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An almost sure central limit theorem of products of partial sums for ρ^--mixing sequences

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Abstract

Let $\{X_n, n \geq 1\}$ be a strictly stationary ρ^--mixing sequence of positive random variables with $EX_1 = \mu > 0$ and $Var(X_1) = \sigma^2 < \infty$. Denote $S_n = \sum_{i=1}^{n} X_i$ and $\gamma = \frac{\sigma}{\mu}$ the coefficient of variation. Under suitable conditions, by the central limit theorem of weighted sums and the moment inequality we show that

$$\forall x = \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\left(\prod_{i=1}^{k} \frac{S_i}{i\mu}\right)^{\frac{1}{\gamma\sigma k}}\right\} = F(x) \quad a.s.,$$

where $\sigma_k^2 = Var\left(S_{k,k}\right)$, $S_{k,k} = \sum_{i=1}^{k} b_{i,k} Y_i$, $b_{i,k} = \sum_{j=i}^{k} \frac{1}{j}$, $i \leq k$ with $b_{i,k} = 0, i > k$, $Y_i = \frac{X_{i-\mu}}{\sigma}$, $F(x)$ is the distribution function of the random variable $e^{\sqrt{2}\mathcal{N}}$, and $\mathcal{N}$ is a standard normal random variable.

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1 Introduction and main results

For a random variable X, define $\|X\|_p = (E|X|^p)^{1/p}$. For two nonempty disjoint sets $S,T \subset N$, we define dist(S,T) to be $\min\{|j - k|; j \in S, k \in T\}$. Let $\sigma(S)$ be the σ-field generated by $\{X_k, k \in S\}$, and define $\sigma(T)$ similarly. Let $\mathscr{C}$ be a class of functions which are coordinatewise increasing. For any real number x, x^+, and x^- denote its positive and negative part, respectively, (except for some special definitions, for examples, $\rho^-(s)$, $\rho^-(S,T)$, etc.). For random variables X, Y, define

$$\rho^-(X,Y) = 0 \vee \sup \frac{Cov(f(X),g(Y))}{(Varf(X))^{\frac{1}{2}}(Varg(Y))^{\frac{1}{2}}},$$

where the sup is taken over all $f, g \in \mathscr{C}$ such that $E(f(X))^2 < \infty$ and $E(g(Y))^2 < \infty$.

A sequence $\{X_n, n \geq 1\}$ is called negatively associated (NA) if for every pair of disjoint subsets S, T of N,

$$Cov\left\{f(X_i, i \in S), g\left(X_j, j \in T\right)\right\} \leq 0,$$

whenever $f, g \in \mathscr{C}$.

A sequence $\{X_n, n \geq 1\}$ is called ρ^*-mixing if

$$\rho * (s) = \sup \left\{ \rho (S, T); S, T \subset N, \operatorname{dist}(S, T) \geq s \right\} \to 0 \quad as \quad s \to \infty,$$

where

$$\rho(S, T) = \sup \left\{ \left| E(f - Ef)(g - Eg) / \left(\|f - Ef\|_2 \cdot \|g - Eg\|_2 \right) \right| ; f \in L_2(\sigma(S)), g \in L_2(\sigma(T)) \right\}.$$

A sequence $\{X_n, n \geq 1\}$ is called ρ^--mixing, if

$$\rho^-(s) = \sup \left\{ \rho^-(S, T); S, T \subset N, \operatorname{dist}(S, T) \geq s \right\} \to 0 \quad as \quad s \to \infty.$$

where,

$$\rho^-(S, T) = 0 \vee \sup\{ \frac{Cov \left\{ f(X_i, i \in S), g(X_i, j \in T) \right\}}{\sqrt{Var\left\{ f(X_i, i \in S) \right\} Var\left\{ g(X_i, j \in T) \right\}}}; f, g \in C\}.$$

The concept of ρ^--mixing random variables was proposed in 1999 (see [1]). Obviously, ρ^--mixing random variables include NA and ρ^*-mixing random variables, which have a lot of applications, their limit properties have aroused wide interest recently, and a lot of results have been obtained, such as the weak convergence theorems, the central limit theorems of random fields, Rosenthal-type moment inequality, see [1-4]. Zhou [5] studied the almost sure central limit theorem of ρ^--mixing sequences by the conditions provided by Shao: on the conditions of central limit theorem, and if $\varepsilon_0 > 0$, $Var\left(\sum_{i=1}^{n} \frac{1}{i} f\left(\frac{S_i}{\sigma_i} \right) \right) = O\left(\log^{2-\varepsilon_0} n \right)$, where f is Lipschitz function. In this article, we study the almost sure central limit theorem of products of partial sums for ρ^--mixing sequence by the central limit theorem of weighted sums and moment inequality.

Here and in the sequel, let $b_{k,n} = \sum_{i=k}^{n} \frac{1}{i}, k \leq n$ with $b_{k,n} = 0$, $k > n$. Suppose $\{X_n, n \geq 1\}$ be a strictly stationary ρ^--mixing sequence of positive random variables with $EX_1 = \mu > 0$ and $Var(X_1) = \sigma^2 < \infty$. Let $\tilde{S}_n = \sum_{k=1}^{n} Y_k$ and $S_{n,n} = \sum_{k=1}^{n} b_{k,n} Y_k$, where $Y_k = \frac{X_k - \mu}{\sigma}, k \geq 1$. Let $\sigma_n^2 = Var\left(S_{n,n} \right)$, and C denotes a positive constant, which may take different values whenever it appears in different expressions. The following are our main results.

Theorem 1.1 Let $\{X_n, n \geq 1\}$ be a defined as above with $0 < E|X_1|^r < \infty$ for a certain $r > 2$, denote $S_n = \sum_{i=1}^{n} X_i$ and $\gamma = \frac{\sigma}{\mu}$ the coefficient of variation. Assume that

(a_1) $\sigma_1^2 = EX_1^2 + 2 \sum_{n=2}^{\infty} Cov(X_1, X_n) > 0$,

(a_2) $\sum_{n=2}^{\infty} |Cov(X_1, X_n)| < \infty$,

(a_3) $\rho^-(n) = O(\log^{-\delta} n)$, $\exists\ \delta > 1$,

(a_4) $\inf_{n \in N} \frac{\sigma_n^2}{n} > 0$.

Then

$$\forall x \quad \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \frac{S_{k,k}}{\sigma_k} \leq x \right\} = \Phi(x) \quad a.s. \tag{1.1}$$

Here and in the sequel, $I\{\cdot\}$ denotes indicator function and $\Phi(\cdot)$ is the distribution function of standard normal random variable $\mathcal{N}$.

Theorem 1.2 Under the conditions of Theorem 1.1, then

$$\forall x = \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \left(\prod_{i=1}^{k} \frac{S_i}{i\mu} \right)^{\frac{1}{\gamma\sigma k}} \leq x \right\} = F(x) \quad a.s. \tag{1.2}$$

Here and in the sequel, $F(\cdot)$ is the distribution function of the random variable $e^{\sqrt{2}\mathcal{N}}$.

2 Some lemmas

To prove our main results, we need the following lemmas.

Lemma 2.1 [3] Let $\{X_n, n \geq 1\}$ be a weakly stationary ρ^--mixing sequence with $EX_n = 0, \quad 0 < EX_1^2 < \infty$, and

(i) $\sigma_1^2 = EX_1^2 + 2\sum_{n=2}^{\infty} Cov(X_1, X_n) > 0$,

(ii) $\sum_{n=2}^{\infty} |Cov(X_1, X_n)| < \infty$,

then

$$\frac{ES_n^2}{n} \to {\sigma_1}^2, \quad \frac{S_n}{\sigma_1\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \quad as\ n \to \infty.$$

Lemma 2.2 [4] For a positive real number $q \geq 2$, if $\{X_n, n \geq 1\}$ is a sequence of ρ^--mixing random variables with $EX_i = 0$, $E|X_i|^q < \infty$ for every $i \geq 1$, then for all $n \geq 1$, there is a positive constant $C = C(q, \rho^-(\cdot))$ such that

$$E\left(\max_{1\leq j\leq n} |S_j|^q \right) \leq C \left\{ \sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2 \right)^{\frac{q}{2}} \right\}.$$

Lemma 2.3 [6] $\sum_{i=1}^{n} b_{i,n}^2 = 2n - b_{1,n}$.

Lemma 2.4 [[3], Theorem 3.2] Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of centered random variables with $EX_{ni}^2 < \infty$ for each $i = 1,2,...,n$. Assume that they are ρ^--mixing. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers with $a_{ni} = \pm 1$ for $i = 1, 2,..., n$. Denote $\sigma_n^2 = Var\left(\sum_{i=1}^{n} a_{ni} X_{ni} \right)$ and suppose that

$$\sup_n \frac{1}{\sigma_n^2} \sum_{i=1}^{n} EX_{ni}^2 < \infty,$$

and

$$\lim_{n\to\infty}\sup \frac{1}{\sigma_n^2} \sum_{\substack{i,j: |i-j| \geq A \\ 1\leq i,j\leq n}} Cov(X_{ni}, X_{nj})^- \to 0 \quad as \quad A \to \infty,$$

and the following Lindeberg condition is satisfied:

$$\frac{1}{\sigma_n^2}\sum_{i=1}^{n} EX_{ni}^2 I\{|X_{ni}| \geq \varepsilon\sigma_n\} \to 0 \ \ as \ \ n \to \infty$$

for every $\varepsilon > 0$. Then

$$\frac{1}{\sigma_n}\sum_{i=1}^{n} a_{ni}X_{ni} \xrightarrow{d} \mathcal{N}(0,1) \ \ as \ \ n \to \infty.$$

Lemma 2.5 Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of ρ^--mixing random variables with $EX_n = 0$ and $\sum_{n=2}^{\infty} |Cov(X_1, X_n)| < \infty$, $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers such that $\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty$ and $\max_{1\leq i\leq n} |a_{ni}| \to 0$ as $n \to \infty$. If $Var\left(\sum_{i=1}^{n} a_{ni}X_i\right) = 1$ and $\{X_n^2\}$ is an uniformly integrable family, then

$$\sum_{i=1}^{n} a_{ni}X_i \xrightarrow{d} \mathcal{N}(0,1) \ \ as \ \ n \to \infty.$$

Proof Notice that

$$\sum_{i=1}^{n} a_{ni}X_i = \sum_{i=1}^{n} \frac{a_{ni}}{|a_{ni}|} |a_{ni}| X_i =: \sum_{i=1}^{n} b_{ni}Y_{ni},$$

where $b_{ni} = \frac{a_{ni}}{|a_{ni}|}$ and $Y_{ni} = |a_{ni}|X_i$. Then $\{Y_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of ρ^--mixing centered random variables with $EY_{ni}^2 = a_{ni}^2 EX_i^2 < \infty$ and $b_{ni} = \pm 1$ for $i = 1, 2,..., n$ and $\sigma_n^2 = Var\left(\sum_{i=1}^{n} b_{ni}Y_{ni}\right) = 1$. Note that $\{X_n^2\}$ is an uniformly integrable family, we have

$$\sup_n \frac{1}{\sigma_n^2}\sum_{i=1}^{n} EY_{ni}^2 = \sup_n \sum_{i=1}^{n} a_{ni}^2 EX_i^2 \leq \sup_n \sum_{i=1}^{n} a_{ni}^2 \cdot \sup_i EX_i^2 < \infty,$$

and

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{\substack{i,j:|i-j|\geq A \\ 1\leq i,j\leq n}} Cov(Y_{ni}, Y_{nj})^- \\
&= \limsup_{n\to\infty} \sum_{\substack{i,j:|i-j|\geq A \\ 1\leq i,j\leq n}} Cov(|a_{ni}| X_i, |a_{nj}| X_j)^- \\
&\leq \limsup_{n\to\infty} \sum_{\substack{i,j:|i-j|\geq A \\ 1\leq i,j\leq n}} |a_{ni}| \cdot |a_{nj}| \cdot |Cov(X_i, X_j)| \\
&\leq C\left(\limsup_{n\to\infty}\left(\sum_{\substack{i,j:|i-j|\geq A \\ 1\leq i,j\leq n}} |a_{ni}|^2 \cdot |Cov(X_i, X_j)| + \sum_{\substack{i,j:|i-j|\geq A \\ 1\leq i,j\leq n}} |a_{ni}|^2 \cdot |Cov(X_i, X_j)|\right)\right) \\
&\leq C \sup_n \sum_{i=1}^n |a_{ni}|^2 \cdot \sum_{i>A} |Cov(X_1, X_i)| \\
&\to 0 \quad as \quad A \to \infty,
\end{aligned}$$

and $\forall\ \varepsilon > 0$, we get

$$\begin{aligned}
&\frac{1}{\sigma_n^2} \sum_{i=1}^n EY_{ni}^2 I\{|Y_{ni}| \geq \varepsilon\sigma_n\} \\
&= \sum_{i=1}^n a_{ni}^2 EX_i^2 I\{|a_{ni}| \cdot |X_i| \geq \varepsilon\} \\
&\leq \sup_n \sum_{i=1}^n a_{ni}^2 \cdot EX_1^2 I\{|a_{ni}| \cdot |X_1| \geq \varepsilon\} \\
&\leq \sup_n \sum_{i=1}^n a_{ni}^2 \cdot EX_1^2 I\left\{\max_{1\leq i\leq n} |a_{ni}| \cdot |X_1| \geq \varepsilon\right\} \to 0 \quad as\ n \to \infty,
\end{aligned}$$

thus the conclusion is proved by Lemma 2.4.

Lemma 2.6 [2] Suppose that $f_1(x)$ and $f_2(y)$ are real, bounded, absolutely continuous functions on R with $|f'_1(x)| \leq C_1$ and $|f'_2(y)| \leq C_2$. Then for any random variables X and Y,

$$|Cov(f_1(X), f_2(Y))| \leq C_1 C_2 \left\{-Cov(X, Y) + 8\rho^-(X, Y)\|X\|_{2,1}\|Y\|_{2,1}\right\},$$

where $\|X\|_{2,1} = \int_0^\infty P^{\frac{1}{2}}(|X| > x)\,dx\cdot$

Lemma 2.7 Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of ρ^--mixing random variables with $EX_1 = 0,\ \ 0 < EX_1^2 < \infty$ and

$$0 < \sigma_1^2 = EX_1^2 + 2\sum_{n=2}^\infty Cov(X_1, X_n) < \infty,$$

$$\sum_{n=2}^\infty |Cov(X_1, X_n)| < \infty,$$

then for $0 < p < 2$, we have

$$n^{-\frac{1}{p}} S_n \to 0 \ \ a.s. \ \ as \ \ n \to \infty.$$

Proof By Lemma 2.1, we have

$$\lim_{n\to\infty} \frac{ES_n^2}{n} = \sigma_1^2. \tag{2.1}$$

Let $n_k = k^{\alpha}$, where $\alpha > \max\left\{1, \frac{p}{2-p}\right\}$. By (2.1), we get

$$\sum_{k=1}^{\infty} P\left\{|S_{nk}| \geq \varepsilon n_k^{\frac{1}{p}}\right\} \leq \sum_{k=1}^{\infty} \frac{ES_{n_k}^2}{\varepsilon^2 n_k^{\frac{2}{p}}} \leq \sum_{k=1}^{\infty} \frac{C}{\varepsilon^2 k^{\alpha\left(\frac{2}{p}-1\right)}} < \infty.$$

From Borel-Cantelli lemma, it follows that

$$n_k^{-\frac{1}{p}} S_{n_k} \to 0 \ \ a.s. \ \ as \ \ k \to \infty. \tag{2.2}$$

And by Lemma 2.2, it follows that

$$\sum_{k=1}^{\infty} P\left\{\max_{n_k \leq n < n_{k+1}} \frac{\left|S_n - S_{n_k}\right|}{n^{\frac{1}{p}}} \geq \varepsilon\right\} \leq \sum_{k=1}^{\infty} \frac{E \max_{n_k \leq n < n_{k+1}} \left|S_n - S_{n_k}\right|^2}{\varepsilon^2 n_k^{\frac{2}{p}}}$$

$$= \sum_{k=1}^{\infty} \frac{E \max_{n_k \leq n < n_{k+1}} \left|\sum_{i=n_{k+1}}^{n} X_i\right|^2}{\varepsilon^2 n_k^{\frac{2}{p}}} \leq C \sum_{k=1}^{\infty} \frac{(n_{k+1} - n_k)}{\varepsilon^2 n_k^{\frac{2}{p}}} \leq C \sum_{k=1}^{\infty} \frac{1}{k^{\alpha\left(\frac{2}{p}-1\right)}} < \infty.$$

By Borel-Cantelli lemma, we conclude that

$$\max_{n_k \leq n < n_{k+1}} \frac{\left|S_n - S_{n_k}\right|}{n^{\frac{1}{p}}} \to 0 \ \ a.s. \ \ as \ \ n \to \infty. \tag{2.3}$$

For every n, there exist n_k and n_{k+1} such that $n_k \leq n < n_{k+1}$, by (2.2) and (2.3), we have

$$\begin{aligned} \frac{|S_n|}{n^{\frac{1}{p}}} &= \frac{\left|S_n - S_{n_k} + S_{n_k}\right|}{n^{\frac{1}{p}}} \\ &\leq \frac{\left|S_{n_k}\right|}{n_k^{\frac{1}{p}}} + \max_{n_k \leq n < n_{k+1}} \frac{\left|S_n - S_{n_k}\right|}{n^{\frac{1}{p}}} \to 0 \ \ a.s. \ \ as \ \ n \to \infty. \end{aligned}$$

The proof is now completed.

3 Proof of the theorems

Proof of Theorem 1.1 By the property of ρ^{-}-mixing sequence, it is easy to see that $\{Y_n\}$ is a strictly stationary ρ^{-}-mixing sequence with $EY_1 = 0$ and $EY_1^2 = 1$. We first prove

$$\frac{S_{n,n}}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad as \quad n \to \infty. \tag{3.1}$$

Let $a_{ni} = \frac{b_{i,n}}{\sigma_n}, 1 \le i \le n, n \ge 1$. Obviously,

$$Var\left(\sum_{i=1}^{n} a_{ni} Y_i\right) = 1.$$

From condition (a_4) in Theorem 1.1 and Lemma 2.3, we have

$$\sup_n \sum_{i=1}^{n} a_{ni}^2 = \sup_n \sum_{i=1}^{n} \frac{b_{i,n}^2}{\sigma_n^2} = \sup_n \frac{2n - b_{1,n}}{\sigma_n^2} \le C \sup_n \frac{2n - b_{1,n}}{n} < \infty,$$

and

$$\max_{1\le i\le n} |a_{ni}| = \max_{1\le i\le n} \frac{b_{i,n}}{\sigma_n} \le \frac{b_{1,n}}{\sigma_n} \le \frac{C \log n}{\sqrt{n}} \to 0 \quad as \quad n \to \infty.$$

By stationarity of $\{Y_n, n \ge 1\}$ and E $|X_1|^2 < \infty$, we know that $\{Y_n^2\}$ is uniformly integrable, and from condition (a_2) in Theorem 1.1, we get $\sum_{n=2}^{\infty} |Cov(Y_1, Y_n)| < \infty$, so applying Lemma 2.5, we have

$$\sum_{i=1}^{n} a_{ni} Y_i \xrightarrow{d} \mathcal{N}(0,1).$$

Notice that

$$\sum_{i=1}^{n} a_{ni} Y_i = \sum_{i=1}^{n} \frac{b_{i,n} Y_i}{\sigma_n} = \frac{S_{n,n}}{\sigma_n},$$

so (3.1) is valid. Let $f(x)$ be a bounded Lipschitz function and have a Radon-Nikodyn derivative $h(x)$ bounded by Γ. From (3.1), we have

$$Ef\left(\frac{S_{n,n}}{\sigma_n}\right) \to Ef(\mathcal{N}(0,1)) \quad as \quad n \to \infty,$$

thus

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} Ef\left(\frac{S_{k,k}}{\sigma_k}\right) - Ef(\mathcal{N}(0,1)) \to 0 \quad as \quad n \to \infty. \tag{3.2}$$

On the other hand, note that (1.1) is equivalent to

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} f\left(\frac{S_{k,k}}{\sigma_k}\right) = \int_{-\infty}^{\infty} f(x) d\Phi(x) = Ef(\mathcal{N}(0,1)) \quad a.s. \tag{3.3}$$

from Section 2 of Peligrad and Shao [7] and Theorem 7.1 on P_{42} from Billingsley [8]. Hence, to prove (3.3), it suffices to show that

$$T_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left[f\left(\frac{S_{k,k}}{\sigma_k}\right) - Ef\left(\frac{S_{k,k}}{\sigma_k}\right) \right] \to 0 \quad a.s. \quad n \to \infty \tag{3.4}$$

by (3.2). Let $\xi_k = f\left(\frac{S_{k,k}}{\sigma_k}\right) - Ef\left(\frac{S_{k,k}}{\sigma_k}\right)$, $1 \le k \le n$m we have

$$\begin{aligned} ET_n^2 &= \frac{1}{\log^2 n} E\left(\sum_{k=1}^{n} \frac{\xi_k}{k}\right)^2 \\ &\le \frac{1}{\log^2 n} \sum_{1 \le k \le l \le n, 2k \ge l} \frac{|E\xi_k \xi_l|}{kl} + \frac{1}{\log^2 n} \sum_{1 \le k \le l \le n, 2k \ge l} \frac{|E\xi_k \xi_l|}{kl} \\ &:= I_1 + I_2. \end{aligned} \tag{3.5}$$

By the fact that f is bounded, we have

$$I_1 \le \frac{C}{\log^2 n} \sum_{k=1}^{n} \sum_{l=k}^{2k} \frac{1}{kl} = \frac{C}{\log^2 n} \sum_{k=1}^{n} \frac{1}{k} \sum_{l=k}^{2k} \frac{1}{l} \le C(\log^{-1} n). \tag{3.6}$$

Now we estimate I_2, if $l > 2k$, we have

$$\begin{aligned} S_{l,l} - S_{2k,2k} &= \left(b_{1,l}Y_1 + b_{2,l}Y_2 + \cdots + b_{l,l}Y_l\right) - \left(b_{1,2k}Y_1 + b_{2,2k}Y_2 + \cdots + b_{2k,2k}Y_{2k}\right) \\ &= \left(b_{2k+1,l}Y_{2k+1} + \cdots + b_{l,l}Y_l\right) + b_{2k+1,l}\tilde{S}_{2k}, \end{aligned}$$

and

$$\begin{aligned} |E\xi_k\xi_l| &= \left|Cov\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l}}{\sigma_l}\right)\right)\right| \\ &\le \left|Cov\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l}}{\sigma_l}\right) - f\left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right)\right)\right| \\ &+ \left|Cov\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right)\right)\right|. \end{aligned}$$

By Lemma 2.3 and condition (a_2) in Theorem 1.1, we have

$$\begin{aligned} Var(S_{k,k}) &= \sum_{i=1}^{k} b_{i,k}^2 EY_i^2 + 2\sum_{j=1}^{k-1} \sum_{i=j+1}^{k} b_{i,k} b_{j,k} Cov\left(Y_i, Y_j\right) \\ &\le \sum_{i=1}^{k} b_{i,k}^2 + 2\sum_{j=1}^{k} b_{j,k}^2 \sum_{i=j+1}^{k} \left|Cov(Y_i, Y_j)\right| \\ &\le Ck, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & Var\left(S_{l,l} - S_{2k,2k} - b_{2k+1,l}\tilde{S}_{2k}\right) \\ &= \sum_{i=2k+1}^{l} b_{i,l}^2 EY_i^2 + 2\sum_{j=2k+1}^{l-1} \sum_{i=j+1}^{l} b_{i,l} b_{j,l} Cov\left(Y_i, Y_j\right) \\ &\le \sum_{i=2k+1}^{l} b_{i,l}^2 + 2\sum_{j=1}^{l} b_{i,l}^2 \sum_{i=j+1}^{l} \left|Cov(Y_i, Y_j)\right| \\ &\le Cl. \end{aligned}$$

By Lemma 2.6, the definition of ρ^{-}-mixing sequence and condition (a_4), we have

$$\begin{aligned}
&\left|Cov\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l}-S_{2k,2k}-b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right)\right)\right| \\
&\leq C\left\{-Cov\left(\tfrac{S_{k,k}}{\sigma_k}, \tfrac{S_{l,l}-S_{2k,2k}-b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right)\right. \\
&\left.+8\rho^{-}\left(\tfrac{S_{k,k}}{\sigma_k}, \tfrac{S_{l,l}-S_{2k,2k}-b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right)\cdot\left\|\tfrac{S_{k,k}}{\sigma_k}\right\|_{2,1}\cdot\left\|\tfrac{S_{l,l}-S_{2k,2k}-b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right\|_{2,1}\right\} \\
&\leq C\rho^{-}(k)\left(Var\left(\tfrac{S_{k,k}}{\sigma_k}\right)\right)^{\frac{1}{2}}\left(Var\left(\tfrac{S_{l,l}-S_{2k,2k}-b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right)\right)^{\frac{1}{2}} \\
&\quad + C\rho^{-}(k)\left\|\tfrac{S_{k,k}}{\sigma_k}\right\|_{2,1}\cdot\left\|\tfrac{S_{l,l}-S_{2k,2k}-b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right\|_{2,1} \\
&\leq C\rho^{-}(k) + C\rho^{-}(k)\left\|\tfrac{S_{k,k}}{\sigma_k}\right\|_{2,1}\cdot\left\|\tfrac{S_{l,l}-S_{2k,2k}-b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right\|_{2,1}.
\end{aligned}$$

By the inequality $\|X\|_{2,1} \leq \frac{r}{r-2}\|X\|_r (r > 2)$ (cf. Zhang [[2], p. 254] or Ledoux and Talagrand [[9], p. 251]), we get

$$\left\|\frac{S_{k,k}}{\sigma_k}\right\|_{2,1} \leq \frac{r}{r-2}\left\|\frac{S_{k,k}}{\sigma_k}\right\|_r = \frac{r}{r-2}\frac{1}{\sigma_k}\left(E\left|S_{k,k}\right|^r\right)^{\frac{1}{r}},$$

and

$$\begin{aligned}
E\left|S_{k,k}\right|^r &= E\left|\sum_{j=1}^{k} b_{j,k}Y_j\right|^r \\
&\leq C\left\{\sum_{j=1}^{k} b_{j,k}^r E\left|X_j\right|^r + \left(\sum_{j=1}^{k} b_{j,k}^2 EX_j^2\right)^{\frac{r}{2}}\right\} \\
&\leq C\left\{k\left(\log^r k\right) + k^{\frac{r}{2}}\right\},
\end{aligned}$$

thus

$$\left\|\frac{S_{k,k}}{\sigma_k}\right\|_{2,1} \leq C\left(\frac{r}{r-2}\cdot\frac{\log k}{k^{\frac{1}{2}-\frac{1}{r}}} + \frac{r}{r-2}\right) < C,$$

similarly,

$$\left\|\frac{S_{l,l}-S_{2k,2k}-b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right\|_{2,1} < C,$$

hence

$$\left|Cov\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l}-S_{2k,2k}-b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right)\right)\right| \leq C\rho^{-}(k).$$

Similarly to (3.7), we have

$$\begin{aligned} Var\left(S_{2k,2k}\right) &= \sum_{i=1}^{2k} b_{i,2k}^2 EY_i^2 + 2\sum_{j=1}^{2k-1}\sum_{i=j+1}^{2k} b_{i,2k}b_{j,2k}Cov\left(Y_i, Y_j\right) \\ &\le \sum_{i=1}^{2k} b_{i,2k}^2 + 2\sum_{j=1}^{2k-1} b_{i,2k}^2 \sum_{i=j+1}^{2k}\left|Cov\left(Y_i, Y_j\right)\right| \\ &\le Ck, \end{aligned}$$

and

$$\begin{aligned} Var\left(\tilde{S}_{2k}\right) &= Var\left(\sum_{i=1}^{2k} Y_i\right) \\ &= \sum_{i=1}^{2k} EY_i^2 + 2\sum_{i=1}^{2k-1}\sum_{j=i+1}^{2k}\left|Cov\left(Y_i, Y_j\right)\right| \\ &= 2k + 2\sum_{i=1}^{2k-1}\sum_{j=2}^{2k-i+1} Cov\left(Y_i, Y_j\right) \\ &\le Ck. \end{aligned}$$

Since f is a bounded Lipschitz function, we have

$$\begin{aligned} &\left|Cov\left(f\left(\frac{S_{k,k}}{\sigma_k}\right), f\left(\frac{S_{l,l}}{\sigma_l}\right) - f\left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l}\right)\right)\right| \\ &\le CE\frac{\left|S_{2k,2k} + b_{2k+1,l}\tilde{S}_{2k}\right|}{\sigma_l} \\ &\le C\frac{\left(Var\left(S_{2k,2k}\right)\right)^{\frac{1}{2}}}{\sigma_l} + C\frac{b_{2k+1,l}\left(Var\left(\tilde{S}_{2k}\right)\right)^{\frac{1}{2}}}{\sigma_l} \\ &\le C\left(\frac{k}{l}\right)^{\frac{1}{2}} + C\left(\frac{k}{l}\right)^{\frac{1}{2}}\log\tfrac{1}{2k} \\ &\le C\left(\frac{k}{l}\right)^{\varepsilon}, \end{aligned}$$

where $0 < \varepsilon < \dfrac{1}{2}$. Hence if $l > 2k$, we have

$$|E\xi_k\xi_l| \le C\left[\rho^-(k) + \left(\frac{k}{l}\right)^{\varepsilon}\right].$$

Thus

$$\begin{aligned} I_2 &\le \frac{C}{\log^2 n}\sum_{l=2}^{n}\sum_{k=1}^{l-1}\frac{1}{k^{1-\varepsilon}l^{1+\varepsilon}} + \frac{C}{\log^2 n}\sum_{l=2}^{n}\frac{1}{l}\sum_{k=1}^{l-1}\frac{\rho^-(k)}{k} \\ &\le \frac{C}{\log^2 n}\sum_{l=2}^{n}\frac{1}{l^{1+\varepsilon}}\frac{(l-1)^{\varepsilon}}{\varepsilon} + \frac{C}{\log^2 n}\sum_{l=2}^{n}\frac{1}{l}\sum_{k=1}^{n}\frac{\log^{-\delta}k}{k} \\ &\le \frac{C}{\log^2 n}\sum_{l=2}^{n}\frac{1}{l} + \frac{C}{\log^2 n}\sum_{l=2}^{n}\frac{1}{l}\sum_{k=1}^{n}\frac{\log^{-\delta}k}{k} \\ &\le C\log^{-1}n. \end{aligned} \tag{3.8}$$

Associated with (3.5), (3.6), and (3.8), we have

$$ET_n^2 \le C\log^{-1}n. \tag{3.9}$$

To prove (3.4), let $n_k = e^{k^{\tau}}$, where $\tau > 1$. From (3.9), we have

$$\sum_{k=1}^{\infty} ET_{n_k}^2 \le C\sum_{k=1}^{\infty}\log^{-1}n_k = C\sum_{k=1}^{\infty}\frac{1}{k^{\tau}} < \infty.$$

Thus $\forall \varepsilon > 0$, we have

$$\sum_{k=1}^{\infty} P\left\{\left|T_{n_k}\right| \ge \varepsilon\right\} \le \sum_{k=1}^{\infty}\frac{ET_{n_k}^2}{\varepsilon^2} < \infty.$$

By Borel-Cantelli lemma, we have

$$T_{n_k \to 0} \;\; a.s. \;\; as \;\; k \to \infty.$$

Note that

$$\frac{\log n_{k+1}}{\log n_k} = \frac{(k+1)^{\tau}}{k^{\tau}} \to 1 \;\; as \;\; k \to \infty.$$

For every n, there exist n_k and n_{k+1} satisfying $n_k < n \le n_{k+1}$, we have

$$\begin{aligned} |T_n| &\le \frac{1}{\log n_k}\left|\sum_{i=1}^{n_k}\frac{\xi_i}{i}\right| + \frac{1}{\log n_k}\sum_{i=n_k}^{n_{k+1}}\frac{|\xi_i|}{i} \\ &\le \left|T_{n_k}\right| + C\left(\frac{\log n_{k+1}}{\log n_k} - 1\right) \to 0 \;\; a.s. \;\; as \;\; n \to \infty, \end{aligned}$$

(3.4) is completed, so the proof of Theorem 1.1 is completed.

Proof of Theorem 1.2 Let $C_i = \dfrac{S_i}{\mu_i}$, we have

$$\frac{1}{\gamma\sigma_k}\sum_{i=1}^{k}(C_i - 1) = \frac{1}{\gamma\sigma_k}\sum_{i=1}^{k}\left(\frac{S_i}{\mu i} - 1\right) = \frac{1}{\sigma_k}\sum_{i=1}^{k} b_{i,k}Y_i = \frac{S_{k,k}}{\sigma_k}.$$

Hence (1.1) is equivalent to

$$\forall x \quad \lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{1}{\gamma\sigma_k}\sum_{i=1}^{k}(C_i - 1) \le x\right\} = \Phi(x) \;\; a.s. \tag{3.10}$$

On the other hand, to prove (1.2), it suffices to show that

$$\forall x \quad \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \frac{1}{\gamma \sigma_k} \sum_{i=1}^{k} \log C_i \le x \right\} = \Phi(x) \quad a.s. \tag{3.11}$$

By Lemma 2.7, for enough large i, for some $\frac{4}{3} < p < 2$ we have

$$|C_i - 1| = \left| \frac{S_i}{\mu i} - 1 \right| \le C i^{\frac{1}{p}-1} \quad a.s.$$

It is easy to know that $\log(1+ x) = x + O(x^2)$ for $|x| < \frac{1}{2}$, thus

$$\left| \sum_{k=1}^{n} \log C_k - \sum_{k=1}^{n} (C_k - 1) \right| \le C \sum_{k=1}^{n} (C_k - 1)^2 \le C \sum_{k=1}^{n} k^{\frac{2}{p}-2} \le C n^{\frac{2}{p}-1} \quad a.s.,$$

and

$$\sum_{k=1}^{n} (C_k - 1) - C n^{\frac{2}{p}-1} \le \sum_{k=1}^{n} \log C_k \le \sum_{k=1}^{n} (C_k - 1) + C n^{\frac{2}{p}-1} \quad a.s.$$

Hence for arbitrary small $\varepsilon > 0$, there is $n_0 = n_0(\omega, \varepsilon)$, such that for every $n > n_0$ and arbitrary x,

$$I\left\{ \frac{1}{\gamma \sigma_k} \sum_{i=1}^{k} (C_i - 1) \le x - \varepsilon \right\} \le I\left\{ \frac{1}{\gamma \sigma_k} \sum_{i=1}^{k} \log C_i \le x \right\} \le I\left\{ \frac{1}{\gamma \sigma_k} \sum_{i=1}^{k} (C_i - 1) \le x + \varepsilon \right\},$$

so by (3.10), we know that (3.11) is true, and (3.11) is equivalent to (1.2), thus the proof of Theorem 1.2 is complete.

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Authors' contributions
XT and YZ carried out the design of the study and performed the analysis. YZ participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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