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On Bernstein type polynomials and their applications

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Abstract

In this study we examine generating functions for the Bernstein type polynomials given in (Simsek in Fixed Point Theory Appl. 2013:80, 2013). We expand these generating functions using the parameters u and v. By applying these generating functions, we obtain some functional equations and partial differential equations. In addition, using these equations, we derive several identities and relations related to these polynomials. Finally, numerical values of these polynomials for selected cases are demonstrated with their plots.

Keywords: Bernstein polynomials; CAD; graphs; Bézier curves; generating functions

1 Introduction

With the advances in computer graphics and CAD, there has been renewed interest by researchers to study Bézier curves and surfaces [1, 2]. According to Goldman [3], free-form curves and surfaces are smooth shapes often describing man-made objects. For instance, the hood of a car, the hull of a ship, and the fuselage of an airplane are all examples of free-form shapes that differ from the typical surfaces as they can be described with a few parameters. On the other hand, free-form shapes such as the hood of car may not easily be described with a few parameters. Therefore, mathematical techniques for describing these surfaces focused on Bernstein polynomials and their various generalizations (*cf.* [1–13]).

Curves obtained by using Bernstein polynomials range from the design of new fonts to the creation of mechanical components and assemblies for large scale industrial design and manufacture. By using the Bernstein polynomials, one can easily find an explicit polynomial representation of a Bézier curves.

In addition to computer graphics, the Bernstein polynomials are also used in the approximation of functions, in statistics, in numerical analysis, in p-adic analysis, and in the solution of differential equations. Therefore, the goal of this paper is to develop a more flexible Bernstein type polynomial using its generating function and visualize the curves obtained with this function over a finite domain with set parameters.

The organization of the paper is as follows.

In Section 2, we give the definition, generating functions, and some properties of the Bernstein type basis functions with respect to u and v. In Section 3, we differentiate the generating function with respect to x and t and obtain select partial differential equations (PDEs). Using these equations, we derive a recurrence relation and derivative formula for

Bernstein type basis functions. Finally curves are plotted using the Bernstein type basis function.

2 Properties of the Bernstein type basis functions

In this section, we give fundamental properties of the Bernstein basis functions and their generating functions. By using generating functions, we derive various functional equations and PDEs. Next, using these equations and PDEs, we obtain several identities related to the Bernstein type basis functions.

2.1 Generating functions

Recently the Bernstein polynomials have been defined and studied in many different ways, for example, by q-series, by complex functions, by p-adic Volkenborn integrals, and many algorithms (*cf.* [1, 2, 4–6]). Here by using an analytic function we construct generating functions for the Bernstein type basis functions related to nonnegative real parameters.

The Bernstein type basis functions $S_k^n(x;b;u,v)$ are defined as follows.

Definition 1 Let u and v be real parameters with $u < v$. Let n, k and b be nonnegative positive integers and let $x \in [u,v]$. Let n be nonnegative integer. The Bernstein type basis functions $S_k^n(x;b;u,v)$ can be defined by

$$S_k^n(x;b;u,v) = 2^{1-b}\binom{n}{b}\left(\frac{x-v}{v-u}\right)^k\left(\frac{x-u}{v-u}\right)^{n-b-k}, \tag{2.1}$$

where

$$n \geq b+k,$$
$$k = 0,1,\ldots,n,$$
$$b = 0,1,\ldots,n$$

and

$$\binom{n}{b} = \frac{n!}{b!(n-b)!}.$$

Remark 1 Substituting $u = -1$ and $v = 0$ into (2.1), we have

$$S_k^n(x;b;-1,0) = Y_k^n(x;b) = 2^{1-b}\binom{n}{b}x^k(1+x)^{n-k-b},$$

where $k = 0,1,\ldots,n$; $b = 0,1,\ldots,n$; $n \geq b+k$, and $x \in [-1,0]$ (*cf.* [7]).

Definition 2 The Bernstein type basis functions can be defined by means of the following generating function:

$$f_{S,k}(x,t;b;u,v) := \sum_{n=0}^{\infty} S_k^n(x;b;u,v)\frac{t^n}{n!}, \tag{2.2}$$

where $k, b = 0,1,\ldots,n$ and $n \geq b+k$.

We construct generating functions for the Bernstein type basis functions explicitly by the following theorem.

Theorem 1 *Let u and v be real parameters with $u < v$. Let $t \in \mathbb{C}$. Let b and k be positive integers and let $x \in [u, v]$. Then we have*

$$f_{S,k}(x,t;b,u,v) = \frac{2(\frac{t}{2})^b(\frac{x-v}{x-u})^k e^{(\frac{x-u}{v-u})t}}{b!}. \tag{2.3}$$

Proof By using (2.1) and (2.2), we have

$$\sum_{n=b}^{\infty} S_k^n(x;b;u,v)\frac{t^n}{n!} = \sum_{n=b}^{\infty} 2^{1-b}\binom{n}{b}\left(\frac{x-v}{v-u}\right)^k\left(\frac{x-u}{v-u}\right)^{n-b-k}\frac{t^n}{n!}.$$

By using the above equation, we get

$$\sum_{n=b}^{\infty} S_k^n(x;b;u,v)\frac{t^n}{n!} = \frac{2^{1-b}}{b!}\left(\frac{x-v}{x-u}\right)^k\sum_{n=b}^{\infty}\left(\frac{x-u}{v-u}\right)^{n-b}\frac{t^{n-b}t^b}{(n-b)!}.$$

The series on the right-hand side is the Taylor series for $e^{(\frac{x-u}{v-u})t}$. Consequently, we obtain (2.3), asserted by Theorem 1. $\square$

Some of the properties of $f_{S,k}(x,t;b,u,v)$ are given as follows:

$$f_{S,k}(v,t;b,u,v) = 0 \quad (k \neq 0),$$
$$f_{S,k}(u,t;b,u,v) = \infty \quad (k \neq 0).$$

Therefore the function $f_{S,k}(x,t;b,u,v)$ is a meromorphic function which has a pole at $x = u$. If $k = 0$, then

$$f_{S,0}(x,t;b,u,v) = \frac{2(\frac{t}{2})^b e^{(\frac{x-u}{v-u})t}}{b!}$$

is an analytic function. Thus using the Taylor expansion of e^{xt}, we get

$$\frac{2(\frac{t}{2})^b e^{(\frac{x-u}{v-u})t}}{b!} = 2^{1-b}\sum_{n=0}^{\infty}\binom{n}{b}\left(\frac{x-u}{v-u}\right)^{n-b}\frac{t^n}{n!}.$$

Therefore,

$$S_0^n(x;b;u,v) = 2^{1-b}\binom{n}{b}\left(\frac{x-u}{v-u}\right)^{n-b}.$$

Substituting $b = 0$ into (2.3), we get

$$f_{S,k}(x,t;0,u,v) = 2\left(\frac{x-v}{x-u}\right)^k e^{(\frac{x-u}{v-u})t},$$

which gives us generating functions for the beta-type polynomials associated with real parameters u and v ($u < v$). Note that $f_{S,k}(x,t;0,-1,0)$ is a generating function for the beta-type polynomials (*cf.* [6, 8]).

2.2 Bernstein type polynomials

A Bernstein type polynomial $P(x;b;u,v)$ is a polynomial represented as the Bernstein type basis function:

$$P(x,a,b,m) = \sum_{k=0}^{n} c_k^n S_k^n(x;b;u,v), \tag{2.4}$$

where c_k^n are real numbers.

Remark 2 If we set $v = 0$ and $u = -1$ in (2.4), and $S_k^n(x;b-1,0) = Y_k^n(x;b)$ then we have

$$P(x) = \sum_{k=0}^{n} c_k^n Y_k^n(x;b)$$

(*cf.* [7]).

Remark 3 A *Bézier type curve* $B(x;b;u,v)$ with control points

$$P_0,\dots,P_n$$

may be defined as follows:

$$B(x;b;u,v) = \sum_{k=0}^{n} P_k S_k^n(x;b;u,v). \tag{2.5}$$

2.3 Sum of the Bernstein type basis functions

Using the same method as proposed in [9], we get the following functional equation:

$$\sum_{b=0}^{\infty} f_{S,k}(x,t;b,u,v) = 2\left(\frac{x-v}{x-u}\right)^k e^{\frac{t}{2}(\frac{2x-3u+v}{v-u})}.$$

From the above equation, we get the sum of the Bernstein basis functions by the following theorem.

Theorem 2

$$\sum_{b=0}^{n} S_k^n(x;b;u,v) = 2^{1-n}\left(\frac{x-v}{x-u}\right)^k \left(\frac{2x-3u+v}{v-u}\right)^n.$$

Similarly, we have

$$\sum_{k=0}^{\infty} f_{S,k}(x,t;b,u,v) = \frac{2(\frac{t}{2})^b e^{(\frac{x-u}{v-u})t}}{b!} \sum_{k=0}^{\infty}\left(\frac{x-v}{x-u}\right)^k.$$

We assume that $|\frac{x-v}{x-u}| < 1$. Thus we have

$$\begin{aligned}\sum_{k=0}^{\infty} f_{S,k}(x,t;b,u,v) &= \frac{2(\frac{t}{2})^b e^{(\frac{x-u}{v-u})t}}{b!} \sum_{k=0}^{\infty}\left(\frac{x-v}{x-u}\right)^k \\ &= \frac{2(\frac{t}{2})^b(\frac{x-u}{v-u})e^{(\frac{x-u}{v-u})t}}{b!}.\end{aligned}$$

From the above equation, we obtain the following theorem.

Theorem 3

$$\sum_{k=0}^{n} S_k^n(x;b;u,v) = 2^{1-b}\binom{n}{b}\left(\frac{x-u}{v-u}\right)^{n+1}.$$

2.4 Alternating sum of the Bernstein type basis functions

Using the same method as proposed in [9], we get the following functional equation:

$$\sum_{b=0}^{\infty}(-1)^b f_{S,k}(x,t;b,u,v) = 2\left(\frac{x-v}{x-u}\right)^k e^{t(\frac{x-u}{v-u}-\frac{1}{2})}.$$

From the above equation, we get the following theorem.

Theorem 4

$$\sum_{b=0}^{n}(-1)^b S_k^n(x;b;u,v) = 2\left(\frac{x-v}{x-u}\right)^k\left(\frac{x-u}{v-u}-\frac{1}{2}\right)^n.$$

3 Differentiating the generating function

In this section, we give derivative of the Bernstein type basis functions. By differentiating the generating function in (2.3) with respect to x, we arrive at the following theorem.

Theorem 5 *We have*

$$\frac{\partial f_{S,k}(x,t;b,u,v)}{\partial x} = \frac{k(v-u)}{(x-u)^2} f_{S,k-1}(x,t;b,u,v) + \frac{t}{v-u} f_{S,k}(x,t;b,u,v).$$

By using Theorem 5, we obtain the derivative of the Bernstein type basis functions by the following theorem.

Theorem 6 *Let u and v be nonnegative real parameters with $u < v$. Let $x \in [u,v]$. Let k and b be nonnegative integers and n be a positive integer with $n \geq k+b$. Then*

$$\frac{dS_k^n(x;b;u,v)}{dx} = \frac{k(v-u)}{(x-u)^2} S_{k-1}^n(x;b;u,v) + \frac{n}{v-u} S_k^{n-1}(x;b;u,v).$$

Remark 4 Substituting $v = 0$ and $u = -1$ into Theorem 6, we have

$$\frac{dY_k^n(x;b)}{dx} = \frac{k}{(x+1)^2} Y_{k-1}^n(x;b) + nY_k^{n-1}(x;b)$$

(*cf.* [7], Theorem 8).

3.1 Recurrence relation

Here, by using higher order derivatives of the generating function with respect to t, we derive a partial differential equation. Using this equation, we shall give a recurrence relation for the Bernstein type basis functions. Here we use the same method as in [9].

Differentiating (2.3) with respect to t, we prove a recurrence relation for the Bernstein type basis functions. By using Leibnitz's formula for the lth derivative, with respect to t, of the product $f_{S,k}(x,t;b,u,v)$ of two functions

$$g(t,x;b,u,v;k) = \frac{2(\frac{t}{2})^b(\frac{x-v}{x-u})^k}{b!} \quad (u < v)$$

and

$$h(t,x;u,v) = e^{(\frac{x-u}{v-u})t},$$

we obtain a higher order partial differential equation as follows:

$$\frac{\partial^l f_{S,k}(x,t;b,u,v)}{\partial t^l} = \sum_{j=0}^{l} \binom{l}{j} \left(\frac{\partial^j g(t,x;b,u,v;k)}{\partial t^j}\right)\left(\frac{\partial^{l-j} h(t,x;u,v)}{\partial t^{l-j}}\right). \tag{3.1}$$

By using (3.1), we arrive at the following theorem.

Theorem 7 *Let $l \in \mathbb{N}$. Then*

$$\frac{\partial^l f_{S,k}(x,t;b,u,v)}{\partial t^l} = \frac{1}{2}\sum_{j=0}^{l} S_0^l(x;j;u,v) f_{S,k}(x,t;b-j,u,v),$$

where $f_{S,k}(x,t;b,u,v)$ and $S_j^l(x;b;u,v)$ are defined in (2.3) *and* (2.1), *respectively.*

Proof The proof of Theorem 7 follows immediately from (3.1). □

Using definition (2.2), (2.1), and Theorem 7, we obtain a recurrence relation for the Bernstein type basis functions by the following theorem.

Theorem 8 *Let u and v be real parameters with $u < v$. Let b be a positive integer and let $x \in [u,v]$. Let k, l, and n be nonnegative integers with $n \geq k+b$ and $b \geq l$. Then*

$$S_k^n(x;b;u,v) = \frac{1}{2}\sum_{j=0}^{l} S_0^l(x;j;u,v) S_k^{n-v}(x;b-j;u,v).$$

Substituting $l = 1$ into Theorem 7, we have the following PDE:

$$\frac{\partial f_{S,k}(x,t;b,u,v)}{\partial t} = \frac{x-u}{v-u} f_{S,k}(x,t;b,u,v) + \frac{1}{2} f_{S,k}(x,t;b-1,u,v).$$

By using the above PDE, we arrive at the following theorem.

Theorem 9 *Let $n \geq 1$. Then we have*

$$S_k^n(x;b;u,v) = \left(\frac{x-u}{v-u}\right) S_k^{n-1}(x;b;u,v) + \frac{1}{2} S_k^{n-1}(x;b-1;u,v).$$

Remark 5 Substituting $v = 0$ and $u = -1$ into Theorem 9, we have

$$Y_k^n(x;b) = (x+1) Y_k^{n-1}(x;b) + \frac{n}{2} Y_k^{n-1}(x;b-1).$$

This recurrence formula is different from that of Theorem 9 in [7].

4 Identities

In this section, we give a functional equation which is related to the generating function in (2.3). By using this functional equation, we derive two identities for the Bernstein type basis functions.

By using (2.3), we derive the following functional equation:

$$f_{S,k}(x,t;b,u,v) e^{-mt(\frac{x-u}{v-u})} = \frac{2(\frac{t}{2})^b (\frac{x-v}{x-u})^k e^{-t(m-1)(\frac{x-u}{v-u})}}{b!}, \tag{4.1}$$

where m is a positive integer.

Combining (2.3) with (4.1), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_k^n(x;b;u,v) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-m)^n \left(\frac{x-u}{v-u}\right)^n \frac{t^n}{n!} \\
&\quad = 2^{1-b} \left(\frac{x-v}{v-u}\right)^k \sum_{n=0}^{\infty} \binom{n}{b} (1-m)^{n-b} \left(\frac{x-u}{v-u}\right)^{n-b} \frac{t^n}{n!}.
\end{aligned}$$

By using the Cauchy product on the right-hand side of the above equation and then equating the coefficients of $\frac{t^n}{n!}$ on both sides of the final equation, we arrive at the following theorem.

Theorem 10 *Let m be a positive integer, then we have*

$$\begin{aligned}
&\sum_{j=0}^{n} \binom{n}{j} S_k^j(x;b;u,v)(-m)^{n-j} \left(\frac{x-u}{v-u}\right)^{n-j} \\
&\quad = (1-m)^{n-b} \left(\frac{x-u}{v-u}\right)^k S_k^n(x;b;u,v).
\end{aligned} \tag{4.2}$$

Substituting $m = 1$ into (4.2), we obtain the following corollary:

Corollary 1

$$\sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j} S_k^j(x;b;u,v) \left(\frac{x-u}{v-u}\right)^{b-j} = \left(\frac{x-u}{v-u}\right)^k S_k^b(x;b;u,v).$$

5 Simulation of the Bernstein type basis functions

Graphics of the Bernstein type polynomials are provided to visualize the shape of polynomials on finite domain. The effects of k, b, and n on the shape of the curve are demonstrated for the given range. These graphics may not only be used in Computer Aided Geometric Design (CAGD) but also in other areas (*cf.* [1–13]).

The figures below are obtained by varying b and k values using (2.1) for x values given between $[-2,2]$. Since $n \geq b+k$, it is written as

$$n = k + b + \text{offset}, \tag{5.1}$$

where offset is valid between 0 and $n-1$. Figures 1-4 show that as the offset increases the amplitude of the plots decreases, while the center of gravity shifts to the left for fixed $b = 1$. In addition an increase in k results in a narrower curve.

Figures 5-8 look similar to the probability distribution functions for $k = 2$. As the offset increases the plots shift from left to right. Furthermore, as b increases the curves distribute more evenly within the plot. Note that for $k = 2$ and offset = 2, the plot looks like a normal distribution plot.

We also plot the surface obtained using (2.1) for fixed $k = 2$ and $b = 1$, respectively. Figure 9 and Figure 10 surfaces obtained by varying b values for k. We note that with its wing like shape the plot shown in Figure 10 may potentially be used in designing airplane wings.

Figure 1 Varying k values for $b = 1$, $n = b + k + 0$.

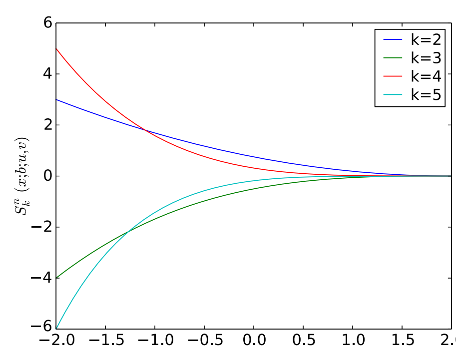


Figure 2 Varying k values for $b = 1$, $n = b + k + 1$.

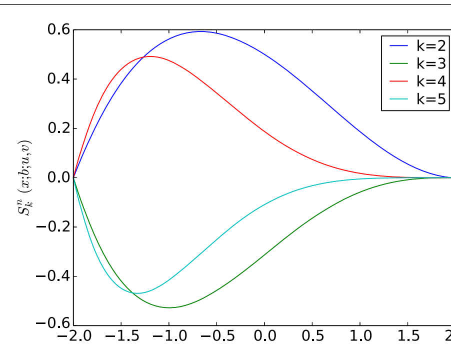


Figure 3 Varying k values for $b = 1$, $n = b + k + 2$.

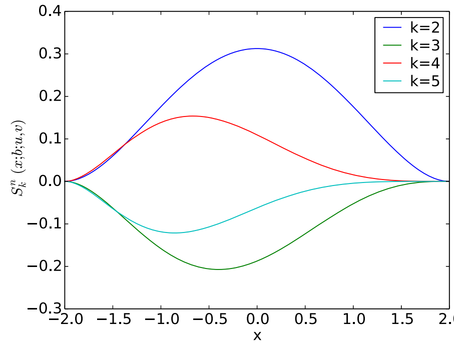


Figure 4 Varying k values for $b = 1$, $n = b + k + 3$.

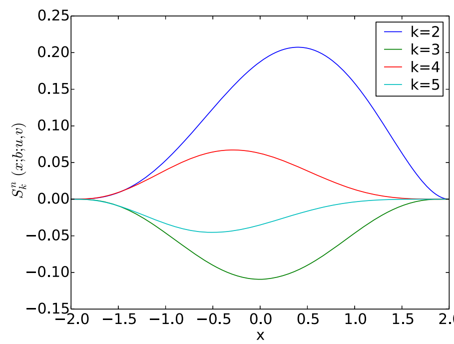


Figure 5 Varying b values for $k = 2$ and $n = b + k + 1$.

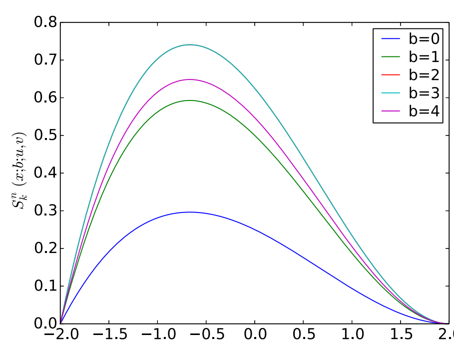
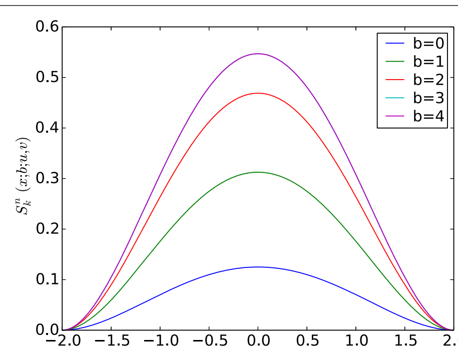


Figure 6 Varying b values for $k = 2$ and $n = b + k + 2$.



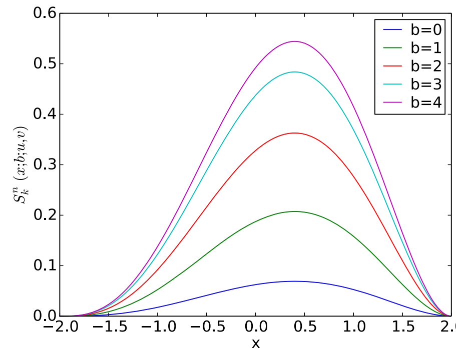


Figure 7 Varying *b* values for $k = 2$ and $n = b + k + 3$.

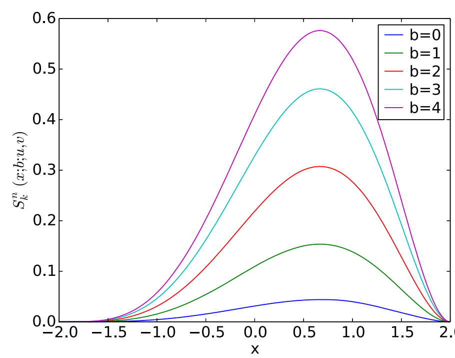


Figure 8 Varying *b* values for $k = 2$ and $n = b + k + 4$.

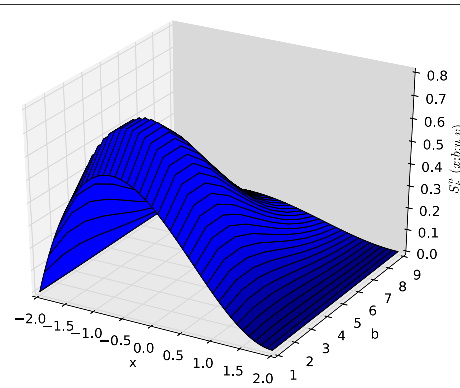


Figure 9 Surface obtained by varying *b* values for $k = 2$.

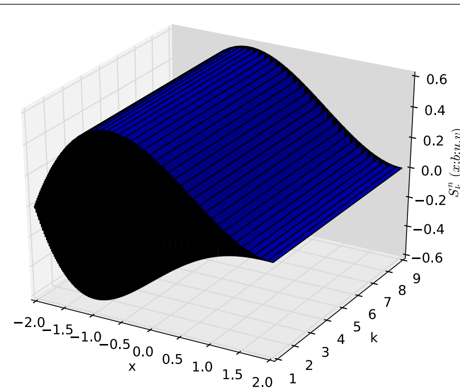


Figure 10 Wing type surface obtained by varying *k* values for $b = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors made equal contributions. Both authors read and approved the final manuscript.

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References

1. Farouki, RT, Goodman, TNT: On the optimal stability of the Bernstein basis. Math. Comput. **65**, 1553-1566 (1996)
2. Goldman, R: Identities for the univariate and bivariate Bernstein basis functions. In: Paeth, A (ed.) Graphics Gems, vol. V, pp. 149-169. Academic Press, San Diego (1995)
3. Goldman, R: An Integrated Introduction to Computer Graphics and Geometric Modeling. CRC Press, New York (2009)
4. Simsek, Y: Generating functions for the Bernstein type polynomials: a new approach to deriving identities and applications for these polynomials. Hacet. J. Math. Stat. **43**(1), 1-14 (2014)
5. Simsek, Y, Acikgoz, M: A new generating function of (q-) Bernstein-type polynomials and their interpolation function. Abstr. Appl. Anal. **2010**, Article ID 769095 (2010)
6. Simsek, Y: q-Beta polynomials and their applications. Appl. Math. Inf. Sci. **7**(6), 2539-2547 (2013)
7. Simsek, Y: A new class of polynomials associated with Bernstein and beta polynomials. Math. Methods Appl. Sci. **37**(5), 676-685 (2014)
8. Simsek, Y: Beta-type polynomials and their generating functions. Appl. Math. Comput. **254**, 172-182 (2015)
9. Simsek, Y: Functional equations from generating functions: a novel approach to deriving identities for the Bernstein basis functions. Fixed Point Theory Appl. **2013**, 80 (2013)
10. Goldman, R: Pyramid Algorithms: A Dynamic Programming Approach to Curves and Surfaces for Geometric Modeling. Morgan Kaufmann, San Diego (2002)
11. Lorentz, GG: Bernstein Polynomials. Chelsea, New York (1986)
12. Simsek, Y: Interpolation function of generalized q-Bernstein type polynomials and their application. In: Curves and Surfaces. Lecture Notes in Computer Science, vol. 6920, pp. 647-662. Springer, Berlin (2011)
13. Bernstein, SN: Démonstration du théorème de Weierstrass fondée sur la calcul des probabilités. Commun. Soc. Math. Kharkow (2) **13**, 1-2 (1912-1913)