *Let* $\psi \in \tilde{\Psi}$ *be symmetric with respect to* $t = 1/2$. *Then* ψ_* *is symmetric with respect to* $t = \dfrac{1}{2}$. *Moreover,*

$$\psi_*(t) = \inf_{0 \le s \le \frac{1}{2}} \frac{(1-s)(1-t) + st}{\psi(s)}$$

for all t *with* $\dfrac{1}{2} \le t \le 1$.

The following inverse Hölder-type inequality holds:

Proposition 9. *Let* $x_1, x_2, y_1, y_2 \in \mathbf{C}$. *If* $\psi \in \tilde{\Psi}$, *then*

$$\left\| (x_1, y_1) \right\|_{\psi} \left\| (x_2, y_2) \right\|_{\psi_*} \le |x_1 x_2| + |y_1 y_2|. \tag{5}$$

Proof. From the definition of the function ψ_* we get that

$$(1-s)(1-t) + st \ge \psi(s)\psi_*(t).$$

Putting in that inequality $s = \dfrac{|y_1|}{|x_1| + |y_1|}, t = \dfrac{|y_2|}{|x_2| + |y_2|}$ and using formula (3) we get (5).

Example 10. *Let* $0 < p < 1$. *The function* ψ_p *belongs to* $\tilde{\Psi}$ *and the related function* ψ_* *is* ψ_q *where* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *In this case inequality* (5) *has a form*

$$|x_1| |x_2| + |y_1| |y_2| \ge (|x_1|^p + |x_2|^p)^{\frac{1}{p}} (|y_1|^q + |y_2|^q)^{\frac{1}{q}}$$

which is the reversed Hölder inequality in the simplest form, see e.g., [12, *p.* 99].

The following result is a variant of the Beckenbach-Dresher inequality, obtained by using the inverse Holder inequality of the concrete space $l_{1/u}$, the Minkowski inequality for the norm $\|.\|_{\psi}$ and the inverse Minkowski inequality (4).

Theorem 11. *Let* $\psi \in \Psi, \tilde{\psi} \in \tilde{\Psi}$. *Let* $f_1 = (a_1^1, a_2^1), f_2 = (a_1^2, a_2^2), g_1 = (|b_1^1|, |b_2^1|), g_2 = (|b_1^2|, |b_2^2|)$ *where* $a_i^j, b_i^j \in \mathbf{C}, i, j = 1, 2$ *and let* $\|g_1\|_{\tilde{\psi}} \ne 0, \|g_2\|_{\tilde{\psi}} \ne 0$.

If $u \ge 1$, *then*

$$\frac{\|f_1 + f_2\|_{\psi}^u}{\|g_1 + g_2\|_{\tilde{\psi}}^{u-1}} \le \frac{\|f_1\|_{\psi}^u}{\|g_1\|_{\tilde{\psi}}^{u-1}} + \frac{\|f_2\|_{\psi}^u}{\|g_2\|_{\tilde{\psi}}^{u-1}}.$$

The inequality holds also when $u < 0, \psi \in \tilde{\Psi}, \tilde{\psi} \in \Psi$. *If* $0 < u < 1, \psi \in \tilde{\Psi}, \tilde{\psi} \in \tilde{\Psi}$, *then the inequality holds in the opposite direction.*

Proof. To prove this inequality in the first case we use the Minkowski inequality for the norm $\|.\|_{\psi}$, the inverse Minkowski inequality (4) for the pseudo-norm $\|.\|_{\tilde{\psi}}$ and the inverse Hölder inequality for the $l_{1/u}$-norm:

$$\frac{\|f_1 + f_2\|_{\psi}^u}{\|g_1 + g_2\|_{\tilde{\psi}}^{u-1}} \le (\|f_1\|_{\psi} + \|f_2\|_{\psi})^u (\|g_1\|_{\tilde{\psi}} + \|g_2\|_{\tilde{\psi}})^{1-u}$$

$$\le \|f_1\|_{\psi}^u \|g_1\|_{\tilde{\psi}}^{1-u} + \|f_2\|_{\psi}^u \|g_2\|_{\tilde{\psi}}^{1-u} = \frac{\|f_1\|_{\psi}^u}{\|g_1\|_{\tilde{\psi}}^{u-1}} + \frac{\|f_2\|_{\psi}^u}{\|g_2\|_{\tilde{\psi}}^{u-1}}.$$

Remark 12. *Note that if* $\begin{smallmatrix}|a_1^1| = x_1, \ |a_2^1| = x_2, \ |a_1^2| = y_1, \ |a_2^2| = y_2, \ g_1 = f_1, \ g_2 = f_2, u = p, \psi = \psi_p, \tilde{\psi} = \\ \psi_{p-1}, 1 \le p \le 2\end{smallmatrix}$

we get the classical Beckenbach inequality (1) *for* $n = 2$.

3 Ψ-Direct sums of spaces and some more generalizations of the Beckenbach-Dresher inequality

The ψ-direct sum $X \oplus_\psi Y$ of the Banach spaces X and Y is a direct sum $X \oplus_\psi Y$ equipped with the norm $\|(x, y)\|_\psi = \|(\|x\|_X, \|y\|_Y)\|_\psi$. This extends the notion of the l_p-sum $X \oplus_p Y$. Recently various geometric properties of ψ- direct sums have been investigated by many authors [11,13-18].

In the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces the Minkowski inequality holds, i.e., we have the following:

Let A and C be Banach spaces and let $f_1 = (a_1^1, a_2^1), f_2 = (a_1^2, a_2^2), a_1^i \in A, a_2^i \in C, i = 1, 2$. Then

$$\|f_1 + f_2\|_{A \oplus_\psi C} \le \|f_1\|_{A \oplus_\psi C} + \|f_2\|_{A \oplus_\psi C}.$$

Let us improve that idea to the sum of ordered spaces.

Let B and D be ordered spaces equiped with pseudo-norms $\|.\|_B$ and $\|.\|_D$ and let $g_1 = (|b_1^1|, |b_2^1|), \ g_2 = (|b_1^2|, |b_2^2|), b_1^i \in B, b_2^i \in D, i = 1, 2$. That means that in B and D the inverse Minkowski inequality holds, i.e.,

$$\||b_1^1| + |b_1^2|\|_B \ge \|b_1^1\|_B + \|b_1^2\|_{B'} \||b_2^1| + |b_2^2|\|_D \ge \|b_2^1\|_D + \|b_2^2\|_D.$$

Let $\tilde{\psi}$ be a concave function from $\tilde{\Psi}$. Let us define $B \oplus_{\tilde{\psi}} D$. $B \oplus_{\tilde{\psi}} D$ is called $\tilde{\psi}$-direct sum of spaces B and D. Using monotonicity of the pseudo-norm $\|.\|_{\tilde{\psi}}$ generated by the function $\tilde{\psi}$ and the fact that the inverse Minkowski inequality holds for this pseudo-norm, we get the following estimates for the $\tilde{\psi}$-direct sum:

$$\begin{aligned}\|g_1 + g_2\|_{B \oplus_{\tilde{\psi}} D} &= \|(\||b_1^1| + |b_1^2|\|_{B'}, \||b_2^1| + |b_2^2|\|_D)\|_{\tilde{\psi}} \\ &\ge \|(\|b_1^1\|_B + \|b_1^2\|_{B'}, \|b_2^1\|_D + \|b_2^2\|_D)\|_{\tilde{\psi}} = \|(\|b_1^1\|_B, \|b_2^1\|_D) + (\|b_1^2\|_{B'}, \|b_2^2\|_D)\|_{\tilde{\psi}} \\ &\ge \|(\|b_1^1\|_{B'}, \|b_2^1\|_D)\|_{\tilde{\psi}} + \|(\|b_1^2\|_{B'}, \|b_2^2\|_D)\|_{\tilde{\psi}} = \|g_1\|_{B \oplus_{\tilde{\psi}} D} + \|g_2\|_{B \oplus_{\tilde{\psi}} D}.\end{aligned}$$

So we have showed that when $\tilde{\psi} \in \tilde{\Psi}$ and the inverse Minkowski inequality holds in B and D, then this inequality holds also for the pseudo-norm of $B \oplus_{\tilde{\psi}} D$.

Our next result is the following inequalities of the Beckenbach-Dresher type:

Theorem 13. *Let* $\psi \in \Psi, \tilde{\psi} \in \tilde{\Psi}$ *and* A, C *be Banach spaces,* B, D *ordered spaces such that the inverse Minkowski inequality holds. Let* $f_1 = (a_1^1, a_2^1), \ f_2 = (a_1^2, a_2^2), \ g_1 = (|b_1^1|, |b_2^1|), \ g_2 = (|b_1^2|, |b_2^2|), \ \|g_1\|_{\tilde{\psi}} \ne 0, \ \|g_2\|_{\tilde{\psi}} \ne 0,$ *for* $a_1^i \in A, \ a_2^i \in C, \ b_1^i \in B, \ b_2^i \in D, \ i = 1, 2$. *If* $u \ge 1$, *then*

$$\frac{\|f_1 + f_2\|_{A \oplus_\psi C}^u}{\|g_1 + g_2\|_{B \oplus_{\tilde{\psi}} D}^{u-1}} \le \frac{\|f_1\|_{A \oplus_\psi C}^u}{\|g_1\|_{B \oplus_{\tilde{\psi}} D}^{u-1}} + \frac{\|f_2\|_{A \oplus_\psi C}^u}{\|g_2\|_{B \oplus_{\tilde{\psi}} D}^{u-1}}.$$

Proof. The proof is similar to the proof of Theorem 11 so we omit the details.

Remark 14. *If* $u < 1$, *the analogue result can be considered*.

If $g_1 = f_1$, $g_2 = f_2$, $u = p$, $\psi = \psi_p$, $\tilde{\psi} = \psi_{p-1}$, $1 \leq p \leq 2$, then we get that

$$\frac{\|a_1^1 + a_1^2\|_A^p + \|a_2^1 + a_2^2\|_C^p}{\| |b_1^1| + |b_1^2| \|_B^{p-1} + \| |b_2^1| + |b_2^2| \|_D^{p-1}} \leq \frac{\|a_1^1\|_A^p + \|a_2^1\|_C^p}{\|b_1^1\|_B^{p-1} + \|b_2^1\|_D^{p-1}} + \frac{\|a_1^2\|_A^p + \|a_2^2\|_C^p}{\|b_1^2\|_B^{p-1} + \|b_2^2\|_D^{p-1}}.$$

Note that if we take $A = B = C = D = \mathbf{R}$ and put $|a_1^1| = x_1$, $|a_2^1| = x_2$, $|a_1^2| = y_1$, $|a_2^2| = y_2$ then we get the classical Beckenbach inequality (1) for $n = 2$.

A natural question arises : can we get some similar generalization of Beckenbach-Dresher inequality for $n > 2$?

We use the construction from [19]. Let Δ_n be a set $\Delta_n = \{(s_1, ..., s_{n-1}) \in \mathbf{R}^{n-1} : s_1 + ... + s_{n-1} \leq 1, s_i \geq 0, 1 \leq i \leq n - 1\}$. In [19] the authors considered the family Ψ_n of all continuous convex functions ψ on Δ_n which satisfy:

$$\psi(0, 0, \ldots, 0) = \psi(0, 1, 0, \ldots, 0) = \cdots = \psi(0, 0, \ldots, 0, 1) = 1. \tag{6}$$

$$\psi(s_1, \ldots, s_{n-1}) \geq (s_1 + \cdots + s_{n-1})\psi\left(\frac{s_1}{s_1 + \cdots + s_{n-1}}, \ldots, \frac{s_{n-1}}{s_1 + \cdots + s_{n-1}}\right)$$

$$\psi(s_1, \ldots, s_{n-1}) \geq (1 - s_k)\psi\left(\frac{s_1}{1 - s_k}, \ldots, \frac{s_{k-1}}{1 - s_k}, 0, \frac{s_{k+1}}{1 - s_k}, \ldots, \frac{s_{n-1}}{1 - s_k}\right)$$

for $k = 1, ..., n - 1$. They showed that the family AN_n of all absolute normalized norms on $\mathbf{C}^n$ and the family Ψ_n are in one-to-one correspondence: if $\|.\| \in AN_n$, then

$$\psi(s_1, \ldots, s_{n-1}) = \| (1 - \sum_{i=1}^{n-1} s_i, s_1, \ldots, s_{n-1}) \|$$

belongs to Ψ_n and for given $\psi \in \Psi_n$ the following norm $\|.\|_\psi$ belongs to AN_n:

$$\|(z_1, \ldots, z_n)\|_\psi = (|z_1| + \ldots + |z_n|)\psi\left(\frac{|z_2|}{|z_1| + \ldots + |z_n|}, \ldots, \frac{|z_n|}{|z_1| + \ldots + |z_n|}\right) \tag{7}$$

if $(z_1, ..., z_n) \neq (0, ..., 0)$ and $\|(0, ..., 0)\|_\psi = 0$.

The set $\tilde{\Psi}_n$ is defined as a set of all positive continuous concave functions $\tilde{\psi}$ on Δ_n, which satisfy condition (6).

Let us define the pseudo-norm $\|.\|_{\tilde{\psi}}$ by the formula given in (7). Consider for simplicity the case $n = 3$. If the function $\tilde{\psi}$ is concave it is not difficult to prove using the same idea as in Proposition 7, that, for $\lambda > 1$, it yields that

$$\frac{\tilde{\psi}(s, t)}{s} \geq \frac{\tilde{\psi}(\lambda s, \lambda t)}{\lambda s} \quad \text{and} \quad \frac{\tilde{\psi}(s, t)}{1 - s} \geq \frac{\tilde{\psi}(\lambda s, \lambda t)}{1 - \lambda s}.$$

In the same way as it was done in Propositions 6 and 7 we can prove monotonicity of the pseudonorm and the inverse Minkowski inequality

$$\left\|(|x_1|, |y_1|, |z_1|) + (|x_2|, |y_2|, |z_2|)\right\|_{\tilde{\psi}} \geq \left\|(|x_1|, |y_1|, |z_1|)\right\|_{\tilde{\psi}} + \left\|((|x_2|, |y_2|, |z_2|)\right\|_{\tilde{\psi}}.$$

As above we could prove that if $B_1, ..., B_n$ have inverse Minkowski property and $\tilde{\psi} \in \tilde{\Psi}_n$, then $(B_1 \oplus B_2 \oplus \ldots \oplus B_n)_{\tilde{\psi}}$ would have inverse Minkowski property. So we can state the following generalization of the previous Theorem:

Theorem 15. *Let $\psi \in \Psi_n$, $\tilde{\psi} \in \tilde{\Psi}_n$ and A_1, A_2, ..., A_n be Banach spaces, B_1, B_2, ..., B_n be ordered spaces in which the inverse Minkowski inequality hold. Let*

$f_1 = (a_1^1, a_2^1, \ldots a_n^1), f_2 = (a_1^2, a_2^2, \ldots, a_n^2), g_1 = (|b_1^1|, |b_2^1|, \ldots, |b_n^1|), g_2 = (|b_1^2|, |b_2^2|, \ldots, |b_n^2|), \|g_1\|_{\tilde{\psi}} \neq 0, \|g_2\|_{\tilde{\psi}} \neq 0$, *where*

$a_i^1, a_i^2 \in A_i, b_i^1, b_i^2 \in B_i$ *for* $i = 1, 2, \ldots, n$. *If* $u \geq 1$, *then*

$$\frac{\|f_1 + f_2\|_{(A_1 \oplus A_2 \oplus \ldots \oplus A_n)_\psi}^u}{\|g_1 + g_2\|_{(B_1 \oplus B_2 \oplus \ldots \oplus B_n)_{\tilde{\psi}}}^{u-1}} \leq \frac{\|f_1\|_{(A_1 \oplus A_2 \oplus \ldots \oplus A_n)_\psi}^u}{\|g_1\|_{(B_1 \oplus B_2 \oplus \ldots \oplus B_n)_{\tilde{\psi}}}^{u-1}} + \frac{\|f_2\|_{(A_1 \oplus A_2 \oplus \ldots \oplus B_n)_\psi}^u}{\|g_2\|_{(B_1 \oplus B_2 \oplus \ldots \oplus B_n)_{\tilde{\psi}}}^{u-1}}.$$

Proof. The proof is completely similar as that before, so we leave out the details.

4 Norm of the Lorentz sequence space and its variants

Let us consider two-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$, where $0 < \omega < 1$, $1 \leq q$. It is $\mathbf{R}^2$ with the norm

$$\|(x, y)\|_{\omega, q} = (x^{*q} + \omega y^{*q})^{\frac{1}{q}}.$$

The corresponding convex function is

$$\psi_{\omega, q}(t) = \begin{cases} ((1 - t)^q + \omega t^q)^{\frac{1}{q}}, & 0 \leq t \leq \frac{1}{2}, \\ (t^q + \omega(1 - t)^q)^{\frac{1}{q}}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The dual norm of $d^{(2)}(\omega, q)$ is completely determined by Mitani and Saito in [10] by finding the corresponding function $\psi_{\omega, q}^*$. Namely, if $0 < \omega < 1$ and $1 < q < \infty$, then we have that

$$\psi_{\omega, q}^*(t) = \begin{cases} ((1 - t)^p + \omega^{1-p} t^p)^{\frac{1}{p}}, & 0 \leq t \leq \frac{\omega}{1 + \omega}, \\ (1 + \omega)^{\frac{1}{p} - 1}, & \frac{\omega}{1 + \omega} \leq t \leq \frac{1}{1 + \omega}, \\ (t^p + \omega^{1-p}((1 - t)^p))^{\frac{1}{p}}, & \frac{1}{1 + \omega} \leq t \leq 1, \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. The dual norm is equal to

$$\|(x, y)\|_{\omega, q}^* = \|(x, y)\|_{\psi_{\omega, q}^*} = \begin{cases} (|x|^p + \omega^{1-p}|y|^p)^{\frac{1}{p}}, & \omega|x| \geq |y|, \\ (1 + \omega)^{\frac{1}{p} - 1}(|x| + |y|), & \omega|x| \leq |y| \leq \omega^{-1}|x|, \\ (|y|^p + \omega^{1-p}|x|^p)^{\frac{1}{p}}, & \omega^{-1}|x| \leq |y|. \end{cases}$$

If $0 < \omega < 1$ and $q = 1$, then

$$\psi_{\omega, 1}^*(t) = \begin{cases} 1 - t, & 0 \leq t \leq \frac{\omega}{1 + \omega}, \\ \frac{1}{1 + \omega}, & \frac{\omega}{1 + \omega} \leq t \leq \frac{1}{1 + \omega}, \\ t, & \frac{1}{1 + \omega} \leq t \leq 1, \end{cases}$$

and

$$\|(x, y)\|_{\omega,1}^* = \|(x, y)\|_{\psi_{\omega,q}^*} = \begin{cases} |x|, & \omega|x| \geq |y|, \\ \dfrac{1}{1+\omega}(|x|+|y|), & \omega|x| \leq |y| \leq \omega^{-1}|x|, \\ |y|, & \omega^{-1}|x| \leq |y|. \end{cases}$$

If $0 < q < 1$, $\omega > 1$, then $\psi_{\omega,q}$ is a concave function from $\tilde{\psi}$ and we have the inverse Minkowski inequality for the corresponding pseudo-norm. Indeed, if $0 \leq t \leq 1/2$, then the function is increasing and concave; if $1/2 \leq t \leq 1$, then it is decreasing and concave (which can be shown by finding the first and second derivatives). Let $0 \leq t_1 \leq 1/2 \leq t_2 \leq 1$. Consider the line connecting the points $(t_1, \psi_{\omega,q}(t_1))$ and $(t_2, \psi_{\omega,q}(t_2))$. For the concavity it is enough to show that the graph of the function is above this line. In fact, this is the case, because if we consider for instance $t_1 \leq t \leq 1/2$, then (because of concavity on this interval) the graph is above the line connecting the points $(t_1, \psi_{\omega,q}(t_1))$ and $(1/2, \psi_{\omega,q}(1/2))$.

Lemma 1 from [20] for the case $1 \leq q < \infty$, $0 < \omega \leq 1$ asserts that

$$\left(\frac{\omega+1}{2}\right)^{\frac{1}{q}}\|a\|_q \leq \|a\|_{\omega,q} \leq \|a\|_q \tag{8}$$

for all $a \in \mathbf{R}^2$. It is easy to obtain similar result for other posibilities of parameters q and ω. For example the following holds:

Lemma 16. *If $\omega \geq 1$ and $q > 0$, then*

$$\|a\|_q \leq \|a\|_{\omega,q} \leq \left(\frac{\omega+1}{2}\right)^{\frac{1}{q}}\|a\|_q \tag{9}$$

for all $a \in \mathbf{R}^2$.

Our next result reads:

Theorem 17. *Let $a, b \in \mathbf{R}^2$. Then the following inequality holds*

$$\|a+b\|_{\omega,q} \leq C\left(\|a\|_{\omega,q} + \|b\|_{\omega,q}\right), \tag{10}$$

where

$$C = \begin{cases} 1, & q \geq 1, \quad 0 < \omega \leq 1, \\ \left(\dfrac{\omega+1}{2}\right)^{\frac{1}{q}}, & q \geq 1, \quad 1 \leq \omega, \\ \dfrac{1}{2}\left(\dfrac{4}{\omega+1}\right)^{\frac{1}{q}}, & 0 < q < 1, \quad 0 < \omega \leq 1, \\ \dfrac{1}{2}(\omega+1)^{\frac{1}{q}}, & 0 < q < 1, \quad \omega \geq 1. \end{cases}$$

Proof. If $q \geq 1$, $0 < \omega \leq 1$, then Kato and Maligranda proved that $C = 1$ in [20].

Let $q \geq 1$, $1 \leq \omega$. Using (9) and the Minkowski inequality for the norm $\|.\|_q$ we have that

$$\|a+b\|_{\omega,q} \leq \left(\frac{\omega+1}{2}\right)^{\frac{1}{q}}\|a+b\|_q \leq \left(\frac{\omega+1}{2}\right)^{\frac{1}{q}}(\|a\|_q + \|b\|_q)$$

$$\leq \left(\frac{\omega+1}{2}\right)^{\frac{1}{q}}(\|a\|_{\omega,q} + \|b\|_{\omega,q}).$$

Let $0 < q < 1$, $0 < \omega \leq 1$. Using inequality between means of order q and 1 and super-additivity of the function $f(s) = s^q$ we have the following:

$$2\|a + b\|_q = 2((a_1 + b_1)^q + (a_2 + b_2)^q)^{\frac{1}{q}} \leq (2(a_1 + b_1 + a_2 + b_2)^q)^{\frac{1}{q}}$$

$$\leq 2^{\frac{1}{q}}\left((a_1^q + a_2)^{\frac{1}{q}} + (b_1^q + b_2^q)^{\frac{1}{q}}\right) = 2^{\frac{1}{q}}(\|a\|_q + \|b\|_q),$$

where $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

Combining the inequality $\|a + b\|_q \leq 2^{\frac{1}{q}-1}(\|a\|_q + \|b\|_q)$ and (8) we get that

$$\|a + b\|_{\omega,q} \leq \|a + b\|_q \leq 2^{\frac{1}{q}-1}(\|a\|_q + \|b\|_q)$$

$$\leq 2^{\frac{1}{q}-1}\left(\frac{2}{\omega + 1}\right)^{\frac{1}{q}}(\|a\|_{\omega,q} + \|b\|_{\omega,q})$$

$$= \frac{1}{2}\left(\frac{4}{\omega + 1}\right)^{\frac{1}{q}}(\|a\|_{\omega,q} + \|b\|_{\omega,q}).$$

Finally, let $0 < q < 1$, $\omega \geq 1$. Using the above-proved inequality and (9) we have that

$$\|a + b\|_{\omega,q} \leq \left(\frac{\omega + 1}{2}\right)^{\frac{1}{q}}\|a + b\|_q \leq 2^{\frac{1}{q}-1}\left(\frac{\omega + 1}{2}\right)^{\frac{1}{q}}(\|a\|_q + \|b\|_q)$$

$$\leq \frac{1}{2}(\omega + 1)^{\frac{1}{q}}(\|a\|_{\omega,q} + \|b\|\,\omega, q).$$

The proof is complete.

Remark 18. *For the second case we see that the quasi-norm constant C is less than or equal to* $((\omega + 1)/2)^{\frac{1}{q}}$. *We will compare with the result from* [21], *Proposition 1, where* $\omega = 2^{\frac{q}{p}-1}, q > q \geq 1$ *and the quasi-norm constant is* $2^{\frac{1}{p}}$. *Since* $((\omega + 1)/2)^{\frac{1}{q}} < 2^{\frac{1}{p}}$ *we see that quasi-norm constant C obtained in this theorem is strictly less than the known constant for that case.*

Remark 19. *As we already mentioned, the norm of dual space of the space* $d^{(2)}(\omega, q)$, $q \geq 1$, $0 < \omega \leq 1$, *is given in* [10].

In the next part of this section we calculate the function $\psi_{*\omega, q}$ for $0 < q < 1$ and $1 \leq \omega$ and the corresponding mapping $\|.\|_{\psi_*}$. Using those results we obtain examples of the inverse Holder inequality, and get some new variations of the Beckenbach-Dresher inequality.

Proposition 20. *If* $0 < q < 1$, $1 \leq \omega$, *then*

$$\psi_{*\omega,q}(t) = \begin{cases} (t^p + \omega^{1-p}(1 - t)^p)^{\frac{1}{p}}, & 0 < t \leq \dfrac{1}{\omega + 1}, \\[2mm] (1 + \omega)^{\frac{1}{p}-1}, & \dfrac{1}{\omega + 1} \leq t \leq \dfrac{\omega}{\omega + 1}, \\[2mm] ((1 - t)^p + \omega^{1-p}t^p)^{\frac{1}{p}}, & \dfrac{\omega}{\omega + 1} \leq t < 1, \end{cases}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. We consider the function

$$f(s) = f(s, t) = \frac{(1 - s)(1 - t) + st}{\psi_{\omega,q}(s)}$$

for fixed t. Let first $t \in \left[\frac{1}{2}, 1\right)$. Here $\psi_{*\omega,q}(t) = \min_{s \in [0, \frac{1}{2}]} f(s)$.
The derivative of f is

$$f'(s) = ((1 - s)^q + \omega s^q)^{-\frac{1}{q} - 1}((1 - s)^{q-1}t - \omega s^{q-1}(1 - t))$$

and $f'(s_0) = 0$ when

$$s_0 = \frac{t^{p-1}}{t^{p-1} + \omega^{p-1}(1 - t)^{p-1}}.$$

If $\dfrac{\omega}{\omega + 1} \le t \le 1$, then $0 \le s_0 \le 1/2$. Therefore it is easy to see that the function f
attains its minimum at $s = s_0$ i.e.,

$$\psi_{*\omega,q}(t) = f(s_0) = \frac{(1 - s_0)(1 - t) + s_0 t}{((1 - s_0)^q + \omega s_0^q)^{\frac{1}{q}}} = ((1 - t)^p + \omega^{1-p}t^p)^{\frac{1}{p}}.$$

If $1/2 \le t \le \dfrac{\omega}{\omega + 1}$, then $s_0 \ge 1/2$. Hence the minimum of f is at $s = 1/2$ i.e.,
$\psi_{*\omega,q}(t) = f\left(\frac{1}{2}\right) = (1 + \omega)^{\frac{1}{p} - 1}$.

Having in mind the symmetry of the function $\psi_{*\omega,q}(t)$ we can end the proof.

In the previous Proposition we consider $t \in (0, 1)$ since for $q \in (0, 1)$ p is negative.
But, it is easy to see that $\psi_{*\omega,q}(0) = \psi_{*\omega,q}(1) = 1$.

Proposition 21. *If $0 < q \le 1, 1 \le \omega$, then*

$$\|(x, \gamma)\| \psi_{*\omega,q} = \begin{cases} (\gamma^{*p} + \omega^{(1-p)}x^{*p})^{\frac{1}{p}} & \dfrac{x*}{\gamma*} \ge \omega, \\ (\omega + 1)^{\frac{1}{p} - 1}(|x| + |\gamma|) & \dfrac{x*}{\gamma*} \le \omega. \end{cases}$$

Proof. Let $x^* \ge \omega y^*$. Without loosing of generality, put $x^* = |x|$. This means that $|x|$
$\ge \omega|y|$, and then $t = \dfrac{|\gamma|}{|x| + |\gamma|} \le \dfrac{|\gamma|}{\omega|\gamma| + |\gamma|} = \dfrac{1}{1 + \omega}$ and

$$\|(x, \gamma)\|_{\psi_{*\omega,q}} = (|x| + |\gamma|)\left(\frac{|\gamma|^p}{(|x| + |\gamma|)^p} + \omega^{(1-p)}\frac{|x|^p}{(|x| + |\gamma|)^p}\right)^{\frac{1}{p}}$$

$$= (|\gamma|^p + \omega^{(1-p)}|x|^p)^{\frac{1}{p}} = (\gamma^{*p} + \omega^{(1-p)}x^{*p})^{\frac{1}{p}}.$$

The case when $x^* = |y|$ is quite analogue. Let $x^* \le \omega y^*$. Let for instance $x^* = |x|$ i.e., $|y| \le |x| \le \omega|y|$. Then $\dfrac{1}{2} \ge t = \dfrac{|\gamma|}{|x| + |\gamma|} \ge \dfrac{|\gamma|}{\omega|\gamma| + |\gamma|} = \dfrac{1}{1 + \omega}$ and

$$\|(x, \gamma)\|_{\psi_{*\omega,q}} = (\omega + 1)^{\frac{1}{p} - 1}(|x| + |\gamma|).$$

Using the inverse Hölder inequality we obtain the following Corollary:

Corollary 22. Let $0 < q \leq 1$, $1 \leq \omega$, $x_1, x_2, y_1, y_2 > 0$.

If $\dfrac{x_2^*}{y_2^*} \geq \omega$, we have that

$$\left(x_1^{*q} + \omega y_1^{*q}\right)^{\frac{1}{q}} \left(y_2^{*p} + \omega^{(1-p)} x_2^{*p}\right)^{\frac{1}{p}} \leq x_1 x_2 + y_1 y_2.$$

If $\dfrac{x_2^*}{y_2^*} \leq \omega$, we find that

$$\left(x_1^{*q} + \omega y_1^{*q}\right)^{\frac{1}{q}} (\omega + 1)^{\frac{1}{p}-1} \left(|x_2| + |y_2|\right) \leq x_1 x_2 + y_1 y_2.$$

Let f_i, g_i be as in Theorem 13, let f_1^*, f_2^* be such that $f_1^* = f_1$ if $\|f_1\|_\psi \geq \|f_2\|_\psi$ and $f_1^* = f_2$ if $\|f_1\|_\psi \leq \|f_2\|_\psi$. Moreover, let g_1^*, g_2^* be such that $g_1^* = g_1$ if $\|g_1\|_{\tilde{\psi}} \geq \|g_2\|_{\tilde{\psi}}$, $g_1^* = g_2$ if $\|g_1\|_{\tilde{\psi}} \leq \|g_2\|_{\tilde{\psi}}$. Hence $\|g_1^*\|_{\tilde{\psi}} \geq \|g_2^*\|_{\tilde{\psi}}$.

We state the following new variant of the Beckenbach-Dresher inequality:

Theorem 23. *Let $u \geq 1$, $1 \leq \omega$. Let $\psi \in \Psi$, $\tilde{\psi} \in \tilde{\Psi}$, f_i, g_i be as in Theorem 13 and*
$\|g_1^*\|_{\tilde{\psi}} \geq \omega^{\frac{1}{u-1}} \|g_2^*\|_{\tilde{\psi}} > 0$. *Then*

$$\frac{\left\|f_1^* + \omega f_2^*\right\|_\psi^u}{\left\|g_1^* + \omega^{\frac{u}{u-1}} g_2^*\right\|_{\tilde{\psi}}^{u-1}} \leq \frac{\left\|f_1\right\|_\psi^u}{\left\|g_1\right\|_{\tilde{\psi}}^{u-1}} + \frac{\left\|f_2\right\|_\psi^u}{\left\|g_2\right\|_{\tilde{\psi}}^{u-1}}.$$

Proof. Using the Minkowski inequality for the norm $\|.\|_\psi$, inverse Minkowski inequality for $\|\cdot\|_{\tilde{\psi}}$ and inverse Hölder inequality i.e., previous corollary for

$$\frac{\left\|f_1^* + \omega f_2^*\right\|_\psi^u}{\left\|g_1^* + \omega^{\frac{u}{u-1}} g_2^*\right\|_{\tilde{\psi}}^{u-1}} \leq \left(\|f_1^*\|_\psi + \omega \|f_2^*\|_\psi\right)^u \left(\|g_1^*\|_{\tilde{\psi}} + \omega^{\frac{u}{u-1}} \|g_2^*\|_{\tilde{\psi}}\right)^{1-u} = \left(x_1^{*q} + \omega y_1^{*q}\right)^{\frac{1}{q}} \left(y_2^{*p} + \omega^{(1-p)} x_2^{*p}\right)^{\frac{1}{p}}$$

$$\quad :$$

$$\leq x_1 x_2 + y_1 y_2 = \|f_1\|_\psi^u \|g_1\|_{\tilde{\psi}}^{1-u} + \|f_2\|_\psi^u \|g_2\|_{\tilde{\psi}}^{1-u} = \frac{\|f_1\|_\psi^u}{\|g_1\|_{\tilde{\psi}}^{u-1}} + \frac{\|f_2\|_\psi^u}{\|g_2\|_{\tilde{\psi}}^{u-1}}.$$

$$\frac{\left\|f_1^* + \omega f_2^*\right\|_\psi^u}{\left\|g_1^* + \omega^{\frac{u}{u-1}} g_2^*\right\|_{\tilde{\psi}}^{u-1}} \leq \left(\|f_1^*\|_\psi + \omega \|f_2^*\|_\psi\right)^u \left(\|g_1^*\|_{\tilde{\psi}} + \omega^{\frac{u}{u-1}} \|g_2^*\|_{\tilde{\psi}}\right)^{1-u} = \left(x_1^{*q} + \omega y_1^{*q}\right)^{\frac{1}{q}} \left(y_2^{*p} + \omega^{(1-p)} x_2^{*p}\right)^{\frac{1}{p}}$$

$$\leq x_1 x_2 + y_1 y_2 = \|f_1\|_\psi^u \|g_1\|_{\tilde{\psi}}^{1-u} + \|f_2\|_\psi^u \|g_2\|_{\tilde{\psi}}^{1-u} = \frac{\|f_1\|_\psi^u}{\|g_1\|_{\tilde{\psi}}^{u-1}} + \frac{\|f_2\|_\psi^u}{\|g_2\|_{\tilde{\psi}}^{u-1}}.$$

Remark 24. *Let $u \geq 1$, $1 \leq \omega$. Let $\psi \in \Psi$, $\tilde{\psi} \in \tilde{\Psi}$ and $0 < \|g_1^*\|_{\tilde{\psi}} \leq \omega^{\frac{1}{u-1}} \|g_2^*\|_{\tilde{\psi}}$. Then*

$$\left\|f_1^* + \omega f_2^*\right\|_\psi^u \left(\|g_1\|_{\tilde{\psi}}^{1-u} + \|g_2\|_{\tilde{\psi}}^{1-u}\right) (\omega + 1)^{-u} \leq \frac{\|f_1\|_\psi^u}{\|g_1\|_{\tilde{\psi}}^{u-1}} + \frac{\|f_2\|_\psi^u}{\|g_2\|_{\tilde{\psi}}^{u-1}}.$$

We get this by replacing x_2, y_2 in the above corollary by $x_2 = \|g_1\|_{\tilde{\psi}}^{1-u}$ $y_2 = \|g_2\|_{\tilde{\psi}}^{1-u}$. If we use the inequality $\left(\dfrac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}} \leq \dfrac{a+b}{2}, \alpha = 1 - u \leq 0 < 1$ we get that

$$\left(\|g_1\|_{\tilde{\psi}}^{1-u} + \|g_2\|_{\tilde{\psi}}^{1-u}\right)(\omega + 1)^{-u} \geq \left(\|g_1\|_{\tilde{\psi}} + \|g_2\|_{\tilde{\psi}}\right)^{1-u} 2^u (\omega + 1)^{-u} \geq \|g_1 + g_2\|_{\tilde{\psi}}^{1-u} 2^u (\omega + 1)^{-u}.$$

Remark 25. *For the case* $\left\|g_1^*\right\|_{\tilde{\psi}} \le \omega^{\frac{1}{u-1}}\left\|g_2^*\right\|_{\tilde{\psi}}$ *we get another variant of the Beckenbach-Dresher inequality, namely*

$$\frac{\left\|f_1^* + \omega f_2^*\right\|_\psi^u}{\left\|g_1 + g_2\right\|_{\tilde{\psi}}^{u-1}}\left(\frac{2}{\omega+1}\right)^u \le \frac{\left\|f_1\right\|_\psi^u}{\left\|g_1\right\|_{\tilde{\psi}}^{u-1}} + \frac{\left\|f_2\right\|_\psi^u}{\left\|g_2\right\|_{\tilde{\psi}}^{u-1}}.$$

Remark 26. *The cases* $u < 0, \psi \in \tilde{\Psi}, \tilde{\psi} \in \Psi$ *and* $u < 1, \psi \in \tilde{\Psi}, \tilde{\psi} \in \tilde{\Psi}$ *can be treated in a similar way.*

For the case $q \ge 1, 0 < \omega \le 1$, $\|.\|_{\omega,q}$ is a norm, so we have the Minkowski and the Hölder inequality (2).

For the case $q \le 1, \omega \le 1$ we have Minkowski inequality with constant $\frac{1}{2}(\omega+1)^{\frac{1}{q}}$, inverse Minkowski, and inverse Holder inequalities.

For the case $q \ge 1, 1 \le \omega$, we have Minkowski inequality with constant $((\omega+1)/2)^{\frac{1}{q}}$. The function $\psi_{\omega,q}$ is not convex. It has relative minimums at the points t_0 and t_1. We could try to improve it constructing a convex function changing it on the interval (t_0, t_1) by replacing it by a constant equal to $\psi_{\omega,q}(t_0) = \psi_{\omega,q}(t_1)$. The corresponding norm is a norm indeed, but some calculations show that actually this is not a new norm, but the norm $\left\|(x,y)\right\|_{\omega^{1-p},p}^* = \left\|(x,y)\right\|_{\psi_{\omega^{1-p},p}^*}$.

For the case $q \le 1, 0 < \omega \le 1$, we have the Minkowski inequality with constant $\frac{1}{2}(4/(\omega+1))^{1/q}$. The function $\psi_{\omega,q}$ is not concave. It has relative maximum at the points t_0 and t_1. We can improve it constructing a concave function, namely changing it on the interval (t_0, t_1) by replacing it by a constant equal to $\psi_{\omega,q}(t_0) = \psi_{\omega,q}(t_1)$. The new function is

$$\psi_{c,\omega,q}(t) = \begin{cases} \left((1-t)^q + \omega t^q\right)^{\frac{1}{q}}, & 0 \le t \le \dfrac{1}{1+\omega^{p-1}}, \\[2mm] \dfrac{\omega^{\frac{1}{q}}}{\left(1+\omega^{p-1}\right)^{\frac{1}{p}}}, & \dfrac{1}{1+\omega^{p-1}} \le t \le \dfrac{\omega^{p-1}}{1+\omega^{p-1}}, \\[2mm] \left(t^q + \omega(1-t)^q\right)^{\frac{1}{q}}, & \dfrac{\omega^{p-1}}{1+\omega^{p-1}} \le t \le 1, \end{cases}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and

$$\left\|(x,y)\right\|_{c,\omega,q} = \begin{cases} \left(|x|^\omega + \omega|y|^q\right)^{\frac{1}{q}}, & \omega^{p-1}|x| \ge |y|, \\[2mm] \dfrac{\omega}{\left(1+\omega^{p-1}\right)^{q-1}}(|x|+|y|), & \omega^{1-1}|x| \le |y| \le \omega^{p-1}|x|, \\[2mm] \left(|y|^q + \omega|x|^q\right)^{\frac{1}{q}}, & \omega^{p-1}|x| \le |y|. \end{cases}$$

Remark 27. *Analogous results connected to the function* $\psi_{p,q,\lambda}$ *are given in the article* [22].

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Authors' contributions
All authors conceived of the study and carried out the proof. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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