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Periodic solutions of a class of nonautonomous second-order Hamiltonian systems with nonsmooth potentials

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Abstract

This paper is concerned with nonautonomous second-order Hamiltonian systems with nondifferentiable potentials. By using the nonsmooth critical point theory for locally Lipschitz functionals, we obtain some new existence results for the periodic solutions.

Keywords: periodic solutions; second-order Hamiltonian systems; saddle point theorem; discontinuous nonlinearities; locally Lipschitz continuous

1 Introduction and main results

In this paper we consider the following second-order differential inclusions systems:

$$\begin{cases} \ddot{u}(t) + A(t)u(t) \in \partial F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1}$$

where $T > 0$, $A(t)$ is a continuous symmetric matrix of order N and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz continuous in x and $\partial F(t, x)$ denotes the Clarke subdifferential of F for x. There have been a lot of contributions on problem (1); see, for example, [1–3] and the references therein.

When $F(t, x)$ is continuously differentiable in x, problem (1) becomes the second-order Hamiltonian system

$$\begin{cases} \ddot{u}(t) + A(t)u(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{2}$$

The smooth system (2) has also been studied in the past decades and many excellent results appeared; see [4, 5]. In those works, the following assumption is necessary:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \le a(|x|)b(t), \qquad |\nabla F(t, x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, where $\mathbb{R}^+$ is the set of all nonnegative real number.

Throughout this paper, we always suppose that $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A′) $F(t,x)$ is integrable in t over $[0,T]$ for each $x \in \mathbb{R}^N$ and locally Lipschitz continuous in x for each $t \in [0,T]$.

Let H_T^1 be the usual Sobolev space with norm

$$\|u\| = \left(\int_0^T \left|u(t)\right|^2 dt + \int_0^T \left|\dot{u}(t)\right|^2 dt \right)^{1/2}.$$

Definition 1.1 We call $u \in H_T^1$ a weak solution of (1) if the following inequality holds:

$$- \int_0^T \left(\dot{u}(t), \dot{v}(t)\right) dt + \int_0^T \left(A(t)u(t), v(t)\right) dt \le \int_0^T F^0\left(t, u(t); v(t)\right) dt, \quad \forall v \in H_T^1, \tag{3}$$

where $F^0(t,x;y)$ denotes the generalized directional derivative of F at x along the direction y.

The main results of this paper are as follows.

Theorem 1.2 *Suppose $F(t,x)$ satisfies* (A′) *and the following conditions*:

(i$_1$) *There exist $f, g \in L^1(0,T;\mathbb{R}^+)$ and $\alpha \in [0,1)$ such that for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$,*

$$\xi \in \partial F(t,x) \quad \Rightarrow \quad |\xi| \le f(t)|x|^\alpha + g(t). \tag{4}$$

(i$_2$) *There exists a $\gamma \in L^1(0,T;\mathbb{R}^+)$ such that for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$,*

$$\frac{F(t,x)}{|x|^{2\alpha}} \le \gamma(t). \tag{5}$$

(i$_3$) *There exists a subset E of $[0,T]$ with* $\operatorname{meas}(E) > 0$ *such that for a.e. $t \in E$,*

$$\frac{F(t,x)}{|x|^{2\alpha}} \to -\infty \quad \text{as } |x| \to \infty. \tag{6}$$

Then problem (1) *possesses at least one weak solution.*

Theorem 1.3 *Suppose $F(t,x)$ satisfies* (A′), (i$_1$) *above and the following conditions*:

(i$_2'$) *There exists $\gamma \in L^1(0,T;\mathbb{R}^+)$ such that*

$$\frac{F(t,x)}{|x|^{2\alpha}} \ge -\gamma(t) \tag{5a}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

(i$_3'$) *There exists a subset E of $[0,T]$ with* $\operatorname{meas}(E) > 0$ *such that*

$$\frac{F(t,x)}{|x|^{2\alpha}} \to +\infty \quad \text{as } |x| \to \infty \tag{6a}$$

for a.e. $t \in E$.

Then problem (1) *possesses at least one weak solution.*

The method in our paper is based on the nonsmooth least action principle and saddle point theorem initiated by Chang [6], different from [1] and [2] based on the nonsmooth mountain pass theorem. Moreover, we consider the case that the growth of the subdifferential $\partial F(t,x)$ in x is sublinear ($\alpha \in [0,1)$), while in [1] and [2] the authors considered the case $\alpha \geq 1$.

Barletta and Papageorgiou [3, Theorem 3.4] proved the existence of solutions for problem (1), where they assumed that
- $F(t,x) \geq -\gamma(t)$, $\forall x \in \mathbb{R}^N$, $t \in [0,T]$,
- $F(t,x) \to +\infty$ as $|x| \to \infty$ and $t \in E$,
- $\dim(\operatorname{span}\{u \in H_T^1 | -\ddot{u} - A(t)u = \lambda u \text{ for some } \lambda < 0\}) = 0$.

By comparison, the first two assumptions are strengthened to (i_2') and (i_3'), but the third one is not necessarily needed in the present paper.

Remark 1.4 If $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, the inequality (4) becomes

$$\bigl|\nabla F(t,x)\bigr| \leq f(t)|x|^\alpha + g(t).$$

Meanwhile, the above inequality (3) takes the form

$$-\int_0^T \bigl(\dot{u}(t),\dot{v}(t)\bigr)\,dt + \int_0^T \bigl(A(t)u(t),v(t)\bigr)\,dt = \int_0^T \bigl(\nabla F\bigl(t,u(t)\bigr);v(t)\bigr)\,dt, \quad \forall v \in H_T^1.$$

Then Theorem 1.2 and Theorem 1.3 generalize Theorem 2 and Theorem 3 of [5], respectively, without assuming the linear second-order system $\ddot{u}(t) + A(t)u(t) = 0$ a.e. $t \in [0,T]$ has a nonzero solution, which is necessary in [5].

Example 1.5 Let

$$F(t,x) = -|\sin \omega t|\bigl(|x|^{1+r} + |x|\bigr)$$

for all $(t,x) \in [0,T] \times \mathbb{R}^N$, where $r \in [0,1)$, $\omega = \frac{2\pi}{T}$. Then F satisfies the conditions of Theorem 1.2, but it is not covered by the results of [1, 2, 4, 5].

2 Basic definitions and preliminary results

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X and X^*. A functional $h : X \to \mathbb{R}$ is called locally Lipschitz continuous if for every $u \in X$ there correspond a neighborhood V_u of u and a constant $L_u \geq 0$ such that

$$\bigl|h(z) - h(w)\bigr| \leq L_u\|z - w\|, \quad \forall z, w \in V_u.$$

If $u, v \in X$, we write $h^0(u;v)$ for the generalized directional derivative of h at the point u along the direction v, *i.e.*,

$$h^0(u;v) := \limsup_{w \to u, t \to 0^+} \frac{h(w + tv) - h(w)}{t}.$$

It is well known that h^0 is upper semicontinuous on $X \times X$ (see [7], Proposition 2.1.1).

For locally Lipschitz continuous functionals $h_1, h_2 : X \to \mathbb{R}$, we have

$$(h_1 + h_2)^0(x; z) \le h_1^0(x; z) + h_2^0(x; z), \quad \forall x, z \in X.$$

The generalized gradient of the function h in u, denoted by $\partial h(u)$, is the set defined by

$$\partial h(u) := \{u^* \in X^* : \langle u^*, v\rangle \le h^0(u; v), \forall v \in X\}.$$

Proposition 2.1.2 of [7] ensures that $\partial h(u)$ turns out nonempty, convex, weak* compact, thus the function $\lambda(x) = \min_{w \in \partial h(x)} \|w\|_{X^*}$ exists and is lower semicontinuous, *i.e.*, $\liminf_{x \to x_0} \lambda(x) \ge \lambda(x_0)$.

If $f, g : X \to X$ be locally Lipschitz continuous, then

$$\partial(f + g)(x) \subset \partial f(x) + \partial g(x).$$

A point $u \in X$ is said to be a critical point of h if

$$h^0(u; v) \ge 0, \quad \forall v \in X.$$

We say the locally Lipschitz functional h satisfies the nonsmooth (PS) condition if any sequence $\{x_n\}$ in X such that $\{h(x_n)\}$ is bounded and $\lambda(x_n) \to 0$ possesses a convergent subsequence.

For more details, we can refer to [8–11]. To prove Theorem 1.2 and Theorem 1.3 in the next section, first we state the following well-known results.

Lemma 2.1 ([7], Theorem 2.3.7) *Let x and y be points in X, and suppose that f is Lipschitz on open set containing the line segment $[x, y]$. Then there exists a point u in (x, y) such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x\rangle.$$

Lemma 2.2 ([6], Theorem 3.3) *Let X be a real Banach space, and let f be a locally Lipschitz function defined on X satisfying the nonsmooth* (*PS*) *condition. Suppose $X = X_1 \oplus X_2$ with a finite-dimensional subspace X_1, and there exist constants $b_1 < b_2$ and a bounded neighborhood N of θ in X_1 such that*

$$f|_{X_2} \ge b_2, \qquad f|_{\partial N} \le b_1.$$

Then f has a critical point.

Lemma 2.3 ([12], Lemma 2) *Suppose that E is a measurable subset of $[0, T]$ and $G(t, x)$ is continuous in x for a.e. $t \in E$. Assume that*

$$G(t, x) \to -\infty \quad \textit{as } |x| \to \infty$$

for a.e. $t \in E$. Then for every $\delta > 0$ there exists a subset E_δ of E with $\operatorname{meas}(E/E_\delta) < \delta$ *such that*

$$G(t, x) \to -\infty \quad \textit{as } |x| \to \infty$$

uniformly for all $t \in E_\delta$.

Lemma 2.4 ([6], Theorem 3.5) *Suppose a locally Lipschitz functional f, defined on a reflexive Banach space X, satisfies the nonsmooth (PS) condition and is bounded from below. Then $c = \inf_X f(x)$ is a critical value of f.*

3 Proof of theorems

For every $u \in H_T^1$, let $\bar{u} = \frac{1}{T}\int_0^T u(t)\,dt$, $\tilde{u}(t) = u(t) - \bar{u}$. Then the following inequalities hold:

$$\begin{aligned} &\|\tilde{u}\|_\infty^2 \leq \frac{T}{12}\int_0^T \left|\dot{u}(t)\right|^2 dt \quad \text{(Sobolev's inequality)}, \\ &\int_0^T \left|\tilde{u}(t)\right|^2 dt \leq \frac{T^2}{4\pi^2}\int_0^T \left|\dot{u}(t)\right|^2 dt \quad \text{(Wirtinger's inequality)}, \\ &\|u\|_\infty \leq C\|u\|, \end{aligned} \tag{7}$$

where $C > 0$ is a constant and $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$.

Define two functionals $\varphi : H_T^1 \to \mathbb{R}$ and $\psi : H_T^1 \to \mathbb{R}$ as follows:

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\int_0^T \left|\dot{u}(t)\right|^2 dt - \frac{1}{2}\int_0^T \bigl(A(t)u(t), u(t)\bigr)\,dt + \int_0^T F\bigl(t, u(t)\bigr)\,dt, \\ \psi(u) &= \int_0^T F\bigl(t, u(t)\bigr)\,dt. \end{aligned}$$

It is easy to verify that they are locally Lipschitz continuous on H_T^1, so it makes sense to consider their generalized directional derivatives φ^0 and ψ^0:

$$\varphi^0(u; v) = \int_0^T \bigl(\dot{u}(t), \dot{v}(t)\bigr)\,dt - \int_0^T \bigl(A(t)u(t), v(t)\bigr)\,dt + \psi^0(u; v), \quad \forall v \in H_T^1.$$

Equation (9) at p.84 of [7] gives

$$\psi^0(u; v) \leq \int_0^T F^0\bigl(t, u(t); v(t)\bigr)\,dt, \quad \forall u, v \in H_T^1.$$

Moreover, by [7], Theorem 2.7.5, one has

$$\partial\psi(u) \subset \int_0^T \partial F\bigl(t, u(t)\bigr)\,dt,$$

i.e., to every $\xi \in \partial\psi(u)$ there corresponds a mapping $t \to q(t)$ from $[0, T]$ to $(H_T^1)^*$ with $q(t) \in \partial F(t, u(t))$ a.e. $t \in [0, T]$ such that for every $v \in H_T^1$,

$$\langle\xi, v\rangle = \int_0^T \bigl(q(t), v(t)\bigr)\,dt.$$

If $u \in H_T^1$ is a critical point of φ, *i.e.*, $\theta \in \partial\varphi(u)$, then

$$\begin{aligned} &\int_0^T \bigl(\dot{u}(t), \dot{v}(t)\bigr)\,dt - \int_0^T \bigl(A(t)u(t), v(t)\bigr)\,dt + \int_0^T F^0\bigl(t, u(t); v(t)\bigr)\,dt \\ &\quad \geq \varphi^0(u; v) \geq 0, \quad \forall v \in H_T^1. \end{aligned}$$

Thus, the critical points of φ correspond to the solutions of problem (3).

In the same time, there exists $q_0(t) \in \partial F(t,u)$ such that for all $v \in H_T^1$,

$$0 = \langle \theta, v \rangle = \int_0^T \big(\dot{u}(t), \dot{v}(t)\big)\,dt - \int_0^T \big(A(t)u(t), v(t)\big)\,dt + \int_0^T \big(q_0(t), v(t)\big)\,dt,$$

it follows easily that $q_0(t) = \ddot{u}(t)$ a.e. $t \in [0,T]$, thus

$$\ddot{u}(t) + A(t)u(t) \in \partial F\big(t, u(t)\big) \quad \text{a.e. on } [0,T],$$

so that u satisfies the system (1) too.

Define the subspaces of H_T^1 by

$$\begin{aligned} W^- &\triangleq \operatorname{span}\big\{u \in H_T^1 \mid -\ddot{u} - A(t)u = \lambda u \text{ for some } \lambda < 0\big\}, \\ V &\triangleq \operatorname{span}\big\{u \in H_T^1 \mid -\ddot{u} - A(t)u = 0\big\}, \\ W^+ &\triangleq \operatorname{span}\big\{u \in H_T^1 \mid -\ddot{u} - A(t)u = \lambda u \text{ for some } \lambda > 0\big\}. \end{aligned}$$

It is easy to verify that W^- is finite-dimensional and there exists $\delta > 0$ such that

$$\begin{aligned} &\int_0^T \big|\dot{u}(t)\big|^2\,dt - \int_0^T \big(A(t)u(t), u(t)\big)\,dt \ge \delta \|u\|^2, \quad \forall u \in W^+, \\ &\int_0^T \big|\dot{u}(t)\big|^2\,dt - \int_0^T \big(A(t)u(t), u(t)\big)\,dt \le -\delta \|u\|^2, \quad \forall u \in W^-. \end{aligned}$$

Decompose H_T^1 as $H_T^1 = W^- \oplus V \oplus W^+$, and denote $W = W^+ + W^-$.

Lemma 3.1 *Suppose that* (5) *and* (6) *hold. Assume that* $u_n = v_n + w_n$, $v_n \in V$, $w_n \in W$ *satisfying* $\|u_n\| \to \infty$ $(n \to \infty)$ *and* $\limsup_{n\to\infty} \frac{\|w_n\|}{\|u_n\|^\alpha} < +\infty$. *Then*

$$\|u_n\|^{-2\alpha} \int_0^T F\big(t, u_n(t)\big)\,dt \to -\infty \quad \text{as } n \to \infty.$$

Proof As the proof of Lemma 3 in [5], for every $\beta > 0$, there exists $m_\beta > 0$ such that

$$\operatorname{meas}\big\{t \in (0,T) \mid \big|v(t)\big| < m_\beta \|v\|\big\} < \beta$$

for all $v \in V$.

Let $B = \{t \in (0,T) \mid |v(t)| \ge m_\beta \|v\|\}$ for all $v \in V$, then we have $\operatorname{meas}((0,T) \setminus B) < \beta$. By (6) and Lemma 2.3, there exists subset E_δ of E with $\operatorname{meas}(E \setminus E_\delta) < \delta$ such that

$$\frac{F(t,x)}{|x|^{2\alpha}} \to -\infty \quad \text{as } |x| \to \infty \tag{8}$$

uniformly for all $t \in E_\delta$. Hence,

$$\operatorname{meas}(B \cap E_\delta) \ge \operatorname{meas}(E_\delta) - \operatorname{meas}\big((0,T) \setminus B\big) \ge \operatorname{meas} E - \delta - \beta > 0 \tag{9}$$

for δ and β small enough.

By (8), for every $\eta > 0$, there exists an $M > 0$ such that

$$\frac{F(t,x)}{|x|^{2\alpha}} \leq -\eta$$

for all $|x| \geq M$ and a.e. $t \in E_\delta$. Furthermore, it follows from (5) that

$$F(t,x) \leq -\eta|x|^{2\alpha} + \gamma_0(t)$$

for all $x \in \mathbb{R}^N$, a.e. $t \in E_\delta$ and some $\gamma_0(t) = M^{2\alpha}(\eta + \gamma(t))$.

From (9) we obtain

$$\int_{B\cap E_\delta} F(t,v)\,dt \leq -\eta m_\beta^{2\alpha}\|v\|^{2\alpha}\operatorname{meas}(B\cap E_\delta) + \int_0^T \gamma_0(t)\,dt.$$

By (5) and (7) we have

$$\int_{[0,T]\setminus(B\cap E_\delta)} F(t,v)\,dt \leq C^{2\alpha}\|v\|^{2\alpha}\int_0^T \gamma(t)\,dt.$$

Since $\|u_n\| \to \infty$ ($n \to \infty$) and $\limsup_{n\to\infty}\frac{\|w_n\|}{\|u_n\|^\alpha} < +\infty$, by a simple computation we obtain $\|v_n\| \to \infty$ as $n \to \infty$, and $V \neq \{0\}$.

Hence, one has

$$\limsup_{v\in V,\|v\|\to\infty}\|v\|^{-2\alpha}\int_0^T F(t,v)\,dt \leq -\eta m_\beta^{2\alpha}\operatorname{meas}(B\cap E_\delta) + C^{2\alpha}\int_0^T \gamma(t)\,dt,$$

which implies that

$$\limsup_{n\to\infty}\|v_n\|^{-2\alpha}\int_0^T F(t,v_n)\,dt \to -\infty \tag{10}$$

by the arbitrariness of η.

By (4), (7), and Lemma 2.1, there exist $s \in [0,1]$ and $\xi_n \in \partial F(t, v_n + sw_n)$ such that

$$\begin{aligned}
&\left|\int_0^T F(t,u_n)\,dt - \int_0^T F(t,v_n)\,dt\right| \\
&\quad= \left|\int_0^T \langle \xi_n, u_n - v_n\rangle\,dt\right| \leq \int_0^T |\xi_n||w_n|\,dt \leq \int_0^T \big(f(t)|v_n + sw_n|^\alpha + g(t)\big)|w_n|\,dt \\
&\quad\leq \int_0^T f(t)\big(|v_n|^\alpha + |w_n|^\alpha\big)|w_n|\,dt + \int_0^T g(t)|w_n|\,dt \\
&\quad\leq \|f\|_{L^1}\big(\|v_n\|_\infty^\alpha + \|w_n\|_\infty^\alpha\big)\|w_n\|_\infty + \|g\|_{L^1}\|w_n\|_\infty \\
&\quad\leq C^{\alpha+1}\|f\|_{L^1}\big(\|v_n\|^\alpha + \|w_n\|^\alpha\big)\|w_n\| + C\|g\|_{L^1}\|w_n\|
\end{aligned} \tag{11}$$

for all n. Since $\limsup_{n\to\infty}\frac{\|w_n\|}{\|u_n\|^\alpha} < +\infty$, one has

$$C_1 \triangleq \limsup_{n\to\infty}\left|\|u_n\|^{-2\alpha}\int_0^T F(t,u_n)\,dt - \|u_n\|^{-2\alpha}\int_0^T F(t,v_n)\,dt\right| < +\infty.$$

It follows from (10) that

$$\limsup_{n\to\infty}\|u_n\|^{-2\alpha}\int_0^T F(t,u_n)\,dt \le \limsup_{n\to\infty}\|u_n\|^{-2\alpha}\int_0^T F(t,v_n)\,dt + C_1$$
$$\le \limsup_{n\to\infty}\|v_n\|^{-2\alpha}\int_0^T F(t,v_n)\,dt + C_1 = -\infty.$$

Hence we obtain

$$\|u_n\|^{-2\alpha}\int_0^T F(t,u_n)\,dt \to -\infty$$

as $n\to\infty$, which completes the proof. □

Lemma 3.2 *Under the conditions* (4), (5) *and* (6), φ *satisfies the nonsmooth* (*PS*) *condition.*

Proof Let $\{u_n\}$ be a sequence in H_T^1 such that $\{\varphi(u_n)\}$ is bounded and $\lambda(u_n) = \min_{x^*\in\partial\varphi(u_n)}\|x^*\| \to 0$ as $n\to\infty$. Put $u_n^*\in\partial\varphi(u_n)$ such that $\|u_n^*\| = \lambda(u_n) = o(1)$, then there exists some integer n_0 such that for each $n \ge n_0$, we have

$$\left|\langle u_n^*, h\rangle\right| \le \|h\| \quad \text{for all } h\in H_T^1.$$

Let $q_n(t)\in\partial(F(t,u_n(t)))$ such that

$$\langle u_n^*, h\rangle = \int_0^T \bigl(\dot u_n(t),\dot h(t)\bigr)\,dt - \int_0^T \bigl(A(t)u_n(t),h(t)\bigr)\,dt + \int_0^T \bigl(q_n(t),h(t)\bigr)\,dt, \quad \forall h\in H_T^1.$$

Firstly, we show that $\{u_n\}$ is bounded. If $\{u_n\}$ is unbounded, without loss of generality we may assume that $\|u_n\|\to\infty$ as $n\to\infty$. Split $u_n = v_n + w_n = v_n + w_n^+ + w_n^- \subset V\oplus W^+\oplus W^-$. It follows from (4) and (7) that

$$\left|\int_0^T \langle q_n(t), w_n^+\rangle\,dt\right| \le \|f\|_{L^1}\|u_n\|_\infty^\alpha \|w_n^+\|_\infty + \|g\|_{L^1}\|w_n^+\|_\infty$$
$$\le C^{\alpha+1}\|f\|_{L^1}\|u_n\|^\alpha\|w_n^+\| + C\|g\|_{L^1}\|w_n^+\|$$

for all n. Hence,

$$\begin{aligned}\|w_n^+\| &\ge \|u_n^*\|\|w_n^+\| \ge \langle u_n^*, w_n^+\rangle\\ &= \int_0^T |\dot w_n^+|^2\,dt - \int_0^T \bigl(A(t)w_n^+, w_n^+\bigr)\,dt + \int_0^T \bigl(q_n(t), w_n^+(t)\bigr)\,dt\\ &\ge \delta\|w_n^+\|^2 - C^{\alpha+1}\|f\|_{L^1}\|u_n\|^\alpha\|w_n^+\| - C\|g\|_{L^1}\|w_n^+\|,\end{aligned}$$

which implies that $\limsup_{n\to\infty}\frac{\|w_n^+\|}{\|u_n\|^\alpha} < +\infty$. In a similar way

$$\left|\int_0^T \bigl(q_n(t), w_n^-(t)\bigr)\,dt\right| \le C^{\alpha+1}\|f\|_{L^1}\|u_n\|^\alpha\|w_n^-\| + C\|g\|_{L^1}\|w_n^-\|$$

for all n. Thus one obtains

$$\begin{aligned}-\|u_n^*\|\,\|w_n^-\| &\leq \langle u_n^*, w_n^-\rangle \\ &= \int_0^T |\dot{w}_n^-|^2\,dt - \int_0^T \bigl(A(t)w_n^-, w_n^-\bigr)\,dt + \int_0^T \bigl(q_n(t), w_n^-(t)\bigr)\,dt \\ &\leq -\delta\|w_n^-\|^2 + C^{\alpha+1}\|f\|_{L^1}\|u_n\|^\alpha\|w_n^-\| + C\|g\|_{L^1}\|w_n^-\|.\end{aligned}$$

This means $\limsup_{n\to\infty}\frac{\|w_n^-\|}{\|u_n\|^\alpha} < +\infty$. Hence we have

$$\limsup_{n\to\infty}\frac{\|w_n\|}{\|u_n\|^\alpha} < +\infty. \tag{12}$$

By the boundedness of $\varphi(u_n)$ and the continuity of $A(\cdot)$, there exists a constant $C_2 \geq 1$ such that

$$\begin{aligned}\int_0^T F(t,u_n)\,dt &= \varphi(u_n) - \frac{1}{2}\int_0^T |\dot{w}_n|^2\,dt + \frac{1}{2}\int_0^T \bigl(A(t)w_n, w_n\bigr)\,dt \\ &\geq -C_2 - \frac{1}{2}\int_0^T |\dot{w}_n|^2\,dt - \frac{1}{2}C_2\int_0^T |w_n|^2\,dt \\ &\geq -C_2 - \frac{1}{2}C_2\|w_n\|^2\end{aligned}$$

for all n. Furthermore, it follows from (12) that

$$\liminf_{n\to\infty}\|u_n\|^{-2\alpha}\int_0^T F(t,u_n)\,dt > -\infty,$$

which contradicts Lemma 3.1. Hence $\{u_n\}$ is bounded in H_T^1, thus there exists an $u \in H_T^1$ such that $u_n \rightharpoonup u$ in H_T^1 and $u_n \to u$ in $C([0,T],\mathbb{R}^N)$, where a subsequence is considered when necessary.

Since H_T^1 is reflexive while $\partial\varphi(u)$ is weak* compact, and the set-valued mapping $u \to \partial\varphi(u)$ is upper semicontinuous, we can find an $u^* \in \partial\varphi(u)$ such that

$$\langle u_n^* - u^*, u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

On the other hand,

$$\begin{aligned}\langle u_n^* - u^*, u_n - u\rangle &= \int_0^T |\dot{u}_n(t) - \dot{u}(t)|^2\,dt \\ &\quad - \int_0^T \bigl(A(t)\bigl(u_n(t) - u(t)\bigr), u_n(t) - u(t)\bigr)\,dt \\ &\quad + \int_0^T \bigl(q_n(t) - q(t), u_n(t) - u(t)\bigr)\,dt,\end{aligned}$$

where $q_n(t) \in \partial F(t,u_n(t))$ and $q(t) \in \partial F(t,u(t))$. From a simple computation we obtain $\int_0^T |\dot{u}_n - \dot{u}|^2\,dt \to 0$ as $n \to \infty$, and hence $u_n \to u$ in H_T^1. Therefore, φ satisfies the nonsmooth (PS) condition. □

Proof of Theorem 1.2 We verify that φ satisfies the other conditions of Lemma 2.2. Firstly we prove that

$$\varphi(u) \to +\infty \quad \text{as } \|u\| \to \infty \text{ in } W^+. \tag{13}$$

By (4), (7), and Lemma 2.1, there exist $s \in [0,1]$ and $\xi \in \partial F(t, su)$ such that

$$\begin{aligned}
&\left|\int_0^T F(t,u)\,dt - \int_0^T F(t,0)\,dt\right| \\
&\quad = \left|\int_0^T \langle \xi, u\rangle\,dt\right| \le \|f\|_{L^1}\|u\|_\infty^\alpha \|u\|_\infty + \|g\|_{L^1}\|u\|_\infty \\
&\quad \le C^{\alpha+1}\|f\|_{L^1}\|u\|^{\alpha+1} + C\|g\|_{L^1}\|u\|
\end{aligned}$$

for all n. Hence we obtain

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,dt - \frac{1}{2}\int_0^T \big(A(t)u(t), u(t)\big)\,dt + \int_0^T F\big(t, u(t)\big)\,dt \\
&\ge \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,dt - \frac{1}{2}\int_0^T \big(A(t)u(t), u(t)\big)\,dt + \int_0^T F(t,0)\,dt \\
&\quad - \left|\int_0^T F\big(t,u(t)\big)\,dt - \int_0^T F(t,0)\,dt\right| \\
&\ge \frac{1}{2}\delta\|u\|^2 - C^{\alpha+1}\|f\|_{L^1}\|u\|^{\alpha+1} - C\|g\|_{L^1}\|u\| + \int_0^T F(t,0)\,dt
\end{aligned}$$

for all $u \in W^+$. Since $\alpha \in [0,1)$, it is clear that (13) holds.

Secondly we show that

$$\varphi(u) \to -\infty \quad \text{as } \|u\| \to \infty \text{ in } W^- \oplus V. \tag{14}$$

Arguing by contradiction, assume that there exist $M \in \mathbb{R}$ and a sequence $\{u_n\} \subset W^- \oplus V$ such that $\|u_n\| \to \infty$ as $n \to \infty$ and

$$\varphi(u_n) \ge M \tag{15}$$

for all n. Write $u_n = v_n + w_n^-$, $v_n \in V$, $w_n^- \in W^-$. We consider the case that $\{u_n\}$ has a subsequence, say $\{u_n\}$, such that $\limsup_{n\to\infty} \frac{\|w_n^-\|}{\|u_n\|^\alpha} < +\infty$. By Lemma 3.1, one has

$$\limsup_{n\to\infty} \|u_n\|^{-2\alpha}\int_0^T F(t,u_n)\,dt = -\infty.$$

Hence

$$\limsup_{n\to\infty} \|u_n\|^{-2\alpha}\varphi(u_n) \le \limsup_{n\to\infty} \|u_n\|^{-2\alpha}\int_0^T F(t,u_n)\,dt = -\infty,$$

which contradicts (15).

Now we consider the case that $\frac{\|w_n^-\|}{\|u_n\|^\alpha} \to +\infty$ as $n \to \infty$, in this case, $W^- \neq \{0\}$ and $\|w_n^-\| \to \infty$ as $n \to \infty$. From (11), one obtains that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T \left|\dot{w}_n^-\right|^2 dt - \frac{1}{2}\int_0^T \left(A(t)w_n^-, w_n^-\right) dt + \int_0^T F(t, u_n)\, dt \\
&\leq -\frac{1}{2}\delta\left\|w_n^-\right\|^2 + \int_0^T F(t, v_n)\, dt + \left|\int_0^T F(t, u_n)\, dt - \int_0^T F(t, v_n)\, dt\right| \\
&\leq -\frac{1}{2}\delta\left\|w_n^-\right\|^2 + C^{\alpha+1}\|f\|_{L^1}\left(\|v_n\|^\alpha + \left\|w_n^-\right\|^\alpha\right)\left\|w_n^-\right\| \\
&\quad + C\|g\|_{L^1}\left\|w_n^-\right\| + \int_0^T F(t, v_n)\, dt
\end{aligned}$$

for all n. It follows from (5) and (7) that

$$\int_0^T F(t, v_n)\, dt \leq C^{2\alpha}\|v_n\|^{2\alpha}\int_0^T \gamma(t)\, dt \leq C^{2\alpha}\|u_n\|^{2\alpha}\int_0^T \gamma(t)\, dt,$$

which implies that

$$\limsup_{n\to\infty} \|u_n\|^{-\alpha}\left\|w_n^-\right\|^{-1}\int_0^T F(t, v_n)\, dt \leq \limsup_{n\to\infty} C^{2\alpha}\frac{\|u_n\|^\alpha}{\|w_n^-\|}\int_0^T \gamma(t)\, dt = 0,$$

hence we obtain

$$\limsup_{n\to\infty} \|u_n\|^{-\alpha}\left\|w_n^-\right\|^{-1}\varphi(u_n) \leq -\frac{1}{2}\delta \liminf_{n\to\infty}\frac{\|w_n^-\|}{\|u_n\|^\alpha} + 2C^{2\alpha+1}\|f\|_{L^1} = -\infty,$$

which contradicts (15) too.

By Lemma 2.2, $\varphi(u)$ has a critical point $u \in H_T^1$. The proof is completed.

If $W^- \oplus V = \{0\}$, the solution of system (1) can be obtained according to Lemma 2.4. □

Proof of Theorem 1.3 Similar to the proof of Lemma 3.2 and Theorem 1.2, we can prove that φ satisfies the nonsmooth (PS) condition, and

$$\varphi(u) \to +\infty \quad \text{as } \|u\| \to \infty \text{ in } W^+ \oplus V.$$

If $W^- = \{0\}$, φ has a minimum by Lemma 2.4. In the case of $W^- \neq \{0\}$, we have

$$\varphi(u) \to -\infty \quad \text{as } \|u\| \to \infty \text{ in } W^-.$$

According to Lemma 2.2, φ has a critical point. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YN wrote the first draft and TA corrected and improved the final version. All authors read and approved the final draft.

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