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On the Lipschitz stability of inverse nodal problem for p-Laplacian Schrödinger equation with energy dependent potential

Emrah Yilmaz[1*], Sertac Goktas[2] and Hikmet Koyunbakan[1]

*Correspondence: emrah231983@gmail.com
[1]Department of Mathematics, Firat University, Elazıg, 23119, Turkey
Full list of author information is available at the end of the article

Abstract

In this study, we consider reconstruction and stability issues of an inverse nodal problem for a p-Laplacian Schrödinger equation with energy dependent potential. We solve Lipschitz stability of the inverse nodal problem for this p-Laplacian operator. Furthermore, we show that the space of all potential functions q is homeomorphic to the partition set of all asymptotically equivalent nodal sequences induced by an equivalence relation.

MSC: 34A55; 34L05; 34L20

Keywords: p-Laplacian operator; inverse nodal problem; Lipschitz stability

1 Introduction

Let us consider the following p-Laplacian eigenvalue problem:

$$-\left(u'^{(p-1)}\right)' = (p-1)\left(\lambda^2 - q(x) - 2\lambda r(x)\right)u^{(p-1)}, \quad 0 < x < 1, \tag{1.1}$$

with the Dirichlet conditions

$$u(0) = u(1) = 0, \tag{1.2}$$

where $p > 1$ is a constant, λ is a spectral parameter; $q \in L_2(0,1)$, $r \in W_2^1(0,1)$ are real-valued functions and $u^{(p-1)} = |u|^{(p-1)} \operatorname{sgn} u$ (see [1]).

Uniqueness and reconstruction problems of the p-Laplacian Schrödinger equation with energy dependent potential have been studied in some works (for example, see [1]), just left stability problem is worth considering and undone for the (1.1)-(1.2) eigenvalue problem. In a complete solution of inverse problems, the questions of existence, uniqueness, stability and construction are to be considered. The question of existence and uniqueness is of great importance in testing the assumption behind any mathematical model. If the answer to the uniqueness question is no, then we know that even perfect data do not contain enough information to recover the physical quantity to be estimated. In the question of stability we have to decide whether the solution depends continuously on the data. Stability is necessary if we want to be sure that a variation of the given data in a sufficiently small range leads to an arbitrarily small change in the solution. This concept was introduced by Hadamard in 1902 in connection with the study of boundary value problems for partial differential equations (see [2]). Because of this important reason, we want to deal with a stability issue for problem (1.1)-(1.2).

Notice that equation (1.1) becomes

$$-u'' + [q + 2\lambda r]u = \lambda^2 u \tag{1.3}$$

for $p = 2$ and this equation is known as Schrödinger equation with energy dependent potential (or diffusion equation, quadratic of differential pencil). Equation (1.3) is very important in both classical and quantum mechanics. For example, such problems arise in solving Klein-Gordon equations which describe the motion of massless particles such as photons. Sturm-Liouville energy dependent equations are also used for modeling vibrations of mechanical systems in viscous media (see [3]). We note that in this type of problems, the spectral parameter λ is related to the energy of the system, and this motivates the terminology 'energy dependent' used for the spectral problem of the form (1.3).

The theory of inverse problems for differential operators occupies an important position in the current development of the spectral theory of linear operators. Inverse problems of spectral analysis consist in the recovery of operators from their spectral data. One takes for the main spectral data, for instance, one, two, or more spectra, the spectral function, the spectrum and the normalizing constants, the Weyl function. Some aspects of spectral problems for the Schrödinger equation with energy dependent potential have been studied by many authors (see [4–13]).

In 1988, McLaughlin [14] posed a new technique to recover the operators. This technique is called inverse nodal problem. Inverse nodal problems consist in recovering operators from given nodes (zeros) of their eigenfunctions. From the physical point of view, this corresponds to finding, *e.g.*, the density of a string or a beam from the zero-amplitude positions of their eigenvibrations. She seems to be the first to consider this sort of inverse problem. Later on, the inverse nodal problem has been studied by many authors (see [15–18]).

Suppose that $\{x_j^n\}_{j=1}^{n-1}$ are the zeros of the eigenfunction $u_n(x)$ which is expressed by (1.1), and denote the nodal set $X_n = \{x_j^n\}_{j=1}^{n-1}$. Define the nodal length $l_j^n = x_{j+1}^n - x_j^n$ for $j = 1, 2, \ldots, n-1$. Using these nodal data, some uniqueness, reconstruction results of the potential function of the Schrödinger equation with energy dependent potential have been solved by many authors (see [19–22]).

In (1.1), we can get the following one-dimensional p-Laplacian Sturm-Liouville eigenvalue problem for the special case $r(x) = 0$:

$$\begin{aligned} &-\left(u'^{(p-1)}\right)' = (p-1)\left(\lambda^2 - q(x)\right)u^{(p-1)}, \\ &u(0) = u(1) = 0, \end{aligned} \tag{1.4}$$

where the eigenvalues of problem (1.4) associated eigenfunctions $u_n(x)$ are countably infinite real and simple [23]. Inverse and stability problems for (1.4) one-dimensional p-Laplacian Sturm-Liouville eigenvalue problem were solved by several authors (see [23–27]).

To say something about the stability of the inverse nodal problem for the given (1.1)-(1.2) eigenvalue problem, we need to introduce a generalized sine function S_p which is the solution of the initial value problem

$$-\left(S_p'^{(p-1)}\right)' = (p-1)S_p^{(p-1)}, \qquad (1.5)$$
$$S_p(0) = 0, \qquad S_p'(0) = 1$$

(see [23, 26, 28]). S_p and S_p' are periodic functions which satisfy the identity

$$\left|S_p(x)\right|^p + \left|S_p'(x)\right|^p = 1$$

for any $x \in \mathbb{R}$. These functions are p-analogues of classical sine and cosine functions in the classical case. It is well known that

$$\pi_p = \frac{2\pi}{p\sin(\frac{\pi}{p})}$$

is the first zero of S_p (see [28–30]). Now, we will give some further properties of S_p by the following lemma.

Lemma 1.1 [23, 28]

(a) *For* $S_p' \neq 0$,

$$\left(S_p'\right)' = -\left|\frac{S_p}{S_p'}\right|^{p-2} S_p.$$

(b)

$$\left(S_p S_p'^{(p-1)}\right)' = \left|S_p'\right|^p - (p-1)S_p^p = 1 - p|S_p|^p = (1-p) + p\left|S_p'\right|^p.$$

This paper is organized as follows. In Section 2, we mention some asymptotic formulas for eigenvalues, nodal parameters and potential function for the (1.1)-(1.2) eigenvalue problem by using the modified Prüfer substitution which were solved in the reference [1]. In Section 3, we define a metric to solve the Lipschitz stability problem for a p-Laplacian Schrödinger equation with energy dependent potential. Eventually, we give some conclusions in Section 4.

2 Asymptotic estimates for eigenvalues, nodal parameters and potential function

In this section, we recall some properties of (1.1) p-Laplacian operator with (1.2) Dirichlet conditions which were solved by Koyunbakan [1]. For this purpose, we can introduce a modified Prüfer substitution as

$$u(x) = c(x)S_p\left(\lambda^{2/p}\theta(x)\right), \qquad (2.1)$$
$$u'(x) = \lambda^{2/p}c(x)S_p'\left(\lambda^{2/p}\theta(x)\right),$$

or

$$\frac{u'(x)}{u(x)} = \lambda^{2/p}\frac{S_p'(\lambda^{2/p}\theta(x))}{S_p(\lambda^{2/p}\theta(x))}, \qquad (2.2)$$

where $c(x)$ and $\theta(x)$ are Prüfer variables. Differentiating both sides of the above equation with respect to x and applying Lemma 1.1, we obtain [1, 23]

$$\theta'(x) = 1 - \frac{q}{\lambda^2} S_p^p - \frac{2}{\lambda} r S_p^p. \tag{2.3}$$

Now, we can establish the estimations of nodal parameters and a reconstruction formula of a potential function for problem (1.1), (1.2).

Theorem 2.1 [1] *The eigenvalues* λ_n *of the Dirichlet eigenvalue problem given in* (1.1), (1.2) *have the form*

$$\lambda_n^{2/p} = n\pi_p + \frac{1}{p(n\pi_p)^{p-1}} \int_0^1 q(t)\,dt + \frac{2}{p(n\pi_p)^{\frac{p-2}{2}}} \int_0^1 r(t)\,dt + O\left(\frac{1}{n^{p/2}}\right)$$

as $n \to \infty$.

Theorem 2.2 [1] *The nodal points and nodal length expansions for problem* (1.1), (1.2) *satisfy*

$$x_j^n = \frac{j}{n} + \frac{2}{(n\pi_p)^{p/2}} \int_0^{x_j^n} r(x) S_p^p\,dx + \frac{1}{(n\pi_p)^p} \int_0^{x_j^n} q(x) S_p^p\,dx + O\left(\frac{1}{n^{p/2+1}}\right),$$

$$l_j^n = \frac{\pi_p}{\lambda_n^{2/p}} + \frac{2}{p\lambda_n} \int_{x_j^n}^{x_{j+1}^n} r(t)\,dt + \frac{1}{p\lambda_n^2} \int_{x_j^n}^{x_{j+1}^n} q(t)\,dt + O\left(\frac{1}{\lambda_n^{\frac{4+p}{p}}}\right),$$

respectively, as $n \to \infty$.

Theorem 2.3 [1] *Let* $q \in L_2(0,1)$, $r \in W_2^1(0,1)$ *and assume that* r *is given a priori on the interval* $[0,1]$. *Then*

$$q(x) = \lim_{n\to\infty} p\lambda_n^2 \left(\frac{\lambda_n^{2/p} l_j^n}{\pi_p} - \frac{2r(x)}{p\lambda_n} - 1 \right)$$

for $x \in (0,1)$, $j = j_n(x) = \max\{j : x_j^n < x\}$.

3 Lipschitz stability of an inverse nodal problem

In this section, we study Lipschitz stability of an inverse nodal problem for (1.1) p-Laplacian operator. Lipschitz stability is about a continuity between two metric spaces. To show this continuity, we will use a homeomorphism between these two metric spaces. Stability problems were studied by many authors (see [31–34]). The method that we have used in the proof of the Lipschitz stability of an inverse nodal problem is similar to the classical Sturm-Liouville problem (see [31]).

Let us define Ω_{dif} and Σ_{dif} by

$$\Omega_{\text{dif}} = \left\{ q \in C^1[0,1] \right\},$$

$$\Sigma_{\text{dif}} = \left\{ X = \left\{ x_k^n \right\} : X \text{ is the nodal set associated with some } q \in \Omega_{\text{dif}} \right\}.$$

We will show that Ω_{dif} and Σ_{dif} are homeomorphic to each other. Hence, when $\overline{X}$ is the nodal set associated with $\overline{q}$ and $\overline{X}$ is close to X in Σ_{dif}, then $\overline{q}$ is close to q in Ω_{dif}, where $\overline{r}$ is so close to r. That is, the inverse nodal problem is Lipschitz stable. Here, we use $L^m(0,1)$ $(m \geq 1)$ for Ω_{dif}. For $m \geq 1$, let

$$S_n^m(X,\overline{X}) = \pi_p^p n^{p+1-\frac{1}{m}} \left[\sum_{k=0}^{n-1} \left| l_k^n - \overline{l}_k^n \right|^m \right]^{\frac{1}{m}} + \frac{2}{p}(n\pi_p)^{\frac{p}{2}} \left[\int_0^1 |\overline{r} - r|^m \, dx \right]^{\frac{1}{m}}, \tag{3.1}$$

where $l_k^n = x_{k+1}^n - x_k^n$ and $\overline{l}_k^n = \overline{x}_{k+1}^n - \overline{x}_k^n$. Define the metric and a pseudometric on Σ_{dif}

$$d_0^m(X,\overline{X}) = \overline{\lim_{n\to\infty}} S_n^m(X,\overline{X}),$$

and

$$d_{\Sigma_{\mathrm{dif}}}^m(X,\overline{X}) = \overline{\lim_{n\to\infty}} \frac{S_n^m(X,\overline{X})}{1 + S_n^m(X,\overline{X})},$$

respectively. If we define $X \sim_m \overline{X}$ iff $d_{\Sigma_{\mathrm{dif}}}^m(X,\overline{X}) = 0$, then $\sim_m$ is an equivalence relation on Σ_{dif} and $d_{\Sigma_{\mathrm{dif}}}^m$ would be a metric for the partition set $\Sigma_{\mathrm{dif}}^* = \Sigma_{\mathrm{dif}} / \sim_m$.

Lemma 3.1 *The function $d_{\Sigma_{\mathrm{dif}}}^m(\cdot,\cdot)$ is a pseudometric on Σ_{dif}.*

Proof It can be proved easily by using a similar method to that in [23]. □

Lemma 3.2 *Let $X, \overline{X} \in \Sigma_{\mathrm{dif}}$. Then*

(a) *The interval $I_{n,k}$ between the points x_k^n and $\overline{x}_k^n$ has length $O(n^{-\frac{p}{2}})$.*

(b) *For all $x \in (0,1)$, we have the inequality $|j_n(x) - \overline{j}_n(x)| \leq 1$ when n is sufficiently large.*

Proof (a) By the asymptotic estimates of the nodal points, we can easily obtain

$$\begin{aligned} |I_{n,k}| &= \left| x_k^n - \overline{x}_k^n \right| \\ &\leq \left| x_k^n - \frac{k}{n} \right| + \left| \frac{k}{n} - \overline{x}_k^n \right| \\ &= O\bigl(n^{-\frac{p}{2}}\bigr) + O\bigl(n^{-\frac{p}{2}}\bigr) \\ &= O\bigl(n^{-\frac{p}{2}}\bigr), \end{aligned}$$

by a similar method as in [31].

(b) We can prove part (b) easily by using a similar method as in [31]. □

Theorem 3.1 *For any of $m \geq 1$, $d_{\Sigma_{\mathrm{dif}}}^m$ is a metric on the space $\Sigma_{\mathrm{dif}} / \sim_m$. Additionally, the metric spaces $(\Omega_{\mathrm{dif}}, \|\cdot\|_m)$ and $(\Sigma_{\mathrm{dif}} / \sim_m, d_{\Sigma_{\mathrm{dif}}}^m)$ are homeomorphic to each other where $\sim_m$ is an equivalence relation induced by $d_{\Sigma_{\mathrm{dif}}}^m$.*

Proof It suffices to indicate that

$$\|q - \overline{q}\|_m = p d_0^m(X,\overline{X}).$$

By Theorem 2.3, we get

$$q(x)-\overline{q}(x) = \lim_{n\to\infty} p(n\pi_p)^p\left[n\left(l^n_{j_n(x)}-\overline{l}^n_{\bar{j}_n(x)}\right)+\frac{2}{p(n\pi_p)^{\frac{p}{2}}}(\overline{r}-r)\right]$$
$$= \lim_{n\to\infty}\left[pn^{p+1}\pi_p^p\left(l^n_{j_n(x)}-\overline{l}^n_{\bar{j}_n(x)}\right)+2(n\pi_p)^{\frac{p}{2}}(\overline{r}-r)\right]$$

for each $x\in(0,1)$. Hence, by Fatou's lemma and the definition of norm on L_m, we have

$$\|q-\overline{q}\|_m \le pn^{p+1}\pi_p^p \varliminf_{n\to\infty}\left\|l^n_{j_n(x)}-\overline{l}^n_{\bar{j}_n(x)}\right\|_m + 2(n\pi_p)^{\frac{p}{2}}\varliminf_{n\to\infty}\left[\int_0^1|\overline{r}-r|^m\right]^{\frac{1}{m}}$$
$$\le p\pi_p^p \varlimsup_{n\to\infty}\left[n^{p+1}\left\|l^n_{j_n(x)}-\overline{l}^n_{j_n(x)}\right\|_m + n^{p+1}\left\|\overline{l}^n_{j_n(x)}-\overline{l}^n_{\bar{j}_n(x)}\right\|_m\right]$$
$$+ 2(n\pi_p)^{\frac{p}{2}}\varliminf_{n\to\infty}\left[\int_0^1|\overline{r}-r|^m\,dx\right]^{\frac{1}{m}}. \tag{3.2}$$

Here, by Lemma 3.2 and Theorem 2.2, we get

$$n^{p+1}\left\|\overline{l}^n_{j_n(x)}-\overline{l}^n_{\bar{j}_n(x)}\right\|_m = n^{p+1}\left[\int_0^1\left|\overline{l}^n_{j_n(x)}-\overline{l}^n_{\bar{j}_n(x)}\right|^m dx\right]^{\frac{1}{m}}$$
$$= n^{p+1}\left[\sum_{k=0}^{n-1}\left|\overline{l}^n_{k+1}-\overline{l}^n_k\right|^m I_{n,k}\right]^{\frac{1}{m}}$$
$$= o(1) \tag{3.3}$$

and

$$n^{p+1}\left\|l^n_{j_n(x)}-\overline{l}^n_{j_n(x)}\right\|_m = n^{p+1}\left[\int_0^1\left|l^n_{j_n(x)}-\overline{l}^n_{j_n(x)}\right|^m dx\right]^{\frac{1}{m}}$$
$$= n^{p+1}\left[\sum_{k=0}^{n-1}\left|l^n_k-\overline{l}^n_k\right|^m l^n_k\right]^{\frac{1}{m}}$$
$$= n^{p+1-\frac{1}{m}}\left[\sum_{k=0}^{n-1}\left|l^n_k-\overline{l}^n_k\right|^m\right]^{\frac{1}{m}}. \tag{3.4}$$

Considering (3.3) and (3.4) in (3.2), we obtain

$$\|q-\overline{q}\|_m \le p\pi_p^p \varlimsup_{n\to\infty} n^{p+1-\frac{1}{m}}\left[\sum_{k=0}^{n-1}\left|l^n_k-\overline{l}^n_k\right|^m\right]^{\frac{1}{m}} + 2(n\pi_p)^{\frac{p}{2}}\varliminf_{n\to\infty}\left[\int_0^1|\overline{r}-r|^m\,dx\right]^{\frac{1}{m}}$$
$$= pd_0^m(X,\overline{X}).$$

Contrarily, using the above derivations

$$\|q-\overline{q}\|_m + o(1)$$
$$= p\pi_p^p n^{p+1}\left\|l^n_{j_n(x)}-\overline{l}^n_{\bar{j}_n(x)}\right\|_m + 2(n\pi_p)^{\frac{p}{2}}\left[\int_0^1|\overline{r}-r|^m\,dx\right]^{\frac{1}{m}}$$

$$\geq p\pi_p^p n^{p+1} \left\| l_{j_n(x)}^n - \bar{l}_{\bar{j}_n(x)}^n \right\|_m + 2(n\pi_p)^{\frac{p}{2}} \left[\int_0^1 |\bar{r} - r|^m \, dx \right]^{\frac{1}{m}} - O\left(n^{\frac{p}{2}+1+\frac{2-p}{m}}\right)$$

$$= p\pi_p^p n^{p+1} \left[\sum_{k=0}^{n-1} \left| l_k^n - \bar{l}_k^n \right|^m l_k^n \right]^{\frac{1}{m}} + 2(n\pi_p)^{\frac{p}{2}} \left[\int_0^1 |\bar{r} - r|^m \, dx \right]^{\frac{1}{m}} - O\left(n^{\frac{p}{2}+1+\frac{2-p}{m}}\right)$$

$$= p\pi_p^p n^{p+1-\frac{1}{m}} \left[\sum_{k=0}^{n-1} \left| l_k^n - \bar{l}_k^n \right|^m \right]^{\frac{1}{m}} + 2(n\pi_p)^{\frac{p}{2}} \left[\int_0^1 |\bar{r} - r|^m \, dx \right]^{\frac{1}{m}} - O\left(n^{\frac{p}{2}+1+\frac{2-p}{m}}\right).$$

Hereby as n approaches infinity,

$$\|q - \bar{q}\|_m \geq p d_0^m(X, \bar{X}).$$

This completes the proof. □

4 Conclusion

In this study, we have emphasized the importance of the stability (specially Lipschitz type stability) for inverse problems. Then, some asymptotic estimates for eigenvalues, nodal parameters and potential function of the (1.1)-(1.2) eigenvalue problem have been recalled. Finally, we have examined the Lipschitz stability of an inverse nodal problem for (1.1) p-Laplacian operator.

Competing interests
The authors declare to have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Firat University, Elazıg, 23119, Turkey. [2]Department of Mathematics, Mersin University, Mersin, 33343, Turkey.

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