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Some properties of an integral operator defined by convolution

Muhammad Arif[1*], Khalida Inayat Noor[2] and Fazal Ghani[1]

* Correspondence: marifmaths@awkum.edu.pk
[1]Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan
Full list of author information is available at the end of the article

Abstract

In this investigation, motivated from Breaz study, we introduce a new family of integral operator using famous convolution technique. We also apply this newly defined operator for investigating some interesting mapping properties of certain subclasses of analytic and univalent functions.

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1. Introduction

Let A denote the class of analytic function satisfying the condition $f(0) = f'(0) - 1 = 0$ in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. By S, C, S^*, C^*, and K we means the well-known subclasses of A which consist of univalent, convex, starlike, quasi-convex, and close-to-convex functions, respectively. The well-known Alexander-type relation holds between the classes C and S^* and C^* and K, that is,

$$f(z) \in C \Leftrightarrow zf'(z) \in S^*,$$

and

$$f(z) \in C^* \Leftrightarrow zf'(z) \in K.$$

It was proved in [1] that a locally univalent function $f(z)$ is close-to-convex, if and only if

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -\pi, \quad z = re^{i\theta}, \tag{1.1}$$

for each $r \in (0,1)$ and every pair θ_1, θ_2 with $0 \le \theta_1 < \theta_2 \le 2\pi$.

Let $P_k(\xi)$ be the class of functions $p(z)$ analytic in $\mathbb{U}$ with $p(0) = 1$ and

$$\int_0^{2\pi} \left|\frac{\operatorname{Re} p(z) - \xi}{1 - \xi}\right| d\theta \le k\pi, \quad z = re^{i\theta}, \; k \ge 2.$$

This class was introduced in [2] and for $k = 2$, $\xi = 0$, the class $p_k(\xi)$ reduces to the class P of functions with positive real part. We consider the following classes:

$$R_k(\xi) = \left\{ f(z) \in A : \frac{zf'(z)}{f(z)} \in P_k(\xi),\ z \in \mathbb{U} \right\}$$
$$T_k(\xi) = \left\{ f(z) \in A : \exists\, g(z) \in C : \frac{f'(z)}{g'(z)} \in P_k(\xi),\ z \in \mathbb{U} \right\}.$$

These classes were studied by Noor [3-5] and Padmanabhan and Parvatham [2]. Also it can easily be seen that $R_2(0) = S^*$ and $T_2(0) = K$, where S^* and K are the well-known classes of starlike and close-to-convex functions.

Using the same method as that of Kaplan [1], Noor [6] extend the result of Kaplan given in (1.1), and proved that a locally univalent function $f(z)$ is in the class T_k, if and only if

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -\frac{k}{2}\pi, \quad z = re^{i\theta}, \tag{1.2}$$

for each $r \in (0,1)$ and every pair θ_1, θ_2 with $0 \le \theta_1 < \theta_2 \le 2\pi$

For any two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n,\ (z \in \mathbb{U}),$$

the convolution (Hadamard product) of $f(z)$ and $g(z)$ is defined by

$$f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n,\ (z \in \mathbb{U}).$$

Using the techniques from convolution theory many authors generalized Breaz operator in several directions, see [7,8] for example. Here, we introduce a generalized integral operator $I_n(f_i, g_i, h_i)(z)$: $A^n \to A$ as follows

$$I_n(f_i, g_i, h_i)(z) = \int_0^z \prod_{i=1}^{n} \left((f_i(t) * g_i(t))'\right)^{\alpha_i} \left(\frac{h_i(t)}{t}\right)^{\beta_i} dt, \tag{1.3}$$

where $f_i(z), g_i(z), h_i(z) \in A$ with $f_i(z) * g_i(z) \ne 0$ and $\alpha_i, \beta_i \ge 0$ for $i = 1, 2, \ldots, n$. The operator $I_n(f_i, g_i, h_i)(z)$ reduces to many well-known integral operators by varying the parameters α_i, β_i and by choosing suitable functions instead of $f_i(z), g_i(z)$. For example,

(i) If we take $g_i(z) = \dfrac{z}{(1-z)}$ for all $1 \le i \le n$, we obtain the integral operator

$$I_n(f_i, h_i)(z) = \int_0^z \prod_{i=1}^{n} (f_i'(t))^{\alpha_i} \left(\frac{h_i(t)}{t}\right)^{\beta_i} dt, \tag{1.4}$$

introduced in [9].

(ii) If we take $\alpha_i = 0$ and $1 \le i \le n$, we obtain the integral

$$I_n(h_i)(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{h_i(t)}{t}\right)^{\beta_i} dt,$$

introduced and studied by Breaz and Breaz [10].

(iii) If we take $g_i(z) = \dfrac{z}{(1-z)}$, $\beta_i = 0$, we obtain the integral operator

$$I_n(f_i)(z) = \int_0^z \prod_{i=1}^n \left(f_i'(t)\right)^{\alpha_i} dt,$$

introduced and studied by Breaz et al. [11].

(iv) If we take $n = 1$, $\alpha_1 = 0$ and $\beta_1 = 1$ in (1.4), we obtain the Alexander integral operator

$$I_n(h_1)(z) = \int_0^z \left(\frac{h_1(t)}{t}\right) dt,$$

introduced in [12].

(v) If we take $n = 1$, $\alpha_1 = 0$ and $\beta_1 = \beta$, we obtain the integral operator

$$I_n(h_1)(z) = \int_0^z \left(\frac{h_1(t)}{t}\right)^{\beta} dt,$$

studied in [13].

In this article, we study the mapping properties of different subclasses of analytic and univalent functions under the integral operator given in (1.3). To prove our main results, we need the following lemmas.

Lemma 1.1 [14]. Let $f(z) \in R_k(\xi)$ for $k \le 2$, $0 \le \xi < 1$. Then with $0 \le \theta_1 < \theta_2 \le 2\pi$ and $z = re^{i\theta}$, $r < 1$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta > -\left(\frac{k}{2}-1\right)(1-\xi)\pi.$$

Lemma 1.2 [15]. If $f(z) \in C$ and $g(z) \in K$, then $f(z)*g(z) \in K$.

2. Main results

Theorem 2.1. Let $f_i(z) \in S^*$, $g_i(z) \in C^*$ and $h_i(z) \in R_k(\xi)$ with $0 \le \xi < 1$, $k \ge 2$ for all $1 \le i \le n$ If

$$\sum_{i=1}^n \left(\alpha_i + \left(\frac{k}{2}-1\right)(1-\rho)\beta_i\right) \le 1, \tag{2.1}$$

then integral operator defined by (1.3) belongs to the class of close-to-convex functions.

Proof. Let $f_i(z) \in S^*$ and $g_i(z) \in C^*$. Then there exists $\phi_i(z) \in C$ such that

$$f_i(z) = z\varphi_i'(z).$$

Now consider

$$f_i(z) * g_i(z) = z\varphi_i'(z) * g_i(z) = \varphi_i(z) * zg_i'(z).$$

Since $g_i(z) \in C^*$, then by Alexander-type relation $zg_i'(z) \in K$. So, by Lemma 1.2, we have

$$\varphi_i(z) * zg_i'(z) \in K,$$

which implies that

$$f_i(z) * g_i(z) \in K$$

and hence, by using (1.1),

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{z\left(f_i(z) * g_i(z)\right)''}{\left(f_i(z) * g_i(z)\right)'}\right\} d\theta > -\pi. \tag{2.2}$$

From (1.3), we obtain

$$I_n\left(f_i, g_i, h_i\right)'(z) = \prod_{i=1}^{n}\left(\left(f_i(z) * g_i(z)\right)'\right)^{\alpha_i}\left(\frac{h_i(z)}{z}\right)^{\beta_i}. \tag{2.3}$$

Differentiating (2.3) logarithmically, we have

$$\begin{aligned} 1 + \frac{I_n\left(f_i, g_i, h_i\right)''(z)}{I_n\left(f_i, g_i, h_i\right)'(z)} &= \sum_{i=1}^{n}\alpha_i \frac{z\left(f_i(z) * g_i(z)\right)''}{\left(f_i(z) * g_i(z)\right)'} + \sum_{i=1}^{\mathrm{n}}\beta_i\left(\frac{zh_i'(z)}{h_i(z)} - 1\right) + 1 \\ &= \sum_{i=1}^{n}\alpha_i\left(1 + \frac{z\left(f_i(z) * g_i(z)\right)''}{\left(f_i(z) * g_i(z)\right)'}\right) + \sum_{i=1}^{\mathrm{n}}\beta_i\left(\frac{zh_i'(z)}{h_i(z)}\right) + 1 - \sum_{i=1}^{\mathrm{n}}\left(\alpha_i + \beta_i\right). \end{aligned}$$

Taking real part and then integrating with respect to θ, we get

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{I_n\left(f_i, g_i, h_i\right)''(z)}{I_n\left(f_i, g_i, h_i\right)'(z)}\right\} d\theta &= \sum_{i=1}^{n}\alpha_i \int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{z\left(f_i(z) * g_i(z)\right)''}{\left(f_i(z) * g_i(z)\right)'}\right) d\theta \\ &+ \sum_{i=1}^{\mathrm{n}}\beta_i \int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{zh_i'(z)}{h_i(z)}\right) d\theta + \left(1 - \sum_{i=1}^{\mathrm{n}}\left(\alpha_i + \beta_i\right)\right)\left(\theta_2 - \theta_1\right). \end{aligned}$$

Using (2.2) and Lemma 1.1, we have

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{I_n\left(f_i, g_i, h_i\right)''(z)}{I_n\left(f_i, g_i, h_i\right)'(z)}\right\} d\theta &> -\pi \sum_{i=1}^{n}\left(\alpha_i + \left(\frac{k}{2} - 1\right)(1-\rho)\beta_i\right) \\ &+ \left(1 - \sum_{i=1}^{\mathrm{n}}\left(\alpha_i + \beta_i\right)\right)\left(\theta_2 - \theta_1\right) \end{aligned}$$

From (2.1), we can easily write

$$\sum_{i=1}^{n}\left(\alpha_i + \beta_i\right) < \sum_{i=1}^{n}\left(\alpha_i + \left(\frac{k}{2} - 1\right)(1-\rho)\beta_i\right) \le 1.$$

This implies that

$$\sum_{i=1}^{n}\left(\alpha_i + \beta_i\right) < 1,$$

so, minimum is for $\theta_1 = \theta_2$, we obtain

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\ \left\{1 + \frac{I_n\left(f_i,\ g_i,\ h_i\right)''(z)}{I_n\left(f_i,\ g_i,\ h_i\right)'(z)}\right\}\ d\theta > -\pi,$$

and this implies that $I_n(f_i,\ g_i,\ h_i)(z) \in\ K$.

For $k = 2$ in Theorem 2.1, we obtain

Corollary 2.3. Let $f_i(z) \in\ S^*$, $g_i(z) \in\ C^*$ and $h_i(z) \in\ S^*(\xi)$ with $0 \leq \xi < 1$, for all $1 \leq i \leq$ n. If

$$\sum_{i=1}^{n} \alpha_i \leq 1,$$

then $I_n(f_i,\ g_i,\ h_i)(z) \in\ K$.

Theorem 2.4. Let $f_i(z) \in\ T_k$ and $h_i(z) \in\ R_k$ for $1 \leq i \leq n$. If $\alpha_i,\ \beta_i \geq 0$ such that $\alpha_i + \beta_i \neq 0$ and

$$\sum_{i=1}^{n} \left(\frac{k}{2}\left(\alpha_i + \beta_i\right) - \beta_i\right) \leq 1, \tag{2.4}$$

then $I_n(f_i,\ h_i)(z)$ defined by (1.4) belongs to the class of close-to-convex functions.

Proof. From (1.4), we have

$$I_n\left(f_i,\ h_i\right)'(z) = \prod_{i=1}^{n} \left(f_i'(z)\right)^{\alpha_i} \left(\frac{h_i(z)}{z}\right)^{\beta_i}. \tag{2.5}$$

Differentiating (2.5) logarithmically, we have

$$1 + \frac{I_n\left(f_i,\ h_i\right)''(z)}{I_n\left(f_i,\ h_i\right)'(z)} = \sum_{i=1}^{n} \alpha_i \left(1 + \frac{z f_i''(z)}{f_i'(z)}\right) + \sum_{i=1}^{\mathrm{n}} \beta_i \left(\frac{z h_i'(z)}{h_i(z)}\right) + 1 - \sum_{i=1}^{\mathrm{n}} \left(\alpha_i + \beta_i\right).$$

Taking real part and then integrating with respect to θ, we get

$$\begin{aligned}\int_{\theta_1}^{\theta_2} \mathrm{Re}\ \left(1 + \frac{I_n\left(f_i,\ h_i\right)''(z)}{I_n\left(f_i,\ h_i\right)'(z)}\right) d\theta &= \sum_{i=1}^{n} \alpha_i \int_{\theta_1}^{\theta_2} \mathrm{Re}\ \left(1 + \frac{z f_i''(z)}{f_i'(z)}\right) d\theta + \sum_{i=1}^{\mathrm{n}} \beta_i \int_{\theta_1}^{\theta_2} \mathrm{Re}\ \left(\frac{z h_i'(z)}{h_i(z)}\right) d\theta \\ &\quad + \left(1 - \sum_{i=1}^{\mathrm{n}} \left(\alpha_i + \beta_i\right)\right) \left(\theta_2 - \theta_1\right). \\ &> -\frac{k\pi}{2} \sum_{i=1}^{n} \alpha_i - \left(\frac{k}{2} - 1\right)\pi \sum_{i=1}^{\mathrm{n}} \beta_i + \left(1 - \sum_{i=1}^{\mathrm{n}} \left(\alpha_i + \beta_i\right)\right) \left(\theta_2 - \theta_1\right),\end{aligned}$$

where we have used Lemma 1.1 and (1.2)

$$= -\sum_{i=1}^{n} \left(\left(\frac{k}{2}\right)\left(\alpha_i + \beta_i\right) - \beta_i\right) + \left(1 - \sum_{i=1}^{\mathrm{n}} \left(\alpha_i + \beta_i\right)\right) \left(\theta_2 - \theta_1\right).$$

From (2.4), we can obtain

$$\sum_{i=1}^{n} \left(\alpha_i + \beta_i\right) < 1.$$

So minimum is for $\theta_1 = \theta_2$, thus we have

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{I_n(f_i,\ h_i)''(z)}{I_n(f_i,\ h_i)'(z)}\right) d\theta > -\pi\ .$$

This implies that $\mathrm{I_n}(f_i,\ h_i)(z) \in\ K$.

For $k = 2$ in Theorem 2.4, we obtain the following result.

Corollary 2.5. Let $f_i(z) \in\ K,\ h_i(z) \in\ S^*$ for $1 \le i \le n$ and

$$\sum_{i=1}^{n}\left(\frac{k}{2}(\alpha_i + \beta_i) - \beta_i\right) \le 1\ ,$$

then $I_n(f_i,\ h_i)(z)$ defined by (1.4) belongs to the class of close-to-convex functions.

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Author details
[1]Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan [2]Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan

Authors' contributions
MA completed the main part of this article, KIN presented the ideas of this article, FG participated in some results of this article. MA made the text file and all the communications regarding the manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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