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Dynamics of a modified Nicholson-Bailey host-parasitoid model

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Abstract

In this paper, we study the qualitative behavior of the following modified Nicholson-Bailey host-parasitoid model:

$$x_{n+1} = \frac{bx_n e^{-ay_n}}{1 + dx_n}, \qquad y_{n+1} = cx_n\left(1 - e^{-ay_n}\right),$$

where a, b, c, d and the initial conditions x_0, y_0 are positive real numbers. More precisely, we investigate the boundedness character, existence and uniqueness of a positive equilibrium point, local asymptotic stability and global stability of the unique positive equilibrium point, and the rate of convergence of positive solutions of the system. Some numerical examples are also given to verify our theoretical results.

MSC: 39A10; 40A05

Keywords: modified Nicholson-Bailey model; boundedness; local stability; global character; rate of convergence

1 Introduction

Many ecological models are governed by differential as well as difference equations. In particular, ecological models with non-overlapping populations are better formulated as discrete dynamical systems compared to the continuous time models. These models have been extensively studied in recent years because of their wide applicability to the study of population dynamics [1, 2]. In fact, in the case of discrete dynamical systems, one has more efficient computational results for numerical simulations and also has rich dynamics as compared to the continuous ones. In recent years, several papers have been published on the mathematical models of biology that discuss the system of difference equations generated from the associated system of differential equations as well as the associated numerical methods [3, 4]. In mathematical biology, the model such as the host-parasitoid has attracted many researchers during the last few decades. Usually, the biologists believe that a unique, positive, locally asymptotically stable equilibrium point in an ecological system is very important [5]. Therefore, it is pertinent to find conditions which may guarantee the global stability of a positive equilibrium point, if it exist, for the given system. See [6] for introduction to mathematical models in biological sciences.

The prime example of an ecologically interesting discrete-time model for interacting populations is the Nicholson-Bailey model for host-parasitoid dynamics. Parasitoids are insect species whose larvae develop by feeding on the bodies of other arthropods, usually killing them. Larvae emerge from the host and develop into free-living adults. The adults then lay their eggs in a subsequent generation of hosts. Most parasitoid larvae require a specific life-stage of the host, so parasitoid and host generations are linked to one another. Consequently host-parasitoid models often use a discrete time step corresponding to the common generation length of host and parasitoid. The classic model was derived by Nicholson and Bailey (1935). The assumptions of the Nicholson-Bailey model are as follows:

(i) Hosts are distributed at random, at density x_n per unit area in generation n.
(ii) Parasitoids search at random and independently, each having an 'area of discovery' a, and lay an egg in each host found.
(iii) Each parasitized host gives rise to one new parasitoid in generation $n+1$.
(iv) Each unparasitized host gives rise to $b>1$ new hosts in generation $n+1$.

Each parasitoid attacks the hosts found in a units of area, so the expected number of hosts attacked by each parasitoid is ax. The expected total number of attacks is axy. The total number of attacks can also be written as the sum over hosts of the number of attacks on each host. All hosts have the same expected number of attacks, so the expected number of attacks on any given host must be ay. Under the assumptions listed above, the number of eggs per host has a Poisson distribution. Consequently, the expected fraction of hosts that are not parasitized is the probability that a Poisson random variable with mean ay takes the value zero. The resulting population dynamics are

$$x_{n+1}=bx_ne^{-ay_n},\qquad y_{n+1}=cx_n\left(1-e^{-ay_n}\right).$$

Here, x_n and y_n represent the densities of the host and parasitoid population at year n. b is the number of offspring of an unparasitized host surviving to the next year. Assuming random encounter between hosts and parasitoids, the probability that a host escapes parasitism can be approximated by e^{-ay_n}, where a is a proportionality constant. Similarly, the probability to become infected is then given by $1-e^{-ay_n}$. The parameter c describes the number of parasitoids that hatch from an infected host.

Now, assume that the host has bounded dynamics in absence of parasitoid, *i.e.*, has self-regulation (density dependence). For example, assume host dynamics are inherently logistic (*e.g.*, the Beverton-Holt model).

Then a modified form of the Nicholson-Bailey host-parasitoid model is

$$x_{n+1}=\frac{bx_ne^{-ay_n}}{1+dx_n},\qquad y_{n+1}=cx_n\left(1-e^{-ay_n}\right),\tag{1}$$

where a, b, c, d and the initial conditions x_0, y_0 are positive real numbers.

In this paper our aim is to study the dynamics of system (1). More precisely, we investigate the boundedness character, existence and uniqueness of a positive equilibrium point, local asymptotic stability and global stability of the unique positive equilibrium point, and the rate of convergence of positive solutions of system (1). To investigate the dynamics we shall use standard results from theory of rational difference equations. However, we shall state the results that we employ and refer the interested readers for a systematic study of rational difference equations to [7–21] and the references therein. In Refs. [22–26] qualitative behavior of some biological models is discussed. The rest of the paper is organized as follows. In Section 2 the required known results about linearized stability are given.

Section 3 discusses the boundedness character of the model. Section 4 is about the existence and uniqueness of the positive equilibrium point. It also contains the local stability of the equilibrium point. Section 5 discusses the global behavior of the equilibrium point. Whereas Section 6 is about the rate of convergence and Section 7 gives the numerical examples of the proved results. In the last section a brief conclusion is given.

2 Linearized stability

Let us consider the two-dimensional discrete dynamical system of the form

$$\begin{aligned} x_{n+1} &= f(x_n, y_n), \\ y_{n+1} &= g(x_n, y_n), \quad n = 0, 1, \ldots, \end{aligned} \tag{2}$$

where $f : I \times J \to I$ and $g : I \times J \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (2) is uniquely determined by initial conditions $(x_0, y_0) \in I \times J$. An equilibrium point of (2) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{y}), \\ \bar{y} &= g(\bar{x}, \bar{y}). \end{aligned}$$

Definition 2.1 Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (2).

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every initial condition (x_0, y_0), $\|(x_0, y_0) - (\bar{x}, \bar{y})\| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\| \cdot \|$ is the usual Euclidian norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\|(x_0, y_0) - (\bar{x}, \bar{y})\| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called a global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(v) An equilibrium point $(\bar{x}, \bar{y})$ is called an asymptotic global attractor if it is a global attractor and stable.

Definition 2.2 Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map $F(x, y) = (f(x, y), g(x, y))$, where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$ is given by

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \binom{x_n}{y_n}$ and F_J is a Jacobian matrix of system (2) about the equilibrium point $(\bar{x}, \bar{y})$.

Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1), then

$$\bar{x} = \frac{b\bar{x}e^{-a\bar{y}}}{1 + d\bar{x}}, \qquad \bar{y} = c\bar{x}\left(1 - e^{-a\bar{y}}\right). \tag{3}$$

The Jacobian matrix of the linearized system of (1) about the fixed point $(\bar{x}, \bar{y})$ is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} \frac{be^{-a\bar{y}}}{(1+d\bar{x})^2} & -\frac{ab\bar{x}e^{-a\bar{y}}}{1+d\bar{x}} \\ c(1 - e^{-a\bar{y}}) & ac\bar{x}e^{-a\bar{y}} \end{pmatrix}.$$

Lemma 2.3 [7] *Consider the second-degree polynomial equation*

$$\lambda^2 - p\lambda - q = 0, \tag{4}$$

where p and q are real numbers.

(i) *If both roots of Equation* (4) *lie in the open unit disk* $|\lambda| < 1$, *then the equilibrium point* $(\bar{x}, \bar{y})$ *is locally asymptotically stable.*

(ii) *If at least one of the roots of Equation* (4) *has absolute value greater than one, then the equilibrium point* $(\bar{x}, \bar{y})$ *is unstable.*

(iii) *A necessary and sufficient condition for both roots of Equation* (4) *to lie inside the open disk* $|\lambda| < 1$ *is*

$$|p| < 1 - q < 2.$$

In this case the locally asymptotically stable equilibrium $(\bar{x}, \bar{y})$ *is also called a sink.*

(iv) *A necessary and sufficient condition for both roots of Equation* (4) *to have absolute value greater than one is*

$$|q| > 1, \qquad |p| < |1 - q|.$$

In this case $(\bar{x}, \bar{y})$ *is a repeller.*

(v) *A necessary and sufficient condition for one root of Equation* (4) *to have absolute value greater than one and for the other to have absolute value less than one is*

$$p^2 + 4q > 0, \qquad |p| > |1 - q|.$$

In this case the unstable equilibrium $(\bar{x}, \bar{y})$ *is called a saddle point.*

(vi) *A necessary and sufficient condition for a root of Equation* (4) *to have absolute value equal to one is*

$$|p| = |1 - q|.$$

In this case the equilibrium $(\bar{x}, \bar{y})$ *is called a non-hyperbolic point.*

3 Boundedness

The following theorem shows that every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1) is bounded.

Theorem 3.1 *Every positive solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *of system* (1) *is bounded.*

Proof Let $\{(x_n, x_n)\}_{n=0}^{\infty}$ be any positive solution of system (1), then

$$x_{n+1} = \frac{bx_n e^{-ay_n}}{1 + dx_n} \leq \frac{bx_n}{dx_n} \leq \frac{b}{d}, \quad n = 0, 1, \ldots. \tag{5}$$

Also

$$y_{n+1} = cx_n\left(1 - e^{-ay_n}\right) \leq cx_n \leq \frac{bc}{d}, \quad n = 0, 1, \ldots. \tag{6}$$

Hence from (5) and (6), we have

$$0 \le x_n \le \frac{b}{d}, \qquad 0 \le y_n \le \frac{bc}{d}, \quad n = 1, 2, \ldots. \tag{7}$$

□

Theorem 3.2 *Let* $\{(x_n, y_n)\}$ *be a positive solution of system* (1). *Then* $[0, \frac{b}{d}] \times [0, \frac{bc}{d}]$ *is an invariant set for system* (1).

Proof It follows from induction. □

4 Existence and uniqueness of a positive equilibrium point and local stability

The following theorem shows the existence and uniqueness of a positive equilibrium point of system (1).

Theorem 4.1 *If* $b > 1$ *and* $d < \frac{ac}{b\ln(\frac{1+b}{b})}$, *then system* (1) *has a unique positive equilibrium point* $(\bar{x}, \bar{y})$ *in* $[0, \frac{b}{d}] \times [0, \frac{bc}{d}]$.

Proof Consider the following system:

$$x = \frac{bxe^{-ay}}{1+dx}, \qquad y = cx\left(1 - e^{-ay}\right). \tag{8}$$

Assume that $(\bar{x}, \bar{y})$ in $[0, \frac{b}{d}] \times [0, \frac{bc}{d}]$, then it follows from (8) that

$$y = \frac{1}{a}\ln\left(\frac{b}{1+dx}\right), \qquad x = \frac{y}{c(1-e^{-ay})}.$$

Define $F(x) = \frac{h(x)}{c(1-e^{-ah(x)})} - x$, where $h(x) = \frac{1}{a}\ln(\frac{b}{1+dx})$ and $x \in [0, \frac{b}{d}]$. It is easy to see that $F(0) = \frac{b\ln b}{ac(b-1)} > 0$ if $b > 1$. Also, $F(\frac{b}{d}) = \frac{b\ln(\frac{1+b}{b})}{ac} - \frac{b}{d} < 0$ if $d < \frac{ac}{b\ln(\frac{1+b}{b})}$. Hence, $F(x)$ has at least one positive solution in the interval $[0, \frac{b}{d}]$. Furthermore, it is easy to show that $F'(x) = \frac{(1-e^{-ah(x)}-ah(x)e^{-ah(x)})h'(x)-c(1-e^{-ah(x)})^2}{c(1-e^{-ah(x)})^2} < 0$, where $h'(x) = -\frac{d}{a(1+dx)}$ for all $x \in [0, \frac{b}{d}]$. Hence, $F(x) = 0$ has a unique positive solution $x \in [0, \frac{b}{d}]$. □

Theorem 4.2 *For the unique positive equilibrium point* $(\bar{x}, \bar{y})$ *in* $[0, \frac{b}{d}] \times [0, \frac{bc}{d}]$ *of system* (1), *the following statements hold true*:

(i) *The unique positive equilibrium point of system* (1) *is locally asymptotically stable if and only if*

$$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2} < 1 - \frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2} < 2.$$

(ii) *The unique positive equilibrium point is a repeller if and only if*

$$\left|\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}\right| > 1$$

and

$$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2} < \left|1-\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}\right|.$$

(iii) *The unique positive equilibrium point is a saddle point if and only if*

$$\left(\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2}\right)^2 + 4\left(\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}\right) > 0$$

and

$$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2} > \left|1-\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}\right|.$$

(iv) *The unique positive equilibrium point is non-hyperbolic if and only if*

$$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2} = \left|1-\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}\right|.$$

Proof (i) The characteristic polynomial of the Jacobian matrix $F_J(\bar{x},\bar{y})$ about the equilibrium point $(\bar{x},\bar{y})$ is given by

$$P(\lambda) = \lambda^2 - \frac{e^{-a\bar{y}}(ac\bar{x}(1+d\bar{x})^2+b)}{(1+d\bar{x})^2}\lambda + \frac{abc\bar{x}e^{-2a\bar{y}}(e^{a\bar{y}}(1+d\bar{x})-d\bar{x})}{(1+d\bar{x})^2}. \tag{9}$$

As pointed out in [27], it is convenient to discuss stability behavior in terms of the quantity r. So, for the equilibrium point $(\bar{x},\bar{y})$ of system (1), we have from system (3)

$$e^{-a\bar{y}} = \frac{1}{b} + dr,$$

where $r = \frac{\bar{x}}{b}$ is the ratio of steady-state $\bar{x}$ with b. Moreover,

$$\bar{y} = cr(b-1-bdr).$$

So, in terms of r, Equation (9) takes the form

$$P(\lambda) = \lambda^2 - \frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2}\lambda - \frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}.$$

Let

$$p = \frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2},$$
$$q = \frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}.$$

Then it follows from Lemma 2.3 that the unique positive equilibrium point $(\bar{x},\bar{y})$ of system (1) is locally asymptotically stable if and only if

$$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2} < 1 - \frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2} < 2.$$

Obviously, one can prove (ii), (iii) and (iv). □

5 Global character

Lemma 5.1 [7] *Let $I = [a,b]$ and $J = [c,d]$ be real intervals, and let $f : I \times J \to I$ and $g : I \times J \to J$ be continuous functions. Consider system* (2) *with initial conditions $(x_0, y_0) \in I \times J$. Suppose that the following statements are true*:

(i) *$f(x,y)$ is non-decreasing in x and non-increasing in y.*
(ii) *$g(x,y)$ is non-decreasing in both arguments.*
(iii) *If $(m_1, M_1, m_2, M_2) \in I^2 \times J^2$ is a solution of the system*

$$m_1 = f(m_1, M_2), \qquad M_1 = f(M_1, m_2),$$
$$m_2 = g(m_1, m_2), \qquad M_2 = g(M_1, M_2)$$

such that $m_1 = M_1$ and $m_2 = M_2$, then there exists exactly one equilibrium point $(\bar{x},\bar{y})$ of system (2) *such that $\lim_{n\to\infty}(x_n, y_n) = (\bar{x},\bar{y})$.*

Theorem 5.2 *Assume that $ac + d > abc$, then the unique positive equilibrium point $(\bar{x},\bar{y})$ in $[0, \frac{b}{d}] \times [0, \frac{bc}{d}]$ of system* (1) *is a global attractor.*

Proof Let $f(x,y) = \frac{bxe^{-ay}}{1+dx}$ and $g(x,y) = cx(1-e^{-ay})$. Then it is easy to see that $f(x,y)$ is non-decreasing in x and non-increasing in y. Moreover, $g(x,y)$ is non-decreasing in both arguments x and y. Let (m_1, M_1, m_2, M_2) be a positive solution of the system

$$m_1 = f(m_1, M_2), \qquad M_1 = f(M_1, m_2),$$
$$m_2 = g(m_1, m_2), \qquad M_2 = g(M_1, M_2).$$

Then one has

$$m_1 = \frac{bm_1e^{-aM_2}}{1+dm_1}, \qquad M_1 = \frac{bM_1e^{-am_2}}{1+dM_1} \tag{10}$$

and

$$m_2 = cm_1\bigl(1-e^{-am_2}\bigr), \qquad M_2 = cM_1\bigl(1-e^{-aM_2}\bigr). \tag{11}$$

From (10) one has

$$m_1 = \frac{be^{-aM_2} - 1}{d}, \qquad M_1 = \frac{be^{-am_2} - 1}{d}. \tag{12}$$

From (11) one has

$$m_2 = \frac{c(be^{-aM_2} - 1)(1 - e^{-am_2})}{d}, \qquad M_2 = \frac{c(be^{-am_2} - 1)(1 - e^{-aM_2})}{d}. \tag{13}$$

From (12) we have

$$m_1 - M_1 = \frac{b}{d} e^{-am_2 - aM_2}\left(e^{am_2} - e^{aM_2}\right). \tag{14}$$

From (13) we have

$$m_2 - M_2 = \frac{c(b-1)}{d} e^{-am_2 - aM_2}\left(e^{am_2} - e^{aM_2}\right). \tag{15}$$

Moreover, one has

$$e^{am_2} - e^{aM_2} = ae^{\gamma}(m_2 - M_2), \tag{16}$$

where

$$\min\{m_2, M_2\} \le \gamma \le \max\{m_2, M_2\}.$$

From (14) and (16), we get

$$m_1 - M_1 = \frac{ab}{d} e^{-am_2 - aM_2 + \gamma}(m_2 - M_2). \tag{17}$$

From (17) it follows that

$$|m_1 - M_1| \le \frac{ab}{d}|m_2 - M_2|. \tag{18}$$

From (15) and (16), we get

$$m_2 - M_2 = \frac{ac(b-1)}{d} e^{-am_2 - aM_2 + \gamma}(m_2 - M_2). \tag{19}$$

From (19) it follows that

$$|m_2 - M_2| \le \frac{ac(b-1)}{d}|m_2 - M_2|. \tag{20}$$

From (20) we get

$$\left(\frac{ac + d - abc}{d}\right)|m_2 - M_2| \le 0,$$

from which it follows that $m_2 = M_2$ and similarly it is easy to show that $m_1 = M_1$. Hence, from Lemma 5.1 the unique positive equilibrium point of system (1) is a global attractor. □

Lemma 5.3 *The unique positive equilibrium point* $(\bar{x},\bar{y})$ *in* $[0,\frac{b}{d}]\times[0,\frac{bc}{d}]$ *of system* (1) *is globally asymptotically stable if and only if*

$$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2}<1-\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}<2.$$

Proof The proof is a direct consequence of Theorems 4.2 and 5.2. □

6 The rate of convergence

In this section we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of system (1).

The following result gives the rate of convergence of solutions of the system of difference equations

$$X_{n+1}=\bigl(A+B(n)\bigr)X_n, \tag{21}$$

where X_n is an m-dimensional vector, $A\in C^{m\times m}$ is a constant matrix, and $B:\mathbb{Z}^+\to C^{m\times m}$ is a matrix function satisfying

$$\bigl\|B(n)\bigr\|\to 0 \tag{22}$$

as $n\to\infty$, where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\bigl\|(x,y)\bigr\|=\sqrt{x^2+y^2}.$$

Proposition 6.1 (Perron's theorem [28]) *Suppose that condition* (22) *holds. If* X_n *is a solution of* (21), *then either* $X_n=0$ *for all large n or*

$$\rho=\lim_{n\to\infty}\bigl(\|X_n\|\bigr)^{1/n}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Proposition 6.2 [28] *Suppose that condition* (22) *holds. If* X_n *is a solution of* (21), *then either* $X_n=0$ *for all large n or*

$$\rho=\lim_{n\to\infty}\frac{\|X_{n+1}\|}{\|X_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Let $\{(x_n,y_n)\}$ be any solution of system (1) such that $\lim_{n\to\infty}x_n=\bar{x}$ and $\lim_{n\to\infty}y_n=\bar{y}$. To find the error terms, one has from system (1)

$$\begin{aligned}x_{n+1}-\bar{x}&=\frac{bx_ne^{-ay_n}}{1+dx_n}-\frac{b\bar{x}e^{-a\bar{y}}}{1+d\bar{x}}\\&=\frac{be^{-ay_n}}{(1+dx_n)(1+d\bar{x})}(x_n-\bar{x})+\frac{b\bar{x}(e^{-ay_n}-e^{-a\bar{y}})}{(1+d\bar{x})(y_n-\bar{y})}(y_n-\bar{y})\end{aligned}$$

and

$$y_{n+1} - \bar{y} = cx_n\left(1 - e^{-ay_n}\right) - c\bar{x}\left(1 - e^{-a\bar{y}}\right)$$
$$= c\left(1 - e^{-ay_n}\right)(x_n - \bar{x}) - \frac{c\bar{x}(e^{-ay_n} - e^{-a\bar{y}})}{y_n - \bar{y}}(y_n - \bar{y}).$$

Let $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, then one has

$$e_{n+1}^1 = a_n e_n^1 + b_n e_n^2$$

and

$$e_{n+1}^2 = c_n e_n^1 + d_n e_n^2,$$

where

$$a_n = \frac{be^{-ay_n}}{(1 + dx_n)(1 + d\bar{x})}, \qquad b_n = \frac{b\bar{x}(e^{-ay_n} - e^{-a\bar{y}})}{(1 + d\bar{x})(y_n - \bar{y})},$$
$$c_n = c\left(1 - e^{-ay_n}\right), \qquad d_n = -\frac{c\bar{x}(e^{-ay_n} - e^{-a\bar{y}})}{y_n - \bar{y}}.$$

Moreover,

$$\lim_{n\to\infty} a_n = \frac{be^{-a\bar{y}}}{(1 + d\bar{x})^2}, \qquad \lim_{n\to\infty} b_n = -\frac{ab\bar{x}e^{-a\bar{y}}}{1 + d\bar{x}},$$
$$\lim_{n\to\infty} c_n = c\left(1 - e^{-a\bar{y}}\right), \qquad \lim_{n\to\infty} d_n = ac\bar{x}e^{-a\bar{y}}.$$

Now the limiting system of error terms can be written as

$$\begin{pmatrix} e_{n+1}^1 \\ e_{n+1}^2 \end{pmatrix} = \begin{pmatrix} \frac{be^{-a\bar{y}}}{(1+d\bar{x})^2} & -\frac{ab\bar{x}e^{-a\bar{y}}}{1+d\bar{x}} \\ c(1 - e^{-a\bar{y}}) & ac\bar{x}e^{-a\bar{y}} \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix},$$

which is similar to the linearized system of (1) about the equilibrium point $(\bar{x}, \bar{y})$.

Using proposition (6.1), one has following result.

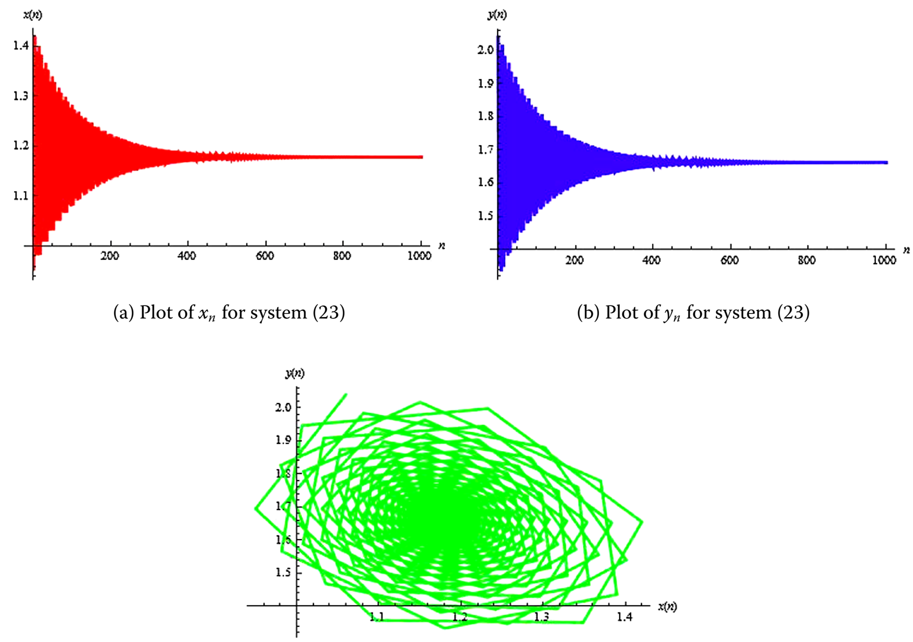
Theorem 6.3 *Assume that* $\{(x_n, y_n)\}$ *is a positive solution of system* (1) *such that* $\lim_{n\to\infty} x_n = \bar{x}$, *and* $\lim_{n\to\infty} y_n = \bar{y}$, *where* $\bar{x}$ *in* $[0, \frac{b}{d}]$ *and* $\bar{y}$ *in* $[0, \frac{bc}{d}]$. *Then the error vector* $e_n = \binom{e_n^1}{e_n^2}$ *of every solution of* (1) *satisfies both of the following asymptotic relations*:

$$\lim_{n\to\infty} \left(\|e_n\|\right)^{\frac{1}{n}} = \left|\lambda_{1,2} F_J(\bar{x}, \bar{y})\right|, \qquad \lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = \left|\lambda_{1,2} F_J(\bar{x}, \bar{y})\right|,$$

where $\lambda_{1,2} F_J(\bar{x}, \bar{y})$ *are the characteristic roots of the Jacobian matrix* $F_J(\bar{x}, \bar{y})$.

7 Examples

In order to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the system of nonlinear difference equations (1). All plots in this section are drawn with Mathematica.



(a) Plot of x_n for system (23)

(b) Plot of y_n for system (23)

(c) An attractor of system (23)

Figure 1 Plots for system (23).

Example 1 Let $a = 0.5$, $b = 9.6$, $c = 2.5$, $d = 2.7$. Then system (1) can be written as

$$x_{n+1} = \frac{9.6x_n e^{-0.5y_n}}{1 + 2.7x_n}, \qquad y_{n+1} = 2.5x_n\left(1 - e^{-0.5y_n}\right), \quad n = 0, 1, \ldots, \tag{23}$$

with initial conditions $x_0 = 1.06$, $y_0 = 2.04$.

In this case the unique positive equilibrium point of system (23) is given by $(\bar{x}, \bar{y}) = (1.17805, 1.66255)$. Moreover, in Figure 1 the plot of x_n is shown in Figure 1(a), the plot of y_n is shown in Figure 1(b) and an attractor of system (23) is shown in Figure 1(c).

Example 2 Let $a = 0.4$, $b = 8.7$, $c = 5.5$, $d = 4.7$. Then system (1) can be written as

$$x_{n+1} = \frac{8.7x_n e^{-0.4y_n}}{1 + 4.7x_n}, \qquad y_{n+1} = 5.5x_n\left(1 - e^{-0.4y_n}\right), \quad n = 0, 1, \ldots, \tag{24}$$

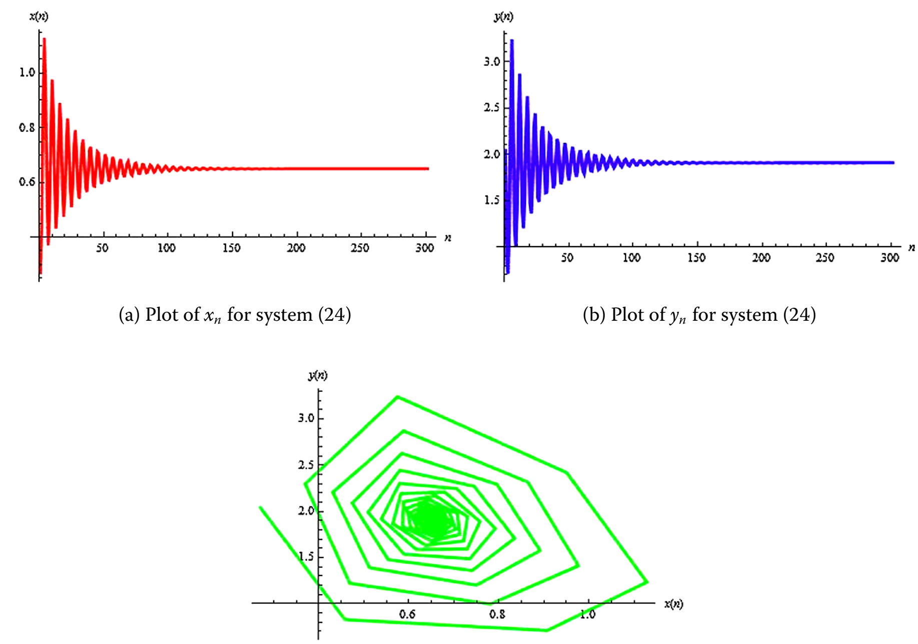
with initial conditions $x_0 = 0.27$, $y_0 = 2.04$.

In this case the unique positive equilibrium point of system (24) is given by $(\bar{x}, \bar{y}) = (0.649926, 1.90865)$. Moreover, in Figure 2 the plot of x_n is shown in Figure 2(a), the plot of y_n is shown in Figure 2(b) and an attractor of system (24) is shown in Figure 2(c).

Example 3 Let $a = 0.4$, $b = 8.8$, $c = 4.5$, $d = 4.8$. Then system (1) can be written as

$$x_{n+1} = \frac{8.8x_n e^{-0.4y_n}}{1 + 4.8x_n}, \qquad y_{n+1} = 4.5x_n\left(1 - e^{-0.4y_n}\right), \quad n = 0, 1, \ldots, \tag{25}$$

with initial conditions $x_0 = 0.28$, $y_0 = 2.04$.



(a) Plot of x_n for system (24) (b) Plot of y_n for system (24)

(c) An attractor of system (24)

Figure 2 Plots for system (24).

In this case the unique positive equilibrium point of system (25) is given by $(\bar{x}, \bar{y}) = (0.753772, 1.61192)$. Moreover, in Figure 3 the plot of x_n is shown in Figure 3(a), the plot of y_n is shown in Figure 3(b) and an attractor of system (25) is shown in Figure 3(c).

Example 4 Let $a = 0.4$, $b = 8.2$, $c = 2.4$, $d = 1.7$. Then system (1) can be written as

$$x_{n+1} = \frac{8.2x_n e^{-0.4y_n}}{1 + 1.7x_n}, \qquad y_{n+1} = 2.4x_n\left(1 - e^{-0.4y_n}\right), \quad n = 0, 1, \ldots, \tag{26}$$

with initial conditions $x_0 = 1.5$, $y_0 = 2.09$.

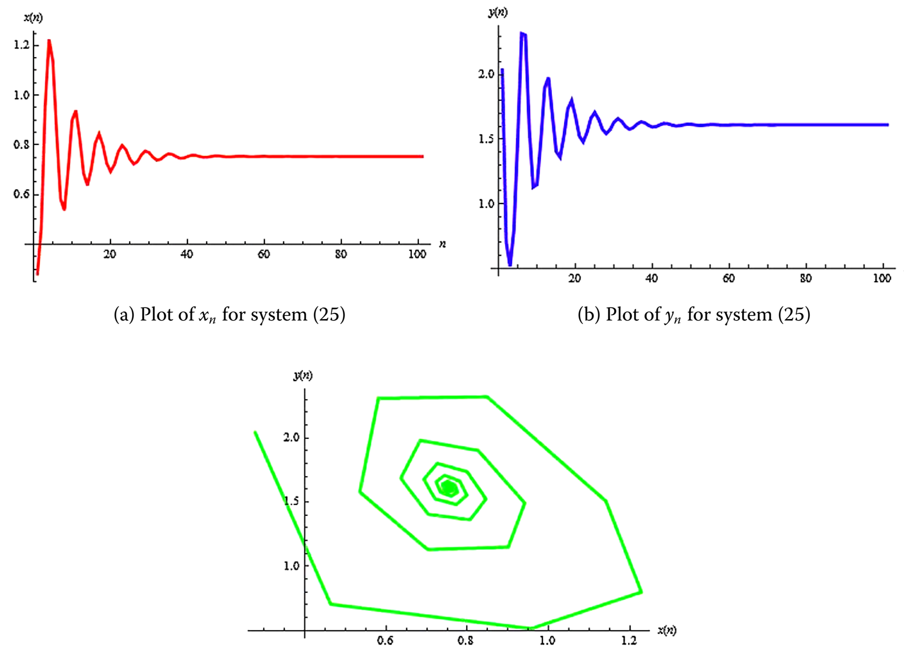
In this case the unique positive equilibrium point of system (26) is unstable. Moreover, in Figure 4 the plot of x_n is shown in Figure 4(a), the plot of y_n is shown in Figure 4(b) and a phase portrait of system (26) is shown in Figure 4(c).

Example 5 Let $a = 0.4$, $b = 8.6$, $c = 3.9$, $d = 2.9$. Then system (1) can be written as

$$x_{n+1} = \frac{8.6x_n e^{-0.4y_n}}{1 + 2.9x_n}, \qquad y_{n+1} = 3.9x_n\left(1 - e^{-0.4y_n}\right), \quad n = 0, 1, \ldots, \tag{27}$$

with initial conditions $x_0 = 0.4$, $y_0 = 2.09$.

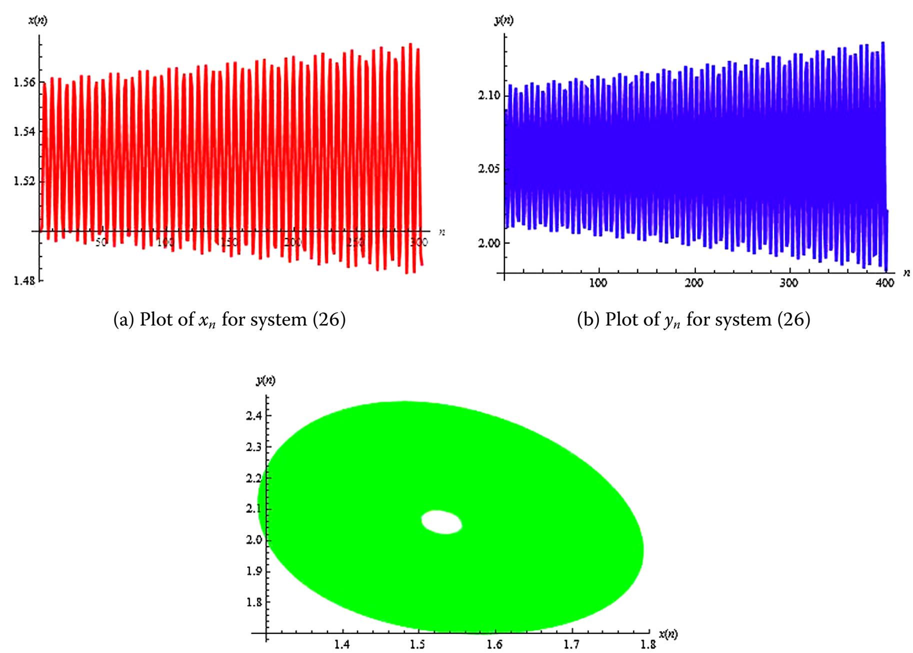
In this case the unique positive equilibrium point of system (27) is unstable. Moreover, in Figure 5 the plot of x_n is shown in Figure 5(a), the plot of y_n is shown in Figure 5(b) and a phase portrait of system (27) is shown in Figure 5(c).



(a) Plot of x_n for system (25) (b) Plot of y_n for system (25)

(c) An attractor of system (25)

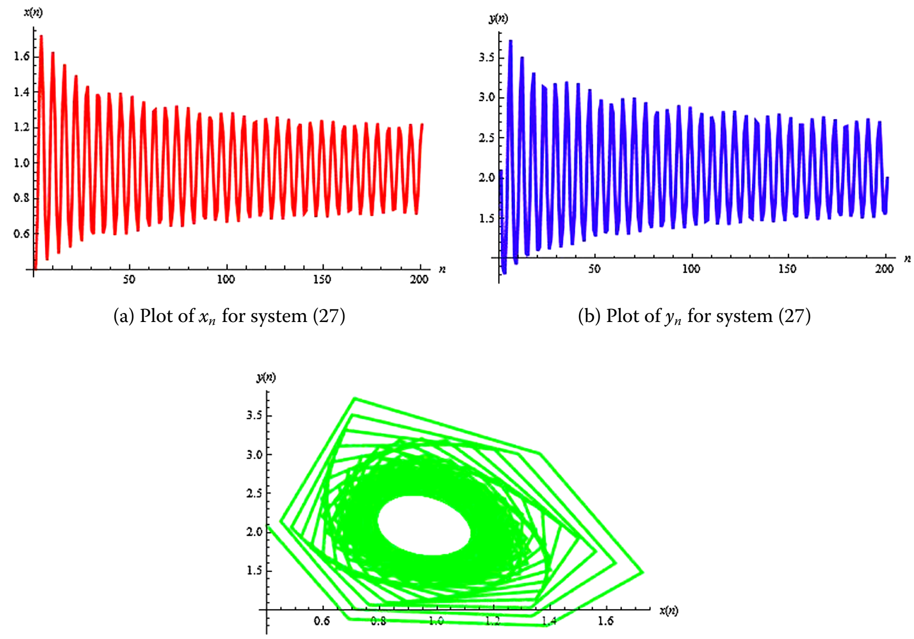
Figure 3 Plots for system (25).



(a) Plot of x_n for system (26) (b) Plot of y_n for system (26)

(c) Phase portrait of system (26)

Figure 4 Plots for system (26).



(a) Plot of x_n for system (27)

(b) Plot of y_n for system (27)

(c) Phase portrait of system (27)

Figure 5 Plots for system (27).

8 Conclusion

This work is related to the qualitative behavior of the modified Nicholson-Bailey host-parasitoid model. We have investigated the existence and uniqueness of positive steady-state of system (1). Under certain parametric conditions, the boundedness of positive solutions is proved. Moreover, we have shown that the unique positive equilibrium $(\bar{x}, \bar{y})$ in the $[0, \frac{b}{d}] \times [0, \frac{bc}{d}]$ point of system (1) is locally asymptotically stable if and only if $\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2} < 1 - \frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2} < 2$ hold true. The main objective of dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining possible global behaviors of the system and determining which initial conditions lead to these long-term behaviors. Furthermore, the rate of convergence of positive solutions of (1) which converge to its unique positive equilibrium point is demonstrated. Finally, some numerical examples are provided to support our theoretical results. These examples are experimental verification of our theoretical discussions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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