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On Poisson–Tweedie mixtures

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Abstract
Poisson-Tweedie mixtures are the Poisson mixtures for which the mixing measure is generated by those members of the family of Tweedie distributions whose support is non-negative. This class of non-negative integer-valued distributions is comprised of Neyman type *A*, back-shifted negative binomial, compound Poisson-negative binomial, discrete stable and exponentially tilted discrete stable laws. For a specific value of the "power" parameter associated with the corresponding Tweedie distributions, such mixtures comprise an additive exponential dispersion model. We derive closed-form expressions for the related variance functions in terms of the exponential tilting invariants and particular special functions. We compare specific Poisson-Tweedie models with the corresponding Hinde-Demétrio exponential dispersion models which possess a comparable unit variance function. We construct numerous local approximations for specific subclasses of Poisson-Tweedie mixtures and identify Lévy measure for all the members of this three-parameter family.

Keywords: Discrete stable distribution, Invariant of the exponential tilting transformation, Lambert *W* function, Large deviations, Lévy measure, Natural exponential family, Poisson-Tweedie mixture, Refined local limit theorem, Variance function, Wright function

Mathematics Subject Classification (2010): 33E20, 60E05, 60E07, 60F05, 60F10

1 Introduction

In this paper, we establish new results of distribution theory and prove new limit theorems of probability theory. Specifically, we investigate and establish numerous properties of the three-parameter family of non-negative integer-valued random variables (or *r.v.*'s) which are hereinafter referred to as the *Poisson-Tweedie* mixtures. This family was considered, among many others, by Kokonendji et al. (2004, Section 3), Jørgensen and Kokonendji (2016), and Bonat et al. (2017). The Poisson-Tweedie mixtures are rigorously introduced by formula (3).

We concentrate on the derivation of *local* limit theorems, which is customary in the case where one deals with *integer-valued* r.v.'s, since in view of the jumps of their cumulative distribution functions, the integral limit theorems for such r.v.'s are usually less accurate, which is due to discontinuities to be taken care of. Moreover, local limit theorems often provide a more detailed picture of the convergence mechanism than their integral counterparts by pointing out at potential singularities. For instance, Remark 4 to Theorems 7 and 8 addresses the singularity at the origin for the local versions of the (integral) theorem on weak convergence for Poisson-Tweedie mixtures given by formula (50) – the fact that can only be revealed through the microscope of local behavior.

Members of the Poisson-Tweedie family defined by formula (3) are often used for modeling *overdispersed* count data, since the variance of a generic member of this class is greater than its mean (compare to formula (4)). At the same time, the simulation studies presented in Bonat et al. (2017, Section 4) pertain to fitting *extended* Poisson-Tweedie regression models to overdispersed and underdispersed data. In turn, a growing interest to this family within the statistics community along with a close connection of the probability function of a general member of this class to the Wright special function (presented by formula (30)) motivated us to consider subtle mathematical properties of the Poisson-Tweedie mixtures. Thus, we concentrate on the theoretical aspects rather than applications of the Poisson-Tweedie mixtures and rely on the above-quoted papers for the *raison d'être*. In many cases, the relationship (30) between Poisson-Tweedie mixtures and Wright function (6) makes it possible to derive the leading error term of our local approximations whose role in applications is yet to be determined.

Next, similar to Kokonendji et al. (2004) we denote this class of Poisson-Tweedie mixtures by $\{\mathcal{PT}_{p,\mu,\lambda}\}$, although a different notation, namely, $\{\mathcal{PT}_p(\theta,\lambda)\}$, is employed by Kokonendji et al. (2004, Section 3). This is because we construct the class of Poisson-Tweedie mixtures by virtue of formula (3) starting from the corresponding members of the *reproductive* Tweedie exponential dispersion models (or *EDM*'s) for which the variance-to-mean relationship is given by formula (2). In contrast, Kokonendji et al. (2004, formula (9)) derived the probability law of a generic member $\mathcal{PT}_p(\theta,\lambda)$ of the Poisson-Tweedie class starting from the corresponding representative of a specific *additive* Tweedie EDM. See Jørgensen (1997, Chapters 3 and 4) respectively, for discussion on these forms of EDM's in the general setting and in the context of Tweedie EDM's. Our notation is more convenient for the derivation of limit theorems.

Each Poisson-Tweedie r.v. $\mathcal{PT}_{p,\mu,\lambda}$ is a particular Poisson mixture for which the mixing measure of the randomized Poisson parameter follows *Tweedie distribution* $Tw_p(\mu,\lambda)$ with the same values of the *"power"* parameter $p \geq 1$, the *scaling* parameter $\lambda \in \mathbf{R}^1_+ := (0,\infty)$, and the *location* parameter μ. The domain Ω_p of the location parameter μ is as follows:

$$\Omega_p := \begin{cases} \mathbf{R}^1_+ & \text{if } p \in [1,2]; \\ (0,\infty] & \text{if } p > 2. \end{cases} \tag{1}$$

Given $p \geq 1$ and $\lambda \in \mathbf{R}^1_+$, the one-parameter class $\{Tw_p(\mu,\lambda), \mu \in \Omega_p\}$ comprises a *natural exponential family* (or *NEF*) of non-negative distributions which is characterized by the *variance function* of a power type. In particular,

$$\mathbf{E}Tw_p(\mu,\lambda) = \mu, \quad \text{and} \quad \mathbf{Var}Tw_p(\mu,\lambda) = \mu^p/\lambda. \tag{2}$$

See Jørgensen (1997, Chapters 2 and 4) for more details on NEF's and Tweedie distributions, respectively. (In this paper, we should exclude the case of *negative values* of the power parameter which correspond to Tweedie laws taking values in the entire real axis $\mathbf{R}^1$, since it is customary to employ *non-negative* probability laws only for constructing Poisson mixtures, compare to formula (3).) Also, the variance-to-mean relationship (2) justifies referring to the totality of Tweedie distributions as the *power-variance family* or the *PVF* (compare to Vinogradov et al. (2012, 2013)).

Subsequently, for arbitrary fixed $p \geq 1$, $\lambda \in \mathbf{R}_+^1$ and $\mu \in \Omega_p$, the probability law of the Poisson-Tweedie mixture $\{\mathcal{PT}_{p,\mu,\lambda}\}$ on $\mathbf{Z}_+ := \{0, 1, 2, \ldots\}$ is such that

$$\mathbf{P}\left\{\mathcal{PT}_{p,\mu,\lambda} = k\right\} = \int_0^\infty e^{-u} \cdot \frac{u^k}{k!} \cdot Tw_p(\mu, \lambda)(du) \quad \text{where} \quad k \in \mathbf{Z}_+. \tag{3}$$

A combination of (2)–(3) yields that $\mathbf{E}(\mathcal{PT}_{p,\mu,\lambda}) = \mu$.

Next, let us discuss overdispersion of the Poisson-Tweedie family. To this end, we point out that Kokonendji et al. (2004, Proposition 2) provides the following *unit variance function* (or *u.v.f.*) of the *additive* Poisson-Tweedie EDM constructed starting from its member $\mathcal{PT}_p(\theta, 1)$, which stipulates the corresponding *variance-to-mean relationship*:

$$\mathbf{V}_p^{\mathcal{PT}}(\mu) = \mu + \mu^p \cdot \exp\{(2-p) \cdot \Phi_p(\mu)\} > \mu, \quad \text{where} \quad \mu > 0. \tag{4}$$

Since by (4) the variance is greater than the mean, all the Poisson-Tweedie mixtures are *overdispersed*. Also, note that Kokonendji et al. (2004, Proposition 2) states that the increasing function $\Phi_p(\mu)$ is *"generally implicit"* being the inverse of function $K'(\mu)$ which they defined by formula (10) therein.

In contrast, our Theorem 1 provides several closed-form expressions which specify the variance-to-mean relationship for all the members of the Poisson-Tweedie family $\left\{\mathcal{PT}_{p,\mu,\lambda}, p \geq 1, \mu \in \Omega_p, \lambda \in \mathbf{R}_+^1\right\}$ introduced by formula (3). The representations given in that theorem involve the invariants (20) of the *exponential tilting* transformation and particular special functions. This approach employs the fact that indexing a specific variance function by invariant(s) of the exponential tilting transformation for a fixed p provides a convenient decomposition of the corresponding two-parameter class of the Poisson-Tweedie distributions into the union of *non-overlapping* NEF's, with each specific NEF corresponding to its own value of the invariant. We defer the consideration of a few special cases and a detailed comparison of Theorem 1 with some related work to Section 4. For instance, Remark 5 addresses a comparison our closed-form representations (37)–(39) with *"generally implicit"* formula (4) and some other related results.

Although the expressions (37)–(39) are interesting in their own right, they can also be used for the derivation of the exact asymptotics of the probabilities of large deviations of partial sums of Poisson-Tweedie r.v.'s in the case where the magnitude of these deviations is at least proportional to the growing number of the summands (compare to Paris and Vinogradov (2015, Corollary 3.8)). See, for example, representation (40) of Theorem 2, which can be regarded as a result of the saddlepoint approximation type. The subsequent local limit Theorems 3 and 4 which pertain to the values of $p \in (1, 2)$ and $p > 2$, respectively, present the exact asymptotics of *superlarge* deviations for the corresponding partial sums of lattice r.v.'s. Theorems 2 and 3 can be regarded as the results of *Cramér*'s type, whereas the mechanisms of formation of the probabilities of large deviations in the cases covered by Theorem 4 are qualitatively different. Specifically, representation (42), which pertains to the *lattice* distributions, is of the same character as numerous results on large deviations for non-lattice r.v.'s presented in Vinogradov (1994, Chapter 5), and (2008b, Theorem 3.6.ii).

Theorems 5 and 6 concern local asymptotics in the case where the corresponding classes of Poisson-Tweedie mixtures converge to a Poisson limit. In this respect, observe that Kokonendji et al. (2004, Table 2) suggests that as $p \to +\infty$, members of a certain subclass of the family of Poisson-Tweedie mixtures tend to a Poisson law. This is clarified by formula (48), which easily follows from Proposition 1. A local version of this assertion is presented as Theorem 6. See also Remark 3.ii and Conjecture 1. In particular, formulas (82) and (84), which were checked numerically, specify the leading error term of the local Poisson approximation applicable in the case of sufficiently large values of the "power" parameter p.

Proposition 3 addresses the behavior of the Poisson-Tweedie mixtures around the points $p = 1$ and 2, whereas Theorems 7 and 8 provide local approximations in the case where the Poisson-Tweedie mixtures $\{\mathcal{PT}_{p,\cdot,\cdot}\}$ converge to a Tweedie distribution with the same p. Since all the Poisson-Tweedie mixtures are infinitely divisible, the above-described limit theorems of Section 3 can be regarded as the results on local asymptotics for the marginals of specific exponential families of (compound Poisson) integer-valued Lévy processes.

Propositions 1 and 2 of Section 2 provide the *probability-generating function* (or the *p.g.f.*) of all the Poisson-Tweedie mixtures and their Lévy measure, respectively.

In Section 4, we compare the Poisson-Tweedie family with a different class of the additive *Hinde-Demétrio* EDM's which correspond to a simpler u.v.f. given by (61). All the proofs are deferred to "Appendix 1" section, whereas "Appendix 2" section presents two relevant conjectures, which are of independent interest.

2 Notation, definitions and basic properties

First, we summarize some notation and terminology that will be used in the sequel. We follow a custom of formulating various statements of distribution theory in terms of the properties of r.v.'s, even when such results pertain only to their *distributions*. In what follows, the symbol "$\stackrel{d}{\to}$" stands for *weak convergence*, whereas log denotes the *natural* logarithm of the real argument. Also, $\mathbf{N}$ and $\mathbb{C}$ stand for the sets of all positive integers and the complex plane, respectively.

We will employ the *Pochhammer symbol* $(a)_j$, which is also known as the *rising factorial*. It is defined for positive integer j by

$$(a)_0 := 1, \quad (a)_j := \frac{\Gamma(a+j)}{\Gamma(a)} = a \cdot (a+1) \cdot (a+2) \cdot \ldots \cdot (a+j-1).$$

In the sequel, an empty sum or product is interpreted as zero or unity, respectively.

We now introduce several special functions and polynomials.

Definition 1 *("Reduced" Wright function, compare to Paris and Vinogradov (*2016*, formula* (1.4)*). Given parameters $\delta \in \mathbb{C}$ and $\rho \in (-1,0) \cup (0,\infty)$, and argument $z \in \mathbb{C}$ with $|z| < \infty$, set*

$$\phi(\rho, \delta; z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\ \Gamma(\rho n + \delta)}\ . \tag{5}$$

Hereinafter, we refer to ϕ as the (complex-valued) "reduced" Wright function.

Definition 2 *(Wright function, compare to Paris and Vinogradov (*2016*, formula* (1.3)*). Given parameters $\delta \in \mathbb{C}$ and $\rho \in (-1,0) \cup (0,\infty)$, and argument $z \in \mathbb{C}$ with $|z| < \infty$, set*

$${}_1\Psi_1(\rho, k; \rho, \delta; z) := \sum_{n=0}^{\infty} \frac{\Gamma(\rho n + k)}{\Gamma(\rho n + \delta)}\ \frac{z^n}{n!}, \tag{6}$$

where real $k \geq 0$. The function ${}_1\Psi_1$ constitutes a particular case of the (complex-valued) Wright function.

Under the restriction $\delta = 0$, the Wright function (6) admits a representation in terms of the *complete Bell polynomials*, which is stipulated by Proposition 5. We introduce them as follows:

Definition 3 *(The complete Bell polynomials). Given $\ell \in \mathbf{N}$, the ℓ^{th} complete Bell polynomial is defined as follows:*

$$\mathbf{B}_\ell\left(z_1, z_2, \ldots, z_\ell\right) := \frac{d^\ell}{dt^\ell} \exp\left(\sum_{j=1}^{\infty} z_j \frac{t^j}{j!}\right)\Bigg|_{t=0}. \tag{7}$$

We will also use *Touchard polynomials* such that for $k \in \mathbf{Z}_+$, the k^{th} Touchard polynomial is as follows:

$$T_k(x) := e^{-x} \cdot \sum_{\ell=0}^{\infty} \frac{\ell^k x^\ell}{\ell!} \tag{8}$$

(see, for example, Paris (2016)). It is well known that the Touchard polynomials can be expressed in terms of the complete Bell polynomials such that for $1 \leq j \leq k$, the argument $z_j = x$: $T_k(x) \equiv \mathbf{B}_k(x, x, \ldots, x)$.

Definition 4 *(The Lambert W function and its principal branch W_p, see Corless et al.* (1996)*).*

(i) *The complex-valued Lambert function $W(z)$ is defined as the multi-valued inverse of the function $y(x) := x \cdot e^x$. Equivalently, it can be defined as the function satisfying the identity $W(z) \cdot e^{W(z)} \equiv z$, where $z \in \mathbb{C}$. Its Taylor series around $z = 0$,*

$$W(z) = \sum_{\ell=1}^{\infty} w_\ell \cdot z^\ell, \tag{9}$$

has the radius of convergence $1/e$. The coefficients w_ℓ of the Taylor series (9) *are as follows: $w_\ell = (-\ell)^{\ell-1}/\ell!$.*

(ii) *The series* (9) *can be extended to a holomorphic function on $\mathbb{C}$ with a branch cut along $(-\infty, -1/e]$. This function defines the principal branch $W_p(z)$ of $W(z)$.*

In the sequel, we will need to employ specific solutions to the next two equations. First, given real $r > 0$ and $w > 0$, there exists the unique solution greater than 1, which we hereinafter denote by $t_{s0}(w)$, to the following equation:

$$t_s^r \cdot (t_s - 1) = w. \tag{10}$$

A modification of Paris and Vinogradov (2016, formula (4.4) and footnote 2) implies that this solution admits the following representation in terms of the "reduced" Wright function ϕ introduced by formula (5):

$$t_{s0}(w) = \left\{w - \log\left(\int_0^\infty \frac{e^{-wy}}{y(1+y)} \phi\left(-\frac{r}{r+1}, 0; -\frac{1+y}{y^{r/(r+1)}}\right) dy\right)\right\}^{1/(r+1)}. \tag{11}$$

(Note that w in formulas (10)–(11) is the same as w on the right-hand side of the formula in Paris and Vinogradov (2016, footnote 2), which is equal to $1/(\rho u)$ on the right-hand side of formula (4.4) of this reference). Given $r > 0$, it can be shown that

$$t_{s0} \sim w^{1/(1+r)} \text{ as } w \to +\infty, \text{ and } t_{s0} \to 1 \text{ as } w \downarrow 0. \tag{12}$$

Now, consider the equation

$$y^{1+\rho} = a(1-y) \tag{13}$$

when $\rho \in (-1, 0)$ and real $a > 0$ is a constant (compare to Paris and Vinogradov (2016, Proposition 5)). Then the unique root $y_s(a)$ on $(0, 1)$ admits the following closed-form representation in terms of the "reduced" Wright function ϕ for $\rho \in (-1/2, 0)$:

$$y_s(a) = \left\{ -\log \left(\int_0^\infty \frac{e^{-ay}}{y(1-y)} \phi \left(-(1+\rho), 0; \frac{y-1}{(ay)^{\rho+1}} \right) dy \right) \right\}^{1/(\rho+1)}. \tag{14}$$

When $\rho = -1/2$, it is straightforward to verify that $y_s(a) = \left(2a/\left(1+\sqrt{1+4a^2}\right)\right)^2$, which is consistent with formula (14). We do not yet have a proof of (14) when $\rho \in (-1, -1/2)$, although we verified the veracity of (14) numerically. Also, given $\rho \in (-1, 0)$, one ascertains that

$$y_s \sim a^{1/(1+\rho)} \text{ as } a \downarrow 0, \text{ and } y_s \to 1 \text{ as } a \uparrow +\infty. \tag{15}$$

Although it is natural to define Poisson-Tweedie mixtures starting from Tweedie distributions by formula (3), which in turn imposes the $\{p, \mu, \lambda\}$-parameterization, but a different parameterization motivated by their connection to the Wright function (6) and stipulated by formulas (16)–(18) is more suitable for studying properties of these mixtures. Hence, we now consider the following closely related triplet $\{\rho_p, \theta_p, \mathcal{A}_p\}$ of the *transformed* parameters, but the latter two of them have additional restrictions on the domain of p. Set

$$\rho_p := (2-p)/(p-1) \in (-1, +\infty] \quad \text{if} \quad p \in [1, +\infty). \tag{16}$$

Given $p > 1$, $\mu \in \Omega_p$ and $\lambda \in \mathbf{R}^1_+$, we introduce the following *"exponential tilting"* parameter:

$$\theta_p (= \theta(p, \mu, \lambda)) := \frac{1}{p-1} \lambda \mu^{1-p}. \tag{17}$$

By (1), $\theta_p > 0$ if $p \in (1, 2]$, and $\theta_p \geq 0$ if $p > 2$ (with $\theta_p = 0$ corresponding to $\mu = \infty$). Also, for $p \in [1, +\infty) \setminus \{2\}$, set

$$\mathcal{A}_p \left(= \mathcal{A}_p(\mu, \lambda)\right) := \frac{1}{2-p} \lambda \mu^{2-p}. \tag{18}$$

By Vinogradov et al. (2012, formula (3.9)), the product $\mathcal{A}_p(\theta_p)^{\rho_p}$ does not depend on μ such that given $p \in (1, 2) \cup (2, +\infty)$,

$$\mathcal{Z}_{p,\infty} := \mathcal{A}_p(\theta_p)^{\rho_p} = \frac{(p-1)^{(2-p)/(1-p)}}{2-p} \lambda^{1/(p-1)}. \tag{19}$$

Also, by analogy to Paris and Vinogradov (2015, formula (3.1)), set

$$\mathcal{Z} \; (= \mathcal{Z}_p) := \begin{cases} \mathcal{A}_p(\theta_p/(\theta_p+1))^{\rho_p} \\ \quad \text{if} \;\; p \in (1, +\infty) \setminus \{2\}; \\ \mathcal{A}_1 \cdot e^{-1/\lambda} \;\; \text{if} \;\; p = 1; \\ \lambda \;\; \text{if} \;\; p = 2. \end{cases} \tag{20}$$

It can be shown that for a fixed $p \geq 1$, the quantity $\mathcal{Z}_p$ is an *invariant of the exponential tilting transformation* for the corresponding class of Poisson-Tweedie mixtures with such

p. This means that given $p \geq 1$, all the members of the class of Poisson-Tweedie distributions characterized by the same value of $\mathcal{Z}$ comprise their own *natural exponential family* (or *NEF*). See Jørgensen (1997, Chapter 2) for more details on NEF's.

It is clear that (20) yields that for arbitrary fixed $\mu \in \mathbf{R}_+^1$ and $\lambda \in \mathbf{R}_+^1$,

$$\mathcal{Z}_p = \mathcal{A}_p \left(\theta_p / \left(\theta_p + 1\right)\right)^{\rho_p} \to \mathcal{A}_1 \cdot e^{-1/\lambda} = \mathcal{Z}_1 \text{ as } p \downarrow 1. \tag{21}$$

Now, for a fixed $p > 2$ consider the "boundary" case $\mu = +\infty$ (which pertains to the *discrete stable distributions*). Then a combination of (19) with (20) yields that for arbitrary fixed $p > 2$ and $\lambda \in \mathbf{R}_+^1$,

$$\mathcal{Z}_p \to \mathcal{Z}_{p,\infty} \text{ as } \mu \to +\infty. \tag{22}$$

In view of (16), if $p \in (1,2)$ or $p > 2$ then $\rho_p > 0$ or $\rho_p \in (-1,0)$, respectively. For these parameter values, the representations for the probability distributions of Tweedie models in terms of the reduced Wright function $\phi(\rho_p, 0; \cdot)$ introduced by (5) are given in Vinogradov et al. (2012, formulas (3.14) and (3.25)). In particular, in the case where $p > 2$, the probability density function (or *p.d.f.*) $f_{p,\mu,\lambda}(x)$ of $Tw_p(\mu,\lambda)$ is as follows:

$$f_{p,\mu,\lambda}(x) = x^{-1} \cdot \phi\left(\rho_p, 0, \mathcal{Z}_{p,\infty} x^{\rho_p}\right) \cdot e^{-\theta_p x - \mathcal{A}_p} \text{ if } x \geq 0. \tag{23}$$

Here, the function $u^{-1} \cdot \phi(\rho, 0, -\mathcal{C} \cdot u^{\rho})$ is extended at zero as $u \downarrow 0$ by continuity, where $\mathcal{C} \in \mathbf{R}_+^1$ is a constant. For $p > 2$, the law of the r.v. $Tw_p(\mu,\lambda)$ is obtained from that of stable r.v. $Tw_p(\infty,\lambda)$ by the exponential tilting transformation.

Similar to (23), in the case where $p \in (1,2)$, the density $f_{p,\mu,\lambda}(x)$ of the absolutely continuous component of compound Poisson-gamma r.v. $Tw_p(\mu,\lambda)$ admits the following representation:

$$f_{p,\mu,\lambda}(x) = x^{-1} \cdot \phi\left(\rho_p, 0, \mathcal{Z}_{p,\infty} x^{\rho_p}\right) \cdot e^{-\theta_p x - \mathcal{A}_p} \text{ if } x \in \mathbf{R}_+^1.$$

In addition, this Poisson-gamma r.v. $Tw_p(\mu,\lambda)$ is such that

$$\mathbf{P}\{Tw_p(\mu,\lambda) = 0\} = \exp\{-\mathcal{A}_p\}. \tag{24}$$

The subclass $\left\{Tw_1(\mu,\lambda), \mu \in \mathbf{R}_+^1, \lambda \in \mathbf{R}_+^1\right\}$ is comprised of the scaled Poisson laws. Also, we parameterize the gamma family $\left\{Tw_2(\mu,\lambda), \mu \in \mathbf{R}_+^1, \lambda \in \mathbf{R}_+^1\right\}$ in a manner for the p.d.f. of its member to have the following form:

$$f_{2,\mu,\lambda}(x) := \frac{(\lambda/\mu)^{\lambda}}{\Gamma(\lambda)} \cdot x^{\lambda-1} \cdot \exp\left\{-(\lambda/\mu) \cdot x\right\}, \text{ where } x > 0. \tag{25}$$

Next, we will consider the probability function for Poisson-Tweedie mixtures. It is known that the probability function of a generic Poisson–Tweedie mixture with $p = 1$ is expressed in terms of the *Touchard polynomials*, introduced by (8), such that for arbitrary fixed $\mu \in \mathbf{R}_+^1$, $\lambda \in \mathbf{R}_+^1$, and $k \in \mathbf{Z}_+$,

$$\mathbf{p}_k := \mathbf{P}\left\{\mathcal{PT}_{1,\mu,\lambda} = k\right\} = \frac{e^{\mathcal{Z} - \mathcal{A}_1}}{\lambda^k \cdot k!} \cdot T_k(\mathcal{Z}). \tag{26}$$

The totality of the subclass of Poisson–Tweedie mixtures with $p = 2$ is the family of negative binomial distributions which start from zero such that for arbitrary fixed $\mu \in \mathbf{R}_+^1$, $\lambda \in \mathbf{R}_+^1$, and $k \in \mathbf{Z}_+$,

$$\mathbf{p}_k := \mathbf{P}\left\{\mathcal{PT}_{2,\mu,\lambda} = k\right\} = \frac{(\lambda)_k \cdot (\theta_2 + 1)^{-k}}{k!} \left(1 - 1/(\theta_2 + 1)\right)^{\lambda}. \tag{27}$$

Set $\pi_\eta(\ell) := \mathbf{P}\{\mathcal{P}oiss(\eta) = \ell\} = e^{-\eta}\eta^\ell/\ell!$, where $\ell \in \mathbf{Z}_+$. Also, given $p \in (1, +\infty)$, $\mu \in \Omega_p$, and $\lambda \in \mathbf{R}^1_+$, formula (3) implies that

$$\mathbf{P}\left\{\mathcal{PT}_{p,\mu,\lambda} = k\right\} = \int_0^\infty \pi_u(k) \cdot Tw_p(\mu, \lambda)(du)$$
$$= \begin{cases} \int_0^\infty e^{-u} \cdot \frac{u^k}{k!} \cdot f_{p,\mu,\lambda}(u) \cdot du \\ \quad \text{if } p \in (1, 2) \text{ and } k \in \mathbf{N}, \text{ or } p \geq 2 \text{ and } k \in \mathbf{Z}_+; \\ e^{-\mathcal{A}_p} + \int_0^\infty e^{-u} \cdot f_{p,\mu,\lambda}(u) \cdot du \text{ if } p \in (1, 2) \text{ and } k = 0. \end{cases} \tag{28}$$

Subsequently, formula (28) yields that for each $p \in (1, +\infty) \setminus \{2\}$,

$$\mathbf{p}_0 := \mathbf{P}\left\{\mathcal{PT}_{p,\mu,\lambda} = 0\right\} = \mathcal{P}_{p,\mu,\lambda}(0) = e^{\mathcal{Z} - \mathcal{A}_p}. \tag{29}$$

Also, representation (28) implies that for each integer $k \in \mathbf{N}$ if $p \in (1, 2)$, and each integer $k \in \mathbf{Z}_+$ if $p > 2$,

$$\begin{aligned} \mathbf{p}_k := \mathbf{P}\left\{\mathcal{PT}_{p,\mu,\lambda} = k\right\} &= \frac{e^{-\mathcal{A}_p}(\theta_p + 1)^{-k}}{k!} \cdot \sum_{\ell=0}^\infty \frac{\Gamma(\rho_p \ell + k)}{\Gamma(\rho_p \ell) \cdot \ell!} \mathcal{Z}^\ell \\ &= \frac{e^{-\mathcal{A}_p} \cdot (\theta_p + 1)^{-k}}{k!} \cdot {}_1\Psi_1(\rho_p, k; \rho_p, 0; \mathcal{Z}). \end{aligned} \tag{30}$$

Remark 1 *Suppose that* $\left\{\mathcal{PT}^{(1)}_{p,\mu,\lambda}, \ldots, \mathcal{PT}^{(n)}_{p,\mu,\lambda}\right\}$ *are the i.i.d.r.v.'s whose common distribution* $\mathcal{PT}_{p,\mu,\lambda}$ *belongs to the family of Poisson-Tweedie mixtures. Then the* n^{th} *partial sum* $\mathcal{S}_n := \mathcal{PT}^{(1)}_{p,\mu,\lambda} + \ldots + \mathcal{PT}^{(n)}_{p,\mu,\lambda}$ *is also a Poisson-Tweedie mixture with the same* $p \geq 1$*. It can be shown that for* $p = 1$ *and* $p > 1$*, quantities* λ *and* θ_p*, respectively, remain intact, while for* $p \in [1, +\infty) \setminus \{2\}$*,* $\mathcal{A}_p$ *is to be multiplied by n. A combination of these comments with* (20) *implies that for a fixed* $p \geq 1$*, the invariants* $\mathcal{Z}$ *of the exponential tilting transformation given by* (20) *and the limit in* (22) *are to be multiplied by n. Also, a combination of these observations with* (30) *stipulates that for* $k \in \mathbf{N}$ *if* $p \in (1, 2)$*, and for* $k \in \mathbf{Z}_+$ *if* $p > 2$*,*

$$\mathbf{P}\left\{\mathcal{PT}^{(1)}_{p,\mu,\lambda} + \ldots + \mathcal{PT}^{(n)}_{p,\mu,\lambda} = k\right\} = e^{-n\mathcal{A}_p} \frac{(\theta_p + 1)^{-k}}{k!} {}_1\Psi_1(\rho_p, k; \rho_p, 0; n\mathcal{Z}). \tag{31}$$

Definition 5 *(See Nagaev (*1998*, Definition* 2*)). A generic r.v.* $\mathcal{N}$ *which takes values on the lattice* $\{f + \ell h\}$ *(with real* $f \geq 0$*, span* $h \in \mathbf{R}^1_+$*, and* $\ell \in \mathbf{Z}$*) is said to belong to class* $(\mathcal{S})$ *if there exists a fixed* $\kappa \in \mathbf{R}^1_+$ *such that for* $\ell \in \{f + \ell h\}$*, and as* $\ell \to \infty$*,*

$$\mathbf{P}\{\mathcal{N} = \ell\} \sim \exp\left\{-\kappa\ell + \int_{x_0}^\ell g(u)du\right\}.$$

Also, it is assumed that the function $g(\cdot) : \mathbf{R}^1_+ \to \mathbf{R}^1$ *is such that* ***(i)*** *there exists* $x_0 \in \mathbf{R}^1_+$ *such that* $\forall x \geq x_0 > 0$*, function* $g(x) > 0$*;* ***(ii)*** $g(\infty) = 0$*;* ***(iii)*** $g''(x) \downarrow$*;* ***(iv)*** *the product* $x \cdot g(x) \to +\infty$ *as* $x \to +\infty$*, and* ***(v)*** $\forall x \geq x_0$*,* $0 \leq -g''(x)/g'(x) \leq 2/x$*.*

Proposition 1 *The p.g.f.* $\mathcal{P}(u)$ $(= \mathcal{P}_{p,\mu,\lambda}(u))$ *of the Poisson-Tweedie mixture* $\mathcal{PT}_{p,\mu,\lambda}$ *admits the following representations:*

(i) *For* $p \in (1, +\infty) \setminus \{2\}$*,* $\mu \in \mathbf{R}^1_+$*,* $\lambda \in \mathbf{R}^1_+$*,* $u < \theta_p + 1$ *if* $p \in (1, 2)$*, and* $u \leq \theta_p + 1$ *if* $p > 2$*,*

$$\mathcal{P}(u) = \exp\left\{\mathcal{A}_p\left(\left(1 + \frac{1-u}{\theta_p}\right)^{-\rho_p} - 1\right)\right\} = e^{\mathcal{Z}_{p,\infty}\left\{(1+\theta_p-u)^{-\rho_p} - \theta_p^{-\rho_p}\right\}}. \tag{32}$$

(ii) For $p > 2$, $\mu = +\infty$, $\lambda \in \mathbf{R}^1_+$, and $u \leq 1$,

$$\mathcal{P}(u) = \exp\left\{\mathcal{Z}_{p,\infty}(1-u)^{-\rho_p}\right\}. \tag{33}$$

(iii) In the case where $p = 1$, $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, and for $u \in \mathbf{R}^1$,

$$\mathcal{P}(u) = \exp\left\{\mathcal{A}_1 \cdot \left\{e^{\lambda^{-1}(u-1)} - 1\right\}\right\}. \tag{34}$$

(iv) For $p = 2$, $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, and $u < \theta_2 + 1$,

$$\mathcal{P}(u) = (1 + 1/\theta_2 - u/\theta_2)^{-\lambda}. \tag{35}$$

Proposition 2 *(Lévy measure for Poisson-Tweedie mixtures). The Lévy representation for the cumulant-generating function of r.v. $\mathcal{PT}_{p,\mu,\lambda}$ does not contain the drift and diffusion components. For $p \in (1, +\infty) \setminus \{2\}$, $\mu \in \Omega_p$, and $\lambda \in \mathbf{R}^1_+$, its Lévy measure $\nu_{p,\mu,\lambda}(\cdot)$ is concentrated on $\mathbf{N}$, where it admits the following representation:*

$$\nu_{p,\mu,\lambda}(\{k\}) = \mathcal{Z}_p \cdot (\theta_p + 1)^{-k} \cdot (\rho_p)_k / k!, \quad \text{where} \;\; k \in \mathbf{N}. \tag{36}$$

Remark 2 *For $p = 3/2$, formula* (36) *is consistent with Vinogradov (*2007*, formula* (3.4)*). Thus, $\nu_{3/2,\mu,\lambda}(\{k\}) = \mathcal{Z}_{3/2}(\theta_{3/2} + 1)^{-k}$ for $k \geq 1$.*

3 Main results

The first result of this section concerns the closed-form representations for the variance functions of specific NEF's comprised of particular Poisson-Tweedie mixtures. Note that by (20), $\mathcal{Z}_p > 0$ if $p \in [1, 2]$, whereas $\mathcal{Z}_p < 0$ if $p > 2$.

Theorem 1 *Given real $p \geq 1$ and an admissible value $\mathcal{Z}_p$ of the exponential tilting invariant, consider the NEF comprised of the Poisson-Tweedie mixtures characterized by such values of p and $\mathcal{Z}_p$ with domain Ω_p of the location parameter μ. Then*

(i) In the case where $1 < p < 2$ and $\mathcal{Z}_p > 0$, the variance function of such NEF is as follows:

$$\mathbf{V}_{\mathcal{Z}_p}(\mu) = \mu + \frac{\mu^2}{(2-p)\mathcal{Z}_p \cdot t_{s0}(\mu/(\rho_p \mathcal{Z}_p))^{\rho_p}}, \tag{37}$$

where the argument $\mu \in \mathbf{R}^1_+$ and $t_{s0}(\mu/(\rho_p\mathcal{Z}_p))$ is obtained from (10)–(11) by setting $r = \rho_p$ and $w = \mu/(\rho_p\mathcal{Z}_p)$.

(ii) Given $p > 2$ and $\mathcal{Z}_p < 0$, the variance function of such NEF admits the following representation:

$$\mathbf{V}_{\mathcal{Z}_p}(\mu) = \mu + \frac{\mu^2}{(2-p)\mathcal{Z}_p} \cdot y_s\left(\rho_p \mathcal{Z}_p/\mu\right)^{\rho_p}, \tag{38}$$

where the argument $\mu \in (0, +\infty]$ and $y_s(\rho_p\mathcal{Z}_p/\mu)$ is derived from (14) by setting $\rho = \rho_p$ and $a = \rho_p\mathcal{Z}_p/\mu$.

(iii) For $p = 1$ and $\mathcal{Z}_1 > 0$, the variance function of such NEF comprised of specific Neyman type A distributions is as follows:

$$\mathbf{V}_{\mathcal{Z}_1}(\mu) = \mu \cdot (1 + W_p(\mu/\mathcal{Z}_1)), \;\; \text{where } \mu \in \mathbf{R}^1_+. \tag{39}$$

Next, we proceed with three local large deviation limit theorems for n^{th} partial sums of the i.i.d.r.v.'s whose common distribution belongs to the Poisson-Tweedie family. The first of them employs the above variance function $\mathbf{V}_{\mathcal{Z}_p}(\cdot)$ given by (37).

Theorem 2 *Fix* $p \in (1,2)$, $\mu \in \mathbf{R}_+^1$, $\lambda \in \mathbf{R}_+^1$, *and real* $\epsilon > 0$. *Suppose that the integers* k *and* n *are such that* $n \to +\infty$ *and* $k \geq (\mu + \epsilon) \cdot n$. *Then*

$$\begin{aligned}\mathbf{P}\left\{\mathcal{PT}_{p,\mu,\lambda}^{(1)} + \ldots + \mathcal{PT}_{p,\mu,\lambda}^{(n)} = k\right\} &\sim \frac{1}{\sqrt{2\pi k \cdot \mathbf{V}_{\mathcal{Z}_p}(k/n)}} \\ &\times \exp\left\{-n \int_\mu^{k/n} \frac{k/n - t}{\mathbf{V}_{\mathcal{Z}_p}(t)} \cdot dt\right\}.\end{aligned} \tag{40}$$

Theorem 3 *For fixed* $p \in (1,2)$, $\mu \in \mathbf{R}_+^1$ *and* $\lambda \in \mathbf{R}_+^1$, *and for integer values of* k *and* n *such that* $n \to +\infty$ *and* $k/n \to +\infty$,

$$\begin{aligned}\mathbf{P}\left\{\mathcal{PT}_{p,\mu,\lambda}^{(1)} + \ldots + \mathcal{PT}_{p,\mu,\lambda}^{(n)} = k\right\} &\sim e^{-n\mathcal{A}_p} \cdot \frac{(\theta_p + 1)^{-k}}{\sqrt{2\pi(1+\rho_p)k}} \\ \times \left(\rho_p n \mathcal{Z}_p \cdot k^{\rho_p}\right)^{\frac{1}{2(1+\rho_p)}} &\cdot \exp\left\{\frac{1+\rho_p}{\rho_p} \cdot \left(\rho_p n \mathcal{Z}_p \cdot k^{\rho_p}\right)^{1/(1+\rho_p)}\right\}.\end{aligned} \tag{41}$$

Theorem 4 *For fixed* $p > 2$, $\mu \in \Omega_p$ *and* $\lambda \in \mathbf{R}_+^1$, *and for integer values of* k *and* n *such that* $n \to +\infty$ *and* $k \cdot n^{1/\rho_p} \to +\infty$,

$$\mathbf{P}\left\{\mathcal{PT}_{p,\mu,\lambda}^{(1)} + \ldots + \mathcal{PT}_{p,\mu,\lambda}^{(n)} = k\right\} \sim n \cdot e^{-(n-1)\mathcal{A}_p} \cdot \mathbf{P}\left\{\mathcal{PT}_{p,\mu,\lambda} = k\right\}. \tag{42}$$

Next, we will proceed with two assertions on the local asymptotics for Poisson-Tweedie mixtures which are related to Poisson convergence. First, we derive the leading error term for the large-λ asymptotics of these mixtures for which $\mu < \infty$. To this end, observe that Jørgensen (1997, Proposition 4.12) ascertains that given $p \in [1, +\infty)$ and $\mu \in \mathbf{R}_+^1$, and as $\lambda \to \infty$,

$$\mathcal{PT}_{p,\mu,\lambda} \xrightarrow{\text{d}} \mathcal{P}oiss(\mu). \tag{43}$$

The following assertion generalizes Paris and Vinogradov (2015, Theorem 3.13) and refines the local counterpart of the Poisson convergence result (43).

Theorem 5 *Fix* $p \in [1, +\infty)$, $\mu \in \mathbf{R}_+^1$, *and* $\ell \in \mathbf{Z}_+$. *Then*

$$\mathbf{P}\left\{\mathcal{PT}_{p,\mu,\lambda} = \ell\right\} = \pi_\mu(\ell) \cdot \left\{1 + \frac{(\mu - \ell)^2 - \ell}{2 \cdot \lambda \mu^{2-p}} + \mathcal{O}(1/\lambda^2)\right\} \text{ as } \lambda \to +\infty. \tag{44}$$

The following result is related to Kokonendji et al. (2004, Table 2).

Theorem 6 *Suppose that* $p \to +\infty$, *and there exists real constant* $\mathcal{Z} < 0$ *such that the parameters* $\mu = \mu_p$ *and* $\lambda = \lambda_p$ *vary in such a manner that the following two conditions are met:*

$$\lambda_p^{1/(p-1)} \to |\mathcal{Z}|, \tag{45}$$

and

$$\lambda_p = o\left(p \cdot \mu_p^{p-1}\right). \tag{46}$$

Then for a fixed $\ell \in \mathbf{Z}_+$,

$$\mathbf{P}\{\mathcal{PT}_{p,\mu_p,\lambda_p} = \ell\} \to \pi_{|\mathcal{Z}|}(\ell). \tag{47}$$

Remark 3 *(i) The "integral" version of Theorem* 6 *easily follows from* (32)–(33). *Thus, one ascertains that under the fulfillment of all the assumptions of Theorem* 6,

$$\mathcal{PT}_{p,\mu_p,\lambda_p} \xrightarrow{\text{d}} \mathcal{P}oiss(|\mathcal{Z}|). \tag{48}$$

It is easily seen that under the fulfillment of (45)*, the condition that* $\mu_p > |\mathcal{Z}| + \epsilon$ *for all sufficiently large p is sufficient for* (46)*, whereas the condition* $\mu_p > |\mathcal{Z}| - \epsilon$ *is necessary. (Here,* $\epsilon > 0$ *is an arbitrary small real.)*

(ii) In the case of discrete stable distributions per se, i.e., when $\mu = \mu_p \equiv +\infty$ *and condition* (46) *is fulfilled automatically, it is plausible to derive the leading error term in the local Poisson convergence result* (47)*. By* (33)*, the discrete stable distributions with the same fixed value of* $\mathcal{Z}_{p,\infty}(= \mathcal{Z}) < 0$ *converge weakly as* $\rho_p \downarrow -1$ *to a Poisson distribution with mean* $|\mathcal{Z}_{p,\infty}|$*. (By convention, Poisson r.v.'s are often included to the class of discrete stable r.v.'s.) See also Conjecture* 1*.*

Now, fix $p_0 \geq 1$, $\mu_0 \in \Omega_{p_0}$, and $\lambda_0 \in \mathbf{R}^1_+$. Consider a certain subfamily of Poisson-Tweedie mixtures $\mathcal{PT}_{p,\mu,\lambda}$, where $p \geq 1$, $\mu \in \Omega_p$, and $\lambda \in \mathbf{R}^1_+$ are such that $p \to p_0$, $\mu \to \mu_0$, and $\lambda \to \lambda_0$. A subsequent combination of representations (32)–(35) with Vinogradov (2004, Proposition 1.1 and Theorem 2.6) and Panjer and Willmot (1992, formula (8.2.3)) yields the continuity of the family of the Poisson-Tweedie mixtures with respect to parameters p, μ and λ in Lévy metric such that

$$\mathcal{PT}_{p,\mu,\lambda} \xrightarrow{\mathrm{d}} \mathcal{PT}_{p_0,\mu_0,\lambda_0}. \tag{49}$$

The next assertion can be regarded in some sense as a *local counterpart* of (49).

Proposition 3 ***(i)*** *For arbitrary fixed* $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, $k \in \mathbf{Z}_+$*, and as* $p \downarrow 1$*,*

$$\mathbf{P}\{\mathcal{PT}_{p,\mu,\lambda} = k\} \to \mathbf{P}\{\mathcal{PT}_{1,\mu,\lambda} = k\}.$$

(ii) *For arbitrary fixed* $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, $k \in \mathbf{Z}_+$*, and as* $p \to 2$*,*

$$\mathbf{P}\{\mathcal{PT}_{p,\mu,\lambda} = k\} \to \mathbf{P}\{\mathcal{PT}_{2,\mu,\lambda} = k\}.$$

We now proceed with two refined local limit theorems for Poisson-Tweedie mixtures when $p > 1$. They are related to the following result on weak convergence of scaled Poisson-Tweedie mixtures to the corresponding Tweedie distribution which can be derived from a combination of Kokonendji et al. (2004, Propositions 2 and 6) with Jørgensen et al. (2009, formula (5.2)). It can be expressed as follows:

$$\mathcal{C}^{-1} \cdot \mathcal{PT}_{p,\mathcal{C}\mu,\mathcal{C}^{p-2}\lambda} \xrightarrow{\mathrm{d}} Tw_p(\mu,\lambda) \text{ as } \mathcal{C} \to +\infty. \tag{50}$$

The case $p = 1$ is to be treated separately, since by (39), $\mathbf{V}_{\mathcal{Z}_1}(\mu) \sim \mu \cdot \log \mu$ as $\mu \to +\infty$ rather than just to $Const \cdot \mu$ (compare to formula (62)).

The following assertion, which can be regarded as a refinement of the local version of (50) in the case where $p = 2$, constitutes a result of the *Yaglom-theorem* type on gamma convergence (compare to Jørgensen et al. (2009, pp. 411-412)).

Theorem 7 *Set* $\mathcal{D}_{2,\mu,\lambda}(u) := \theta_2^2 \cdot u$ *and* $\mathcal{E}_{2,\mu,\lambda}(u) := \theta_2^3 u(3\theta_2 u - 8)$*. Suppose that real* $u > 0$ *is fixed, and that* $u \cdot \mathcal{C}$ *takes on positive integer values. Then given real* $\mu > 0$ *and* $\lambda > 0$*, one ascertains that as* $\mathcal{C} \to +\infty$*,*

$$\mathbf{P}\left\{\mathcal{PT}_{2,\mathcal{C}\mu,\lambda} = u\mathcal{C}\right\} = \frac{f_{2,\mu,\lambda}(u)}{\mathcal{C}}\left(1 + \frac{\mathcal{D}_{2,\mu,\lambda}(u)}{2\mathcal{C}} + \frac{\mathcal{E}_{2,\mu,\lambda}(u)}{6\mathcal{C}^2} + \mathcal{O}\left(\frac{1}{\mathcal{C}^3}\right)\right).$$

The next Theorem 8, which is of the same spirit as the above Theorem 7, generalizes Paris and Vinogradov (2015, Theorem 3.10), where the special case $p = 3/2$ was considered. But first, we need to introduce the following function of argument $u \in \mathbf{R}^1_+$, which is expressed in terms of the "reduced" Wright function as follows:

$$\begin{aligned}\mathcal{D}_{p,\mu,\lambda}(u) := {} & \frac{\phi\left(\rho_p, 2\rho_p, \mathcal{Z}_{p,\infty}u^{\rho_p}\right)}{\phi\left(\rho_p, 0, \mathcal{Z}_{p,\infty}u^{\rho_p}\right)} \cdot \frac{\left(\rho_p \mathcal{Z}_{p,\infty}u^{\rho_p}\right)^2}{u} \\ & + \frac{\phi\left(\rho_p, \rho_p, \mathcal{Z}_{p,\infty}u^{\rho_p}\right)}{\phi\left(\rho_p, 0, \mathcal{Z}_{p,\infty}u^{\rho_p}\right)} \cdot \left(\rho_p - 1 - 2\theta_p u\right) \cdot \frac{\rho_p \mathcal{Z}_{p,\infty}u^{\rho_p}}{u} + \theta_p^2 u.\end{aligned} \tag{51}$$

Theorem 8 *Fix $p \in (1, +\infty) \setminus \{2\}$, $\mu \in \Omega_p$, $\lambda \in \mathbf{R}^1_+$, and the value of the argument $u \in \mathbf{R}^1_+$. Suppose that the real-valued parameter $\mathcal{C}$ is such that $u\mathcal{C}$ is an integer. Consider the one-parameter family $\{\mathcal{PT}_{p,\mathcal{C}\mu,\mathcal{C}^{p-2}\lambda}\}$ of the Poisson-Tweedie mixtures, which is indexed by $\mathcal{C}$. Then as $\mathcal{C} \to +\infty$,*

$$\mathbf{P}\left\{\mathcal{PT}_{p,\mathcal{C}\mu,\mathcal{C}^{p-2}\lambda} = u \cdot \mathcal{C}\right\} = \frac{1}{\mathcal{C}} \cdot f_{p,\mu,\lambda}(u) \cdot \left(1 + \frac{\mathcal{D}_{p,\mu,\lambda}(u)}{2\mathcal{C}} + \mathcal{O}\left(1/\mathcal{C}^2\right)\right). \tag{52}$$

Remark 4 ***(i)*** *Since by* (29)*,*

$$\mathbf{P}\left\{\mathcal{PT}_{p,\mathcal{C}\mu,\mathcal{C}^{p-2}\lambda} = 0\right\} = e^{\mathcal{Z}(\mathcal{C}) - \mathcal{A}_p}, \tag{53}$$

one ascertains that in the case where $p > 2$ (or $\rho_p \in (-1, 0)$), the expression (53) *approaches* 0 *faster than any negative power of $\mathcal{C}$. Hence, taking limit as $\mathcal{C} \to +\infty$ eliminates the point mass at the origin in this case.*

At the same time, in the case where $p \in (1, 2)$ the weak-convergence result (50) *implies that the limiting Tweedie distribution $Tw_p(\mu, \lambda)$ has a positive mass* $\exp\{-\mathcal{A}_p\}$ *at zero (compare to* (24)*). Here,* (53) *stipulates that as $\mathcal{C} \to +\infty$,*

$$\mathbf{P}\left\{\mathcal{PT}_{p,\mathcal{C}\mu,\mathcal{C}^{p-2}\lambda} = 0\right\} = e^{-\mathcal{A}_p} \cdot \left(1 + \mathcal{Z}_{p,\infty} \cdot \mathcal{C}^{-\rho_p} + \mathcal{O}\left(\mathcal{C}^{-\min(2\rho_p, \rho_p + 1)}\right)\right).$$

*However, it appears that merging the cases of $u > 0$ and $u = 0$ into a unified assertion only makes sense for $p = 3/2$, which was already dealt with in Paris and Vinogradov (*2015*, Theorem* 3.10*). This is partly due to the fact that the behavior of the function $f_{p,\mu,\lambda}(u)$ at zero can have one of three qualitatively different types, which pertain to the values of $p \in (1, 3/2)$, $p = 3/2$, and $p \in (3/2, 2)$ (see Vinogradov et al. (*2012*, formula* (3.27)*) for more detail). Moreover, the behavior of function $\mathcal{D}_{p,\mu,\lambda}(u)$ (which is defined by formula* (51)*) is even more diverse. Specifically, it can be shown that as $u \downarrow 0$,*

$$\begin{aligned}\mathcal{D}_{p,\mu,\lambda}(u) = {} & \left(\rho_p^2 \mathcal{Z}_{p,\infty} u^{\rho_p - 1} + \rho_p \cdot (\rho_p - 1)/u - 2\theta_p \rho_p\right) \\ & \times (1 + o(1)) + \theta_p^2 u \to \begin{cases} +\infty & \text{if } p \in (1, 3/2); \\ 4\lambda^2 - 2\theta_{3/2} & \text{if } p = 3/2; \\ -\infty & \text{if } p \in (3/2, 2). \end{cases}\end{aligned}$$

Similar to the above, in the case where $p = 2$ we apply (27) *to obtain that*

$$\mathbf{P}\left\{\mathcal{PT}_{2,\mathcal{C}\mu,\lambda} = 0\right\} \sim \theta_2^{\lambda} \cdot \mathcal{C}^{-\lambda} \text{ as } \mathcal{C} \to +\infty. \tag{54}$$

Since the behavior of the gamma density $f_{2,\mu,\lambda}(u)$ at zero is similar to that of $f_{p,\mu,\lambda}(u)$ and determined by a particular value of the shape parameter λ, we elected to exclude the value of argument $u = 0$ from consideration in Theorem 7 *and give it as a separate formula* (54)*.*

(ii) *There exists a function $\{\mathcal{E}_{p,\mu,\lambda}(u),\ u > 0\}$ which constitutes the next term in the expansion* (52) *over negative powers of $\mathcal{C}$. Similar to function $\mathcal{D}_{p,\mu,\lambda}(u)$, it admits a representation in terms of the "reduced" Wright function which is analogous to, but more* *complicated than, the expression that emerges on the right-hand side of formula* (51)*. Hence, it is too long to be included here. Compare to Paris and Vinogradov (*2015*, Theorem* 3.10*) where this function is given in the special case where* $p = 3/2$*.*

(iii) *The first-order error terms which emerge in Theorems* 7 *and* 8 *are consistent in the sense that* $\mathcal{D}_{2,\mu,\lambda}(u) = \lim_{p\to 2} \mathcal{D}_{p,\mu,\lambda}(u)$*.*

4 Special cases of Theorem 1, Hinde-Demétrio EDM's and discussion

First, we will present a few special cases of Theorem 1 as well as discuss its relationship to Kokonendji et al. (2004) and other works in a series of remarks.

Remark 5 *(i) The closed-form representations* (37)–(39) *are consistent with formula* (4) *that contains the function* $\Phi_p(\mu)$ *defined implicitly in Kokonendji et al. (*2004*, Proposition* 2*). For instance, in the case where* $p = 1$*, one can employ several assertions given in Kokonendji et al.* (2004) *and express the solution to equation* $e^x + x = b$ *in terms of the Lambert W function (see Corless et al. (*1996*, p.* 332*)) to ascertain that the u.v.f.* (4) *of the additive Poisson-Tweedie EDM that is obtained starting from r.v.* $\mathcal{PT}_1(\theta, 1)$ *can be rewritten as follows:*

$$\mathbf{V}_1^{\mathcal{PT}}(\mu) = \mu \cdot \left(1 + W_p\left(\mu \cdot e^{1-\theta}\right)\right). \tag{55}$$

A subsequent combination of representation (20) *for* $\mathcal{Z}_1$ *with Jørgensen (*1997*, Subsection* 3.3.3*) yields that representation* (55) *is consistent with* (39)*. Also, representation* (39) *is consistent with Vinogradov (*2013*, Theorem* 5.1*).*

(ii) In the case where $p = 3/2$*, Paris and Vinogradov (*2016*, formula* (2.1)*) yields that formula* (30) *coincides with Paris and Vinogradov (*2015*, formulas* (3.2)–(3.3)*). For* $p = 3/2$*, Paris and Vinogradov (*2015*, formula* (3.15)*) implies that in this special case, our representation* (37) *can be simplified as follows:* $V_{\mathcal{Z}_{3/2}}(\mu) = \mu \cdot \sqrt{4\mathcal{Z}_{3/2}^{-1} \cdot \mu + 1}$*. For* $\mathcal{Z}_{3/2} = 4$*, this formula is consistent with Jørgensen (*1997*, p.* 170*).*

(iii) In the case where $p = 3$*, Paris and Vinogradov (*2016*, formula* (2.4)*) implies that representation* (30) *is equivalent to Vinogradov (*2008a*, formula* (14)*)), since formula* (20) *stipulates that* $\mathcal{Z}_3 = -\sqrt{2\lambda + \lambda^2/\mu^2}$*. A subsequent combination of this fact with* (38) *yields that*

$$V_{\mathcal{Z}_3}(\mu) = \mu \cdot \left\{1 + \frac{\mu}{\mathcal{Z}_3^2} \cdot \left(\sqrt{\mu^2 + \mathcal{Z}_3^2} + \mu\right)\right\}. \tag{56}$$

In the case where $\mathcal{Z}_3 = -\sqrt{2}$*, representation* (56) *is consistent with Kokonendji and Khoudar (*2004*, formula* (3.10)*).*

Remark 6 *(i) It follows from a combination of formulas* (12)*,* (15) *and* (20) *that*

$$(2-p) \cdot \mathcal{Z}_p \cdot t_{s0}\left(\mu/\left(\rho_p \mathcal{Z}_p\right)\right)^{\rho_p} \to \lambda \quad as\ p \uparrow 2; \tag{57}$$

$$(2-p)\mathcal{Z}_p \cdot y_s\left(\rho_p \mathcal{Z}_p/\mu\right)^{-\rho_p} \to \lambda \quad as\ p \downarrow 2. \tag{58}$$

Also, formulas (57)–(58) *are consistent with formulas* (74)–(75) *of the next "Appendix 1" section and the* L^2 *analogue of the weak convergence result* (49)*. In addition, the convergence result*

$$(2-p) \cdot \mathcal{Z}_p \cdot t_{s0}\left(\mu/\left(\rho_p \mathcal{Z}_p\right)\right)^{\rho_p} \to \mu/W_p\left(\mu/\mathcal{Z}_1\right) \quad as\ p \downarrow 1 \tag{59}$$

(which is consistent with formulas (21), (37), (39) *and the* L^2 *analogue of* (49)*) is equivalent to Conjecture* 2*.*

(ii) A combination of (12), (15) *and* (37)–(38) *with some algebra implies that for an arbitrary fixed* $p \in (1,2) \cup (2,+\infty)$ *and as* $\mu \to \infty$*,*

$$\mathbf{V}_{\mathcal{Z}_p}(\mu) \sim |p-2|^{1-p} \cdot |p-1|^{p-2} \cdot |\mathcal{Z}_p|^{1-p} \cdot \mu^p. \tag{60}$$

A combination of (60) *with Jørgensen (*1997*, Theorem* 4.5*) justifies the validity of* (50)*.*

Also, a combination of formulas (12), (15) *and* (37)–(39) *implies that given* $p \in [1,+\infty)$*,* $\mathbf{V}_{\mathcal{Z}_p}(\mu) \sim \mu$ *as* $\mu \downarrow 0$*, which is consistent with* (43) *(see also Jørgensen (*1997*, Proposition* 4.12 *and the last formula of Section* 4.6*)).*

Next, let us compare the Poisson-Tweedie family with a different class of the additive *Hinde-Demétrio* EDM's which correspond to a simpler u.v.f. defined by (61). Specifically, given $p \in \{0\} \cup [1,+\infty)$, set

$$\mathbf{V}_p^{\mathcal{HD}}(\mu) = \mu + \mu^p \tag{61}$$

(see, for example, Kokonendji et al. (2004, Theorem 5)). As per the follow-up paper by Kokonendji et al. (2007, p. 278), *"the origin of the Hinde-Demétrio family could be considered as an approximation (in terms of the unit variance function) to the Poisson-Tweedie family."* In this respect, we should point out that in the case where $p = 1$, even the similarity between their u.v.f.'s does not hold, which in turn necessitates a modification of Kokonendji et al. (2004, Proposition 6.ii) in this particular case. See Remark 7 and formula (62) specifically for more details.

Remark 7 *A combination of formulas* (55) *and* (61) *with the logarithmic asymptotics of the principal branch* W_p *of the Lambert function at infinity (see Corless et al.* (1996)*) implies that as* $\mu \to +\infty$*,*

$$\mu \cdot \log\, \mu \sim \mathbf{V}_1^{\mathcal{PT}}(\mu) \not\sim \mathbf{V}_1^{\mathcal{HD}}(\mu) = 2\mu. \tag{62}$$

*This corrects Kokonendji et al. (*2004*, Proposition* 2*.ii) in the case where* $p = 1$*. Thus, Kokonendji et al. (*2004*, Proposition* 2*.ii) holds for* $p > 1$ *only, in which case it is consistent with formula* (60)*. Moreover, although it is not stated in Kokonendji et al. (*2004*, Proposition* 2*), but for a fixed* $p > 1$ *and as* $\mu \downarrow 0$*, the u.v.f.s* $\mathbf{V}_p^{\mathcal{PT}}(\mu)$ *and* $\mathbf{V}_p^{\mathcal{HD}}(\mu)$ *are also equivalent to each other, since they are both locally Poisson at zero.*

We stress that although the u.v.f.'s for members of the Hinde-Demétrio class are simpler than those for the Poisson-Tweedie EDM's, the probability function for the Poisson-Tweedie mixtures appears to have a much simpler structure for $p \in (1,2) \cup (2,+\infty)$. Thus, even in the case where $p = 3$ which corresponds to the strict arcsine distributions introduced by Letac and Mora (1990, Example C), for which the range is also $\mathbf{Z}_+$, the probability function exhibits an unusual behavior. We clarify this by considering a subclass of the strict arcsine distributions which corresponds to the "border" value $\mu = +\infty$ of the location parameter. (The remaining members of this class are easily derived from them by exponential tilting). Similar to Letac and Mora (1990, formula (4.14)) or Kokonendji and Khoudar (2004, formulas (1.1)–(1.2)), given $a \in \mathbf{R}^1_+$ we introduce the *strict arcsine* r.v.'s $\mathcal{SA}_{a,\infty}$ on $\mathbf{Z}_+$ as follows:

$$\mathbf{P}\left\{\mathcal{SA}_{a,\infty} = n\right\} := \begin{cases} \frac{\prod_{k=0}^{n-1}(a^2+4k^2)}{(2n)!} & \text{for even } n; \\ a \cdot \frac{\prod_{k=0}^{n-1}(a^2+(2k+1)^2)}{(2n+1)!} & \text{for odd } n. \end{cases} \tag{63}$$

Proposition 4 *Given* $a \in \mathbf{R}^1_+$,

$$\mathbf{P}\{\mathcal{SA}_{a,\infty} = n\} \sim \frac{a \cdot \sinh(\pi a/2)}{2\sqrt{\pi}}\, n^{-3/2} \quad \textit{if even } n \to +\infty; \tag{64}$$

$$\mathbf{P}\{\mathcal{SA}_{a,\infty} = n\} \sim \frac{a \cdot \cosh(\pi a/2)}{2\sqrt{\pi}}\, n^{-3/2} \quad \textit{if odd } n \to +\infty. \tag{65}$$

A comparison of the fractions which emerge in formulas (64) and (65) stipulates that the power decay of the probability function of r.v. $\mathcal{SA}_{a,\infty}$ as $n \to +\infty$ is of order $-3/2$, but with *different factors of proportionality* for even and odd n.

In contrast to (64)–(65), in the case where $\mu = +\infty$ formula (77) of the next "Appendix 1" section implies that for the corresponding *Poisson-inverse Gaussian subclass* of the Poisson-Tweedie family, for which the power decay of the probability function at $+\infty$ is also of order $-3/2$, the factor of proportionality is identical for both even and odd terms. Apparently, the latter class of the Poisson-inverse Gaussian laws would hence be a more preferred choice for fitting the data than the exponentially tilted strictly arcsine distributions which can be generated from (63).

However, we reckon that the probability function of a general member of the Hinde-Demétrio class (with $p \neq 2$) still deserves being studied, since this might potentially reveal even more surprising properties which could be of interest for probability theory. (It is well known that the classes $\mathcal{PT}_2$ and $\mathcal{HD}_2$ coincide being comprised of negative binomial laws, whereas for other values of p, even the ranges of the corresponding subclasses of Poisson-Tweedie and Hinde-Demétrio families are different). But a comprehensive comparison of these two classes is beyond the scope of this paper.

Appendix 1. Proofs and auxilliary analysis results

Proof of Proposition 1 is obtained by combining formula (3) with Jørgensen (1997, formula (4.16)) and Panjer and Willmot (1992, formula (8.2.3)). □

Proof of Proposition 4 Recall the well-known fact that for any fixed real a and b,

$$\Gamma(z+a)/\Gamma(z+b) = z^{a-b} \cdot \left(1 + (a-b)(a-b+1)/(2z) + \mathcal{O}\left(z^{-2}\right)\right) \sim z^{a-b} \tag{66}$$

as real $z \to +\infty$ (cf. e.g., Askey and Roy (2010, formulas (5.11.12)–(5.11.13))).

For the even terms, the expression which emerges on the right-hand side of (63) can be rewritten as follows:

$$\frac{2^{2n}}{(2n)!}\prod_{k=0}^{n-1}\left(k+\frac{1}{2}ia\right)\left(k-\frac{1}{2}ia\right) = \frac{2^{2n}}{(2n)!}\left(\frac{1}{2}ia\right)_n\left(-\frac{1}{2}ia\right)_n$$
$$= \sqrt{\pi}\frac{\left(\frac{1}{2}ia\right)_n\left(-\frac{1}{2}ia\right)_n}{\Gamma\left(n+\frac{1}{2}\right)\Gamma(n+1)} \sim \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}ia)\Gamma\left(-\frac{1}{2}ia\right)}\, n^{-3/2} = \frac{a\sinh(\pi a/2)}{2\sqrt{\pi}}\, n^{-3/2},$$

upon application of (66). For the odd terms, a similar procedure shows that the right-hand side of (63) becomes

$$\frac{2^{2n}a}{(2n+1)!}\left(\frac{1}{2}+\frac{1}{2}ia\right)_n\left(\frac{1}{2}-\frac{1}{2}ia\right)_n=\frac{a\sqrt{\pi}}{2}\,\frac{\left(\frac{1}{2}+\frac{1}{2}ia\right)_n\left(\frac{1}{2}-\frac{1}{2}ia\right)_n}{\Gamma(n+1)\Gamma\left(n+\frac{3}{2}\right)}$$
$$\sim\frac{a\sqrt{\pi}}{2\Gamma\left(\frac{1}{2}+\frac{1}{2}ia\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}ia\right)}\,n^{-3/2}=\frac{a\cosh(\pi a/2)}{2\sqrt{\pi}}\,n^{-3/2}.$$

□

Proposition 5 *For arbitrary fixed* $\rho\in(-1,0)\cup(0,\infty)$, $z\in\mathbb{C}$, *and* $\ell\in\mathbf{Z}_+$,

$${}_1\Psi_1(\rho,\ell;\rho,0;z)\equiv e^z\cdot\mathbf{B}_\ell\left(z\cdot(\rho)_1,z\cdot(\rho)_2,\ldots,z\cdot(\rho)_\ell\right).$$

Proof of Proposition 5 Let $D\equiv d/dt$ and a prime denote differentiation with respect to z. For brevity we write $B_\ell\equiv\mathbf{B}_\ell(\rho z,(\rho)_2 z,\ldots,(\rho)_\ell z)$ and $\psi_\ell(z)\equiv{}_1\Psi_1(\rho,\ell;\rho,0;z)$. Our aim is to show that, for positive integer ℓ, $z\in\mathbb{C}$ and $\rho\in(-1,0)\cup(0,\infty)$,

$$\psi_\ell(z)=e^zB_\ell. \tag{67}$$

From the definition of the complete Bell polynomials in (7) we see that $\boldsymbol{B}_1(z_1)=z_1$ and $\boldsymbol{B}_2(z_1,z_2)=z_1^2+z_2$. From (6), we have for $\ell=1,2$ that

$$\psi_1(z)=\rho ze^z=e^zB_1,\qquad\psi_2(z)=\{(\rho z)^2+\rho(\rho+1)z\}e^z=e^zB_2.$$

Consequently, the result (67) is true for $\ell=1,2$. We now assume that (67) is true for arbitrary positive integer ℓ and proceed by induction.

From (6), we have for $\rho\in(-1,0)\cup(0,\infty)$

$$\psi_{\ell+1}(z)=\sum_{n=1}^{\infty}\frac{\Gamma(\ell+1+\rho n)}{\Gamma(\rho n)}\frac{z^n}{n!}=\sum_{n=1}^{\infty}(\ell+\rho n)\frac{\Gamma(\ell+\rho n)}{\Gamma(\rho n)}\frac{z^n}{n!}$$
$$=\ell\psi_\ell(z)+\rho z\psi_\ell'(z)=e^z\left\{\ell B_\ell+\rho z\left(B_\ell+B_\ell'\right)\right\}.$$

From the generating function for the complete Bell polynomials in (7) we have

$$B_\ell=D^\ell e^{zF}|_{t=0},\quad\text{and}\quad F:=\sum_{j=1}^{\infty}\frac{(\rho)_jt^j}{j!}=(1-t)^{-\rho}-1 \tag{68}$$

so that $B_\ell'=D^\ell Fe^{zF}|_{t=0}$. Then

$$\rho z(B_\ell+B_\ell')=\rho zD^\ell(1+F)e^{zF}|_{t=0}=\rho zD^\ell(1-t)^{-\rho}e^{zF}|_{t=0}$$
$$=zD^\ell(1-t)(DF)e^{zF}|_{t=0}=D^\ell(1-t)De^{zF}|_{t=0}$$

and

$$\ell B_\ell=\ell D^\ell e^{zF}|_{t=0}=\ell D^{\ell-1}De^{zF}|_{t=0}=D^\ell te^{zF}|_{t=0},$$

since for any function $g(t)$ regular at the origin and with regular derivatives at $t=0$ we have $D^\ell(tg(t))|_{t=0}=\ell D^{\ell-1}g(t)|_{t=0}$. Hence

$$\psi_{\ell+1}(z)=e^z\{D^\ell tDe^{zF}+D^\ell(1-t)De^{zF}\}|_{t=0}=e^zD^{\ell+1}e^{zF}|_{t=0}=e^zB_{\ell+1}.$$

□

Proof of Proposition 2 It easily follows by rewriting the exponent from the middle expression in (32) with subsequent expansion of the function $(1-u/(\theta_p+1))^{-\rho_p}$ in the Taylor series around 0 which can be easily derived from (68). Note that since the signs of $\mathcal{Z}_p$ and $(\rho_p)_k$ (which emerge in (36)) coincide, the Lévy measure $\nu_{p,\mu,\lambda}(\{k\}) > 0$ for each $k \in \mathbf{N}$. $\square$

Proposition 6 *Fix* $\ell \in \mathbf{Z}_+$. *Then*

(i) *In the case where* $z \in \mathbb{C}$ *is fixed and as* $\rho \to +\infty$,

$$\rho^{-\ell} \cdot e^{-z} \cdot {}_1\Psi_1(\rho, \ell; \rho, 0; z) = \rho^{-\ell} \cdot \mathbf{B}_\ell(z \cdot (\rho)_1, z \cdot (\rho)_2, \ldots, z \cdot (\rho)_\ell) \to T_\ell(z). \quad (69)$$

(ii) *Fix* $\mathcal{C} \neq 0$, *and assume that* $\rho \to 0$. *Then*

$${}_1\Psi_1(\rho, \ell; \rho, 0; \mathcal{C}/\rho) \sim e^{\mathcal{C}/\rho} \cdot (\mathcal{C})_\ell. \quad (70)$$

Proof of Proposition 6 (i) By (66), for a fixed $n \in \mathbf{N}$ and as $\rho \to +\infty$, the ratio $\Gamma(\rho n + k)/\Gamma(\rho n) = (\rho n)^k\{1 + \mathcal{O}(1/\rho)\}$. Hence, the sum appearing in the function ${}_1\Psi_1(\rho, k; \rho, 0; z)$ reduces in this limit to $\sum_{n=1}^{\infty} n^k z^n/n! = e^z T_k(z)$.

(ii) We apply the polynomial representation for the Wright function given in Paris and Vinogradov (2016, formulas (2.7)–(2.8)) for $k \in \mathbf{N}$, namely,

$${}_1\Psi_1(\rho, k; \rho, 0; z) = (\rho z)^k e^z \mathbf{h}_{k-1}\left((\rho z)^{-1}\right), \quad (71)$$

where

$$\mathbf{h}_{k-1}(u) = \sum_{n=0}^{k-1} D_n u^n \text{ with } D_n \equiv D_n(\rho, k) = \sum_{\ell=0}^{n} (-1)^\ell \rho^{n-\ell} s_k^{(k-\ell)} S_{k-\ell}^{(k-n)}. \quad (72)$$

Here, $s_k^{(r)}$ and $S_k^{(r)}$ are the Stirling numbers of the first and second kinds, respectively. Then, for $u \in \mathbf{R}^1$ and as $\rho \to 0$,

$$\mathbf{h}_{k-1}(u) \sim \sum_{n=0}^{k-1} s_k^{(k-n)}(-u)^n = (-u)^k \sum_{r=1}^{k} (-u)^{-r} s_k^{(r)} = u^k \frac{\Gamma(k + 1/u)}{\Gamma(1/u)} \quad (73)$$

by application of Bressoud (2010, formula (26.8.7)). A subsequent combination of (73) with (71) implies that given $k \in \mathbf{N}$ with $\rho \to 0$ and $z \sim \mathcal{C}/\rho$, (70) holds. $\square$

Proof of Proposition 3 It easily follows from a combination of formulas (26)–(27) and (29)–(30) for the probability function of the corresponding Poisson-Tweedie mixtures with Proposition 6. In particular, the expressions which emerge on the left-hand sides of formulas (69) and (70) are closely related to the probability function of the Poisson–Tweedie mixtures with $p \in (1, 2) \cup (2, +\infty)$, whereas their limits are employed in formulas (26)–(27) which pertain to the cases where $p = 1$ and 2, respectively. $\square$

Proof of Theorem 1 First, consider a generic member $\mathcal{X}_{p,\mu,\lambda}$ of the family of Poisson-Tweedie mixtures. We combine the Poisson-mixture representation (28) and Panjer and Willmot (1992, formulas (8.2.5)–(8.2.6)) with Jørgensen (1997, formula (3.15)) or Vinogradov (2004, formula (1.6)) and recall that $\mathbf{E}(\mathcal{PT}_{p,\mu,\lambda}) = \mu$ to conclude that

$$\mathbf{Var}(\mathcal{PT}_{p,\mu,\lambda}) = \mu + \mu^p/\lambda. \quad (74)$$

In particular, a combination of (20) with (74) stipulates that for $p = 2$,

$$\mathbf{Var}(\mathcal{PT}_{2,\mu,\lambda}) = \mu + \mu^2/\lambda = \mu + \mu^2/\mathcal{Z}_2. \quad (75)$$

In the remaining cases, one should solve Eq. (20) for λ given the fixed set of values of p, μ and $\mathcal{Z}_p$. In the case where $p = 1$, the solution is found by following along the same lines as the proof of Vinogradov (2013, Theorem 5.1).

Subsequently, the cases where $1 < p < 2$ and $p > 2$ are reduced to solving the Eqs. (10) and (13), respectively, and involve an application of the corresponding analytic results established in Paris and Vinogradov (2016). The details are left to the reader. □

Lemma 1 *Given $p \in (1,2)$, $\mu \in \Omega_p = \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, the corresponding Poisson-Tweedie mixture $\mathcal{X}_{p,\mu,\lambda}$ belongs to class $(\mathcal{S})$ with $f = 0$, $h = 1$, integer $\ell \in \mathbf{Z}_+$, and function*

$$g(x) := \mathcal{A}_p x^{-1/(\rho_p+1)} - (\rho_p + 2)/(2(\rho_p + 1)x) \tag{76}$$

satisfying all the conditions (i)–(v) imposed in Definition 5.

Proof of Lemma 1 Let $p \in (1,2)$, so that $\rho = \rho_p > 0$ and $\mathcal{A}_p > 0$. It is convenient to rewrite the function $g(x)$ introduced by formula (76) as follows:

$$g(x) = \frac{1}{x}\left(\mathcal{A}_p x^{\nu} - \frac{\rho+2}{2(\rho+1)}\right), \text{ where } \nu := \frac{\rho}{\rho+1}.$$

Subsequently, we obtain that

$$g'(x) = -\left(\mathcal{A}_p x^{\nu} - 1 - \frac{1}{2}\rho\right)\frac{1}{(\rho+1)x^2}, \qquad g''(x) = \left(\mathcal{A}_p x^{\nu} - 1 - \rho\right)\frac{\rho+2}{(\rho+1)x^3}\,.$$

It then follows that $g(x) \geq 0$ when $x \geq x_3$, $g''(x) \geq 0$ when $x \geq x_1$ and $-g'(x) \geq 0$ when $x \geq x_2$, where

$$x_1^{\nu} = \frac{\rho+1}{\mathcal{A}_p}, \quad x_2^{\nu} = \frac{\rho+2}{2\mathcal{A}_p}, \quad x_3^{\nu} = \frac{\rho+2}{2(\rho+1)\mathcal{A}_p}.$$

Furthermore, it can be shown that $g'''(x_0) = 0$ and $g^{iv}(x_0) < 0$, where $x_0^{\nu} = 3(\rho+1)^2/((2\rho+3)\mathcal{A}_p)$. It is easily verified that $x_0 > x_1 > x_2 > x_3$.

Then for $x \geq x_0$, we have (i) $g(x) \geq 0$, (ii) $g(\infty) = 0$, (iii) $g''(x) \downarrow$ and (iv) the product $x \cdot g(x) \to +\infty$ as $x \to +\infty$. Finally, we have

$$-\frac{g''(x)}{g'(x)} = \frac{\left(\mathcal{A}_p x^{\nu} - 1 - \rho\right)}{\left(\mathcal{A}_p x^{\nu} - 1 - \frac{1}{2}\rho\right)}\,\frac{\rho+2}{(\rho+1)x}\,.$$

Then for $x \geq x_0$

$$\text{(v)} \qquad 0 < -\frac{g''(x)}{g'(x)} < \frac{\rho+2}{(\rho+1)x} = \left(1 + \frac{1}{\rho+1}\right)\frac{1}{x} < \frac{2}{x}. \square$$

Proof of Theorem 2 First, Lemma 1 justifies the applicability of Nagaev (1998, Theorem 2) on the exact asymptotics of the probabilities of large deviations. The rest follows from a combination of representation (37) for the variance function $\mathbf{V}_{\mathcal{Z}_p}(\cdot)$ with Jørgensen (1997, p. 50 and Exercise 2.25). □

Proof of Theorem 3 The proof of the fact that the asymptotics is given by the expression which emerges on the right-hand side of (41) then easily follows from a combination of (31) with Paris and Vinogradov (2016, formulas (4.8)–(4.9)). □

Lemma 2 *Given $p > 2$, $\mu \in \Omega_p$ and $\lambda \in \mathbf{R}^1_+$, the probability function of r.v. $\mathcal{PT}_{p,\mu,\lambda}$ possesses the following asymptotics as integer $\ell \to +\infty$:*

$$\mathbf{P}\{\mathcal{PT}_{p,\mu,\lambda} = \ell\} \sim e^{-\mathcal{A}_p} \cdot (\theta_p + 1)^{-\ell} \cdot \mathcal{Z} \cdot \ell^{\rho_p - 1}/\Gamma(\rho_p). \tag{77}$$

Proof of Lemma 2 It involves a combination of representation (30) with Paris and Vinogradov (2016, formula (4.14)). In addition, the proof can be derived from Christoph and Schreiber (1998, formula (9)). Moreover, an application of Paris and Vinogradov (2016, formula (4.13)) or Christoph and Schreiber (1998, formula (9)) makes it possible to construct asymptotic expansions of the probability function that emerges on the left-hand side of formula (77). □

Proof of Theorem 4 For $\mu = \infty$, (42) follows with some effort from a combination of Borovkov and Borovkov (2008, p. 167 and Theorem 3.7.1) with (31) and (77). For $\mu < \infty$, (42) is easily reduced to the "boundary" case of $\mu = \infty$. □

Proof of Theorem 5 First consider $p \in (1,2) \cup (2,+\infty)$ with $\ell \in \mathbf{Z}_+$ and let $\lambda \to +\infty$. From (17)–(19), it follows that $\theta_p \to +\infty$ like λ and $\mathcal{Z} \to \pm\infty$ according as $p \in (1,2)$ and $p \in (2,+\infty)$, respectively. We employ representation (30) for $\mathbf{p}_\ell$ and make use of Paris and Vinogradov (2016, formula (3.1)) which stipulates that for integer $\ell > 1$

$$
{}_1\Psi_1\left(\rho,\ell;\rho,0;\mathcal{Z}\right) = (\rho\mathcal{Z})^\ell\, e^{\mathcal{Z}}\left(1+\frac{(\rho+1)\ell(\ell-1)}{2\rho\mathcal{Z}}+\mathcal{O}\left(\mathcal{Z}^{-2}\right)\right) \qquad (\mathcal{Z}\to\pm\infty)
$$

$$
= (\rho\mathcal{Z})^\ell e^{\mathcal{Z}}\left(1+\frac{\ell(\ell-1)}{2\lambda\mu^{2-p}}+\mathcal{O}\left(\lambda^{-2}\right)\right)
$$

since $\rho = \rho_p$ is such that $(\rho+1)(p-1) = 1$, and

$$
\rho\mathcal{Z} = \frac{\rho\lambda\mu^{2-p}}{2-p}(1+1/\theta_p)^{-\rho} = \frac{\lambda\mu^{2-p}}{p-1}\left(1+\mathcal{O}\left(\lambda^{-1}\right)\right).
$$

Then, from (30),

$$
\mathbf{p}_\ell = \frac{e^{\mathcal{Z}-\mathcal{A}_p}}{\ell!}\left(\frac{\rho\mathcal{Z}}{\theta_p+1}\right)^\ell\left(1+\frac{\ell(\ell-1)}{2\lambda\mu^{2-p}}+\mathcal{O}\left(\lambda^{-2}\right)\right), \tag{78}
$$

where $\mathcal{Z} = \mathcal{A}_p(1+1/\theta_p)^{-\rho} = \mathcal{A}_p\left(1-\frac{\rho}{\theta_p}+\frac{\rho(\rho+1)}{2\theta_p^2}+\mathcal{O}\left(\lambda^{-3}\right)\right)$. With $\mu = \rho\mathcal{A}_p/\theta_p$ and θ_p defined in (17), we have

$$
e^{\mathcal{Z}-\mathcal{A}_p} = e^{-\mu}\left(1+\frac{\mu^2}{2\lambda\mu^{2-p}}+\mathcal{O}\left(\lambda^{-2}\right)\right),\quad \frac{\rho\mathcal{Z}}{\theta_p+1} = \mu\left(1-\frac{\mu}{\lambda\mu^{2-p}}+\mathcal{O}\left(\lambda^{-2}\right)\right).
$$

Substitution of these estimates in (78) finally yields the validity of (44).

When $p = 2$, we obtain from (27) with $\theta_2 = \mu/\lambda$ that

$$
\begin{aligned}
\mathbf{p}_\ell &= \frac{(\lambda)_\ell}{\ell!}\cdot\left(\frac{\mu}{\mu+\lambda}\right)^\ell\cdot\left(\frac{\lambda}{\mu+\lambda}\right)^\lambda \\
&= \frac{e^{-\mu}\mu^\ell}{\ell!}\cdot\frac{\lambda^{-\ell}\Gamma(\lambda+\ell)}{\Gamma(\lambda)}\cdot\left(1+\frac{\mu}{\lambda}\right)^{-\ell}\cdot e^\mu\cdot\left(1+\frac{\mu}{\lambda}\right)^{-\lambda}.
\end{aligned}
$$

By formula (66), $\lambda^{-\ell}\Gamma(\lambda+\ell)/\Gamma(\lambda) = 1+\ell(\ell-1)/(2\lambda)+\mathcal{O}\left(\lambda^{-2}\right)$ as $\lambda \to +\infty$. Subsequently, some routine algebra yields that

$$
\mathbf{p}_\ell = \frac{e^{-\mu}\mu^\ell}{\ell!}\left\{\left(1+\frac{\ell(\ell-1)}{2\lambda}\right)\left(1-\frac{\mu\ell}{\lambda}\right)\left(1+\frac{\mu^2}{2\lambda}\right)+\mathcal{O}\left(\lambda^{-2}\right)\right\}
$$

$$
= \frac{e^{-\mu}\mu^\ell}{\ell!}\left\{1+\frac{(\mu-\ell)^2-\ell}{2\lambda}+\mathcal{O}\left(\lambda^{-2}\right)\right\}.
$$

Finally, when $p = 1$ we use (26) with $\mathcal{A}_1 = \mu\lambda$ and $\mathcal{Z} = \mu\lambda e^{-1/\lambda}$ given by (20) to find

$$
\mathbf{p}_\ell = \frac{e^{\mu\lambda(e^{-1/\lambda}-1)}}{\lambda^\ell\ell!}\,T_\ell\left(\mu\lambda e^{-1/\lambda}\right)\frac{e^{-\mu}}{\lambda^\ell\ell!}\left(1+\frac{\mu}{2\lambda}+\mathcal{O}\left(\lambda^{-2}\right)\right)\,T_\ell\left(\mu\lambda-\mu+\mathcal{O}\left(\lambda^{-2}\right)\right).
$$

From Paris (2016, formula (1.4)), the expansion of the ℓ^{th} Touchard polynomial for large values of x is $T_\ell(x) = x^\ell \left\{1 + \ell(\ell-1)/(2x) + \mathcal{O}\left(x^{-2}\right)\right\}$ as $x \to +\infty$, whence

$$\begin{aligned}\mathbf{p}_\ell &= \frac{e^{-\mu}\mu^\ell}{\ell!}\left\{\left(1+\frac{\mu}{2\lambda}\right)\left(1-\frac{\ell}{\lambda}\right)\left(1+\frac{\ell(\ell-1)}{2\mu\lambda}\right)+\mathcal{O}\left(\lambda^{-2}\right)\right\}\\ &= \frac{e^{-\mu}\mu^\ell}{\ell!}\left\{1+\frac{(\mu-\ell)^2-\ell}{2\mu\lambda}+\mathcal{O}\left(\lambda^{-2}\right)\right\}.\end{aligned}$$

The above arguments complete the proof of the validity of (44). □

Proof of Theorem 6 We combine formulas (17)–(20), (30) and (45)–(46) to ascertain that $\theta_p \to 0$, $\mathcal{A}_p \to 0$, $\mathcal{Z}_p \to \mathcal{Z}$ and hence, that

$$\mathbf{P}\{\mathcal{PT}_{p,\mu_p,\lambda_p} = \ell\} \sim \frac{1}{\ell!} \cdot {}_1\Psi_1(\rho_p, \ell, \rho_p, 0; \mathcal{Z}). \tag{79}$$

The rest follows from a combination of (79) with formulas (71)–(72) which stipulate that given $\ell \in \mathbf{Z}_+$ and $\mathcal{Z} \in \mathbf{R}^1 \setminus \{0\}$,

$${}_1\Psi_1(\rho, \ell, \rho, 0; \mathcal{Z}) \to (-\mathcal{Z})^\ell \cdot e^{\mathcal{Z}} \;\text{ as }\; \rho\ (= \rho_p) \downarrow -1. \;\square \tag{80}$$

Proof of Theorem 7 The first step involves a combination of formulas (25) and (28) with subsequent derivation of an integral representation for the probability of interest, where the integral over $\mathbf{R}^1_+$ is similar to those considered in Vinogradov (2008a, formulas (27) and (32)) and Paris and Vinogradov (2015, formula (3.31)).

The second step relies on an application of Paris (2011, p. 14, formula (1.2.22)) for the derivation of an asymptotic expansion of such an integral. This is carried out by a refinement of the Laplace method and is parallel to Paris and Vinogradov (2015, Proof of Theorem 3.10). The details are left to the reader. □

Proof of Theorem 8 First, it follows from the fact that $\mu_{\mathcal{C}} := \mathcal{C}\mu$ and $\lambda_{\mathcal{C}} := \mathcal{C}^{p-2}\lambda$ that $\mathcal{A}_p(\mathcal{C}) \equiv \mathcal{A}_p(1) = \mathcal{A}_p$, and $\theta_p(\mathcal{C}) \equiv \theta_p(1)/\mathcal{C} = \theta_p/\mathcal{C}$. Hence, the probability function of r.v. $\mathcal{PT}_{p,\mathcal{C}\mu,\mathcal{C}^{p-2}\lambda}$ admits representation (30) with the same value of $\mathcal{A}_p(= \mathcal{A}_p(\mathcal{C}))$, the value $\theta_p/\mathcal{C}$ of the exponential tilting parameter, and the following value $\mathcal{Z}^{(\mathcal{C})}$ of the invariant of the exponential tilting transformation:

$$\begin{aligned}\mathcal{Z}^{(\mathcal{C})} &:= \mathcal{A}_p \cdot \left(\theta_p(\mathcal{C})/(\theta_p(\mathcal{C})+1)\right)^{\rho_p} = \mathcal{A}_p \cdot \left(\mathcal{C}^{-1}\cdot\theta_p/\left(\theta_p/\mathcal{C}+1\right)\right)^{\rho_p}\\ &= \mathcal{Z}_{p,\infty}\cdot\left(\theta_p+\mathcal{C}\right)^{-\rho_p}.\end{aligned} \tag{81}$$

In particular, (81) yields that as $\mathcal{C} \to +\infty$,

$$\mathcal{Z}^{(\mathcal{C})} \sim \mathcal{Z}_{p,\infty}\cdot\mathcal{C}^{-\rho_p}\begin{cases}\downarrow 0 & \text{if } \rho_p > 0;\\ \to -\infty & \text{if } \rho_p \in (-1,0).\end{cases}$$

Next, combine the Poisson-mixture representation (28) with Paris and Vinogradov (2016, formula (1.5)). The asymptotics of the integral over $\mathbf{R}^1_+$ which will emerge as a result of an application of the latter structural relationship given by Paris and Vinogradov (2016, formula (1.5)) is evaluated by following along the same lines as Paris and Vinogradov (2015, Proof of Theorem 3.10), where the special case of $p = 3/2$ was considered. Again, it relies on a refinement of the Laplace method described in Paris (2011, p. 14, formula (1.2.22)). The details are left to the reader. □

Appendix 2. Relevant conjectures and their numerical verification

In this section, we will present two conjectures pertaining to the behavior of the Wright function ${}_1\Psi_1(\rho,\ell,\rho,0;\cdot)$ and "reduced" Wright function $\phi(\rho,0;\cdot)$ in the cases where the parameter ρ approaches -1 and $+\infty$, respectively.

First, consider the following function $\mathcal{G}_x(k),\ k\in\mathbf{Z}_+$, with fixed $x\in\mathbf{R}^1\setminus\{0\}$:

$$\mathcal{G}_x(k) := \begin{cases} 0 & \text{if } k=0; \\ \sum_{\ell=1}^{k} x^{-\ell}\cdot\frac{k(k-1)\ldots(k-\ell)}{\ell(\ell+1)} - k & \text{if } k\geq 1. \end{cases} \tag{82}$$

Also, set

$$\lambda_p = \lambda_{\mathcal{Z},p} := |\mathcal{Z}|^{p-1}(p-1)\cdot(1-1/(p-1))^{p-1}. \tag{83}$$

Clearly, $\lambda_p \sim e^{-1}\cdot(p-1)\cdot|\mathcal{Z}|^{p-1}$ as $p\to+\infty$.

The following conjecture can be regarded as a prospective refinement of the local limit Theorem 6 on Poisson convergence.

Conjecture 1 *Given $k\in\mathbf{Z}_+$ and real $\mathcal{Z}<0$,*

$$\mathbf{P}\left\{\mathcal{PT}_{p,\infty,\lambda_p}=k\right\} = \pi_{|\mathcal{Z}|}(k)\cdot\left(1+\mathcal{G}_{|\mathcal{Z}|}(k)/p\right)+\mathcal{O}(1/p^2)\ \ as\ \ p\to+\infty. \tag{84}$$

By (83), formula (84) follows from the next *analysis conjecture* which would refine (80): given $\ell\in\mathbf{Z}_+$ and real $\mathcal{Z}\neq 0$, there exists constant $\mathcal{K}\in\mathbf{R}^1_+$ such that

$$|{}_1\Psi_1\left(\rho,\ell,\rho,0;\mathcal{Z}\right)/\left((-\mathcal{Z})^{\ell}e^{\mathcal{Z}}\right)-1-\mathcal{G}_{-\mathcal{Z}}(\ell)(1+\rho)|/(1+\rho)^2 \to \mathcal{K} \text{ as } \rho\downarrow -1. \tag{85}$$

The veracity of (85) was checked numerically with the help of *Mathematica*. For simplicity, set $\mathcal{Z}=-3$ and $\ell=5$ (with $\epsilon:=1+\rho$ approaching 0).

The computations summarized in Table 1 suggest that for the above $\mathcal{Z}$ and ℓ, the value of constant $\mathcal{K}$ is approximately 3.6419.

The next (previously unknown) hypothesis relates the "reduced" Wright function $\phi(\rho,0;\cdot)$ with $\rho\to+\infty$ to the principal branch W_p of the Lambert W function.

Conjecture 2 *Fix $z\in\mathbf{R}^1_+$ and assume that real $\kappa\downarrow -1$. Then*

$$\begin{aligned}&\log\left\{\int_0^\infty \frac{e^{((1+\kappa)/\kappa)zy}}{y(1+y)}\cdot\phi(\kappa,0;-(1+y)y^{\kappa})dy\right\}\\ &= -\frac{z}{W_p(z)}+z(1+\kappa)+o(1+\kappa)\to -\frac{z}{W_p(z)},\end{aligned} \tag{86}$$

where functions ϕ and W_p are introduced in Definitions 1 and 4, respectively.

As of now, we only have some "probabilistic" arguments in support of the validity of (86), which are given below Table 2. Observe that both the "probabilistic" arguments and

Table 1 The accuracy of the "leading error term" approximation for ${}_1\Psi_1(\rho,5,\rho,0;-3)$ as $\rho\downarrow -1$

$\epsilon=1+\rho$	Left-hand side of (85)
10^{-1}	3.254222
10^{-2}	3.601987
10^{-3}	3.637964
10^{-4}	3.641574
10^{-5}	3.641935
10^{-6}	3.641971

Table 2 The absolute relative error (88) for the approximation (87) in the case where $z = 1$

ρ	Error (88)	ρ	Error (88)
20	8.96×10^{-3}	500	3.62×10^{-4}
50	3.61×10^{-3}	1500	1.21×10^{-4}
100	1.80×10^{-3}	2000	9.04×10^{-5}

the numerical verification of (86) given in Table 2 involve rewriting the integral from (86) in terms of the specific solution to equation (10). Namely, that integral coincides with the integral which emerges on the right-hand side of (11), such that $r := 1/(1+\kappa) - 1 \uparrow +\infty$ and $w = 1/(r \cdot z)$. Also, it can be shown that (86) is equivalent to the fact that given $z > 0$,

$$\log t_{s0}(z/\rho) \sim W_p(z)/\rho \quad \text{as} \quad \rho \uparrow +\infty. \tag{87}$$

Now, we address the veracity of approximation (87). For $z = 1$, Table 2 gives the following absolute relative error for various values of ρ:

$$|\log t_{s0}(z/\rho) - W_p(z)/\rho|/(W_p(z)/\rho). \tag{88}$$

The computations summarized in Table 2 suggest that the error (88) decreases to 0 as ρ increases to $+\infty$.

We now provide the "probabilistic" arguments in support of (86). First, it can be shown that Conjecture 2 is equivalent to (59). Then in view of continuity of the Poisson-Tweedie family in Lévy metric with respect to p (see (49)), it is plausible that their variance functions also converge pointwise as $p \downarrow 1$. Recall that in the cases where $p \in (1, 2)$ and $p = 1$, parts (i) and (iii) of Theorem 1 yield that they are expressed in terms of functions ϕ and W_p, respectively. A subsequent combination of representations (37) and (39) with the anticipated pointwise convergence of the variance functions as $p \downarrow 1$, (21) and some algebra suggests the validity of (59).

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Authors' contributions
VV dealt with the overall presentation, proofs of the probability theory results, and drafted the manuscript. RBP dealt with the asymptotics, proofs of the analysis results, and the numerical verification of two conjectures. Both authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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