Di and Shang *Boundary Value Problems* (2015) 2015:29
DOI 10.1186/s13661-015-0288-6

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Global existence and asymptotic behavior of solutions for the double dispersive-dissipative wave equation with nonlinear damping and source terms

Huafei Di[1,2] and Yadong Shang[1,2*]

*Correspondence: gzydshang@126.com
[1]School of Mathematics and Information Science, Guangzhou University, Guangzhou, 510006, China
[2]Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education, Guangzhou, 510006, China

Abstract

In this paper, we consider the initial boundary value problem of the double dispersive-dissipative wave equation with nonlinear damping and source terms. By the combination of the Galerkin method and the monotonicity-compactness method, the existence of global solutions is obtained with the least amount of a priori estimates. Moreover, the asymptotic behavior of global solutions is investigated under some assumptions on the initial data.

MSC: 35L35; 35L82; 35B40

Keywords: nonlinear wave equation; double dispersive-dissipative; global existence; asymptotic behavior; damping and source terms; monotonicity-compactness

1 Introduction

This paper deals with the initial boundary value problem of the double dispersive-dissipative wave equation with nonlinear damping and source terms

$$u_{tt} - \triangle u - \beta \triangle u_t + \gamma \triangle^2 u - \delta \triangle u_{tt} + g(u_t) = f(u), \quad x \in \Omega, t > 0, \tag{1.1}$$

where $\Omega \subset R^n$ is a suitably smooth bounded domain, β, γ, and δ are some physical parameters. $g(s), f(s)$ are given nonlinear functions.

Equation (1.1) includes many important physical models. For example, in the absence of a dissipative term, double dispersive terms ($\beta = \gamma = \delta = 0$), and a damping term $g(u_t) = 0$, the model reduces to the common semilinear wave equation

$$u_{tt} - \triangle u = f(u), \quad x \in \Omega, t > 0. \tag{1.2}$$

In 1968, Sattinger [1] studied the existence of global weak solutions of (1.2) by using the potential well method. From then on, the potential well theory has become one of the most important methods for studying nonlinear evolution equations. In 1975, Payne and Sattinger [2] made the most important and typical work on the potential well method. They proved the finite time blow up of solutions of (1.2). In 2003, Liu [3] studied the initial boundary value problem of (1.2), where $f(u) = |u|^{p-1}u$. He gave some results on the properties of the family of potential wells. Then, by using them, he obtained some results of the existence and nonexistence of global solutions. In a subsequent article [4], Liu gave a threshold result of the global existence and nonexistence of solutions and proved the global existence of solutions with critical initial conditions $J(u_0) = d$. For the semilinear wave equation (1.2) with nonlinear boundary velocity feedback, it is important to cite [5–7] and the references therein. In 1993, Lasiecka and Tataru [5] obtained the uniform decay rates of the energy for (1.2) under the assumption that the boundary velocity feedback is dissipative. In 2002, Vitillaro [6] studied the local existence, blow up, and global existence of the solutions for (1.2) with nonlinear boundary velocity feedback. In 2004, Cavalcanti et al. [7] proved the existence and uniform decay rates of the energy even if the nonlinear boundary velocity feedback has a polynomial growth near the origin.

In the absence of a dissipative term, double dispersive terms ($\beta = \gamma = \delta = 0$), taking the nonlinear damping term $g(u_t) = a|u_t|^{m-1}u_t$ and the nonlinear source term $f(u) = b|u|^{p-1}u$, the model (1.1) reduces to the wave equation with nonlinear damping and source terms

$$u_{tt} - \triangle u + a|u_t|^{m-1}u_t = b|u|^{p-1}u, \quad x \in \Omega, t > 0. \tag{1.3}$$

In 1994, Georgiev and Todorova [8] investigated the initial boundary value problem of (1.3), where $a = b = 1$, $p, m > 1$. They proved the existence of global solutions under the condition $1 < p \le m$. When $p \ge m > 1$, they obtained the finite time blow up of solutions for sufficiently large initial data. In 1996, Ikehata [9] investigated the initial boundary value problem of (1.3), where $b = 1$ and any $a = \delta > 0$. He proved that $1 \le m < p < \infty$ if $n = 1, 2$, and $1 \le m < p \le \frac{n}{n-2}$ if $n \ge 3$, the problem (1.3) had a global solution for sufficiently small initial data. In 1999, Vitillaro [10] proved the global nonexistence of the solutions with positive initial energy. He also gave the application concerning the classical equations of linear elasticity and the damped clamped plate equation. In 2001, Messaoudi [11] changed the nonlinear term $a|u_t|^{m-1}u_t$, $b|u|^{p-1}u$ into $a|u_t|^{m-2}u_t$, $b|u|^{p-2}u$. He proved that any strong solution, with negative initial energy, blows up in finite time under the condition $p > m$. Messaoudi [12, 13] changed the nonlinear term $a|u_t|^{m-1}u_t$, $b|u|^{p-1}u$ into $au_t(1 + |u_t|^{m-2})$, $b|u|^{p-2}u$. He investigated the global nonexistence and exponential decay of solutions, respectively. For the Cauchy problem associated to (1.3), when Ω is replaced by the entire space R^n, Serrin and Todorova [14] studied the existence of the solutions for the case $p < m$.

When taking into account no effects of dispersion caused by transverse shearing and nonlinear damping, in one dimension, the model (1.1) reduces to the fourth-order wave equation

$$u_{tt} - u_{xx} - u_{xxt} - u_{xxtt} = f(u). \tag{1.4}$$

This type of equation was derived by Hayes and Saccomandi [15] in studying the propagation of transverse homogeneous waves in special incompressible viscoelastic solids. In 2000, Shang [16] studied the initial boundary value problem of the fourth-order nonlinear wave equation (1.4). The existence and uniqueness of a global strong solution for the problem were obtained by means of the Galerkin method. The asymptotic behavior and blow up phenomenon of the solution for the problem were discussed under certain conditions.

In the absence of a dispersive term and dissipation ($\beta = \gamma = 0$), and when the damping term is linear, that is, $g(u_t) = u_t$, the model (1.1) becomes the 1D fourth-order wave equation

$$u_{tt} - u_{xx} + u_t - u_{xxtt} = f(u). \tag{1.5}$$

This type of equation was derived in the study of a longitudinal vibration of a bar. The weak damping term of (1.5) is introduced to model the contacting of the bar with a rough substrate or a viscous external medium. The dispersive term of the above equation is used to explain the lateral inertia of a bar [17].

In the absence of the dispersive term and the damping term, that is, $g(u_t) = 0$ and $\gamma = 0$, the model (1.1) reduces to the fourth-order wave equation ($n \geq 1$)

$$u_{tt} - \triangle u - \triangle u_t - \triangle u_{tt} = f(u), \quad x \in \Omega, t > 0. \tag{1.6}$$

In 2000, Shang [18] investigated the existence, uniqueness, asymptotic behavior, and the blow up phenomenon of the solutions under some specific assumptions on f and for $n = 1, 2, 3$. In 2004, Zhang and Hu [19] proved the existence and the stability of global weak solutions. In 2007, Xie and Zhong [20] studied the initial boundary value problem of (1.6). They established the existence of global attractors in $H_0^1(\Omega) \times H_0^1(\Omega)$, where the nonlinear term f satisfies a critical exponential growth condition. In 2008, Xu et al. [21] investigated the asymptotic behavior of solutions for (1.6) by using the multiplier method. For more results on the long-time behaviors of global strong solutions of the initial boundary value problem of (1.6), the reader is referred to [22–25].

If $\gamma = 0$, $\beta = \delta = 1$, $g(u_t) = u_t$ and $f(u) = |u|^{p-1}u$, the model (1.1) reduces to the fourth-order dispersive-dissipative wave equation

$$u_{tt} - \triangle u + \triangle u_t - \triangle u_{tt} + u_t = |u|^{p-2}u, \quad x \in \Omega, t > 0, \tag{1.7}$$

In 2012, Xu and Yang [26] investigated the initial boundary value problem of (1.7), where $1 < p < \infty$ if $n = 1, 2$, and $1 < p \leq \frac{n+2}{n-2}$ if $n \geq 3$. A blow up result for certain solutions with arbitrary positive initial energy was established.

As far as we know, there have been no results up till now on the initial boundary value problem for a nonlinear wave equation with double dispersive terms $\triangle^2 u$, $\triangle u_{tt}$, the strong dissipation term $\triangle u_t$, the nonlinear damping term $g(u_t)$, and the nonlinear source term $f(u)$. So the aim of the present paper is to solve this open problem.

In this work, we investigate the initial boundary value problem for the double dispersive wave equation with a strong dissipation term, a nonlinear damping and source terms:

$$\begin{cases} u_{tt} - \triangle u - \triangle u_t + \triangle^2 u - \triangle u_{tt} + a|u_t|^{m-2}u_t = b|u|^{p-2}u, & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \\ u = 0, \qquad \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \end{cases} \tag{1.8}$$

where $a, b > 0$, $p, m > 2$, ν represents the unit outward normal to $\partial\Omega$, and Ω is a bounded domain of R^n ($n \geq 1$) with a smooth boundary $\partial\Omega$. First of all, by the combination of the Galerkin method and the monotonicity-compactness method, the existence of global solutions is obtained with the least amount of a priori estimates. Moreover, the asymptotic behavior of global solutions is investigated under some assumptions on the initial data.

This paper is organized as follows. In Section 2, we introduce some notation, basic definitions, and important lemmas for proving the main theorem. In Section 3, the existence of global weak solutions is proved by the Galerkin method and the monotonicity-compactness method. In Section 4, we consider the asymptotic behavior of global solutions under some assumptions on the initial data.

2 Preliminaries

In this section, we introduce some notation, basic definitions, and important lemmas which will be needed in this paper.

For functions $u(x,t)$, $v(x,t)$ defined on Ω, we introduce

$$(u,v)=\int_\Omega uv\,dx,\qquad \|u\|_2=\left(\int_\Omega |u|^2\,dx\right)^{\frac{1}{2}},$$
$$\|u\|_p=\left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}},\qquad \|u\|_{H^m}=\left(\sum_{|\alpha|\le m}\|D^\alpha u\|_2^2\right)^{\frac{1}{2}},$$
$$\|u\|_\infty=\operatorname*{ess\,sup}_{x\in\Omega}\bigl|u(x)\bigr|.$$

To obtain the results of this paper, we will introduce the energy function

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2}\|\triangle u\|_2^2+\frac{1}{2}\|\nabla u_t\|_2^2-\frac{b}{p}\|u\|_p^p. \tag{2.1}$$

For simplicity, we define the weak solutions of (1.8) over the interval $[0,T)$, but it is to be understood throughout that T is either infinity or the limit of the existence interval.

Definition 2.1 We say that $u(x,t)$ is a weak solution of the problem (1.8) on the interval $\Omega\times[0,T)$, if $u\in L^\infty(0,T;H_0^2(\Omega)\cap L^p(\Omega))$, $u_t\in L^\infty(0,T;H_0^1(\Omega))\cap L^m(Q_T)$ satisfy the following conditions:

(i) For any $v\in H_0^2(\Omega)\cap L^m(\Omega)$ and a.e. $0\le t<T$, such that

$$\begin{aligned}&(u_{tt},v)+(\nabla u,\nabla v)+(\nabla u_t,\nabla v)+(\triangle u,\triangle v)+(\nabla u_{tt},\nabla v)\\&\quad+\bigl(a|u_t|^{m-2}u_t,v\bigr)=\bigl(b|u|^{p-2}u,v\bigr);\end{aligned} \tag{2.2}$$

(ii) $u(x,0)=u_0(x)$ in $H_0^2(\Omega)\cap L^p(\Omega)$, $u_t(x,0)=u_1$ in $H_0^1(\Omega)\cap L^m(\Omega)$.

Lemma 2.1 *Let $p,m>2$, $a,b>0$, and u be a solution of* (1.8). *Then $E(t)$ is a non-increasing function, that is,*

$$E'(t)\le 0. \tag{2.3}$$

Moreover, the following energy inequality holds:

$$E(t)+\int_0^t\bigl\|\nabla u_t(\tau)\bigr\|_2^2\,d\tau\le E(0). \tag{2.4}$$

Proof Multiplying (1.8) by u_t, and integrating over Ω, using integration by parts and some manipulation as in [29], we obtain (2.3), (2.4) for any regular solutions. These results remain valid for weak solutions by a simple density argument. We also refer the reader to [5, 7] as regards the method of density arguments.

The following lemma is similar to Lemma 6.1 of Chapter ii of [27] with a slight modification. □

Lemma 2.2 *Assume that the function* u *satisfies* $u \in L^\infty(0,T;H_0^2(\Omega))$, $u_t \in L^\infty(0,T;H_0^1(\Omega)) \cap L^m(Q_T)$, $u(x,0) = u_0$, $u_t(x,0) = u_1$, *and further assume that*

$$\begin{aligned} &u_{tt} - \triangle u - \triangle u_t + \triangle^2 u - \triangle u_{tt} = g, \\ &\quad g \in L^2\bigl(0,T;H^{-1}(\Omega)\bigr) + L^{m'}(Q_T), \end{aligned} \tag{2.5}$$

then for any $t \in (0,T)$, *the following inequality holds*:

$$\begin{aligned} &\|\nabla u\|_2^2 + \|u_t\|_2^2 + \|\triangle u\|_2^2 + \|\nabla u_t\|_2^2 + 2\int_0^t \|\nabla u_t\|_2^2 \, d\tau \\ &\quad \geq \|\nabla u_0\|_2^2 + \|u_1\|_2^2 + \|\triangle u_0\|_2^2 + \|\nabla u_1\|_2^2 + 2\int_0^t (g,u_t)\, d\tau. \end{aligned} \tag{2.6}$$

Proof By choosing the continuous piecewise linear function $\theta_m(t)$ and the regular sequence $\rho_n(t)$, we construct the smooth function $v(t) = ((\theta_m u') * \rho_n * \rho_n)\theta_m$. Then, multiplying (2.5) by $v(t)$ and integrating over $Q = \Omega \times (0,T)$, using integration by parts and some manipulation as in Chapter 2 of [27] as $m,n \to \infty$, we can obtain the inequality (2.6). This method of smooth approximations is called the method of mollifiers. We also refer reader to [27, 28] about the method of mollifiers.

We construct an approximate weak solution of the problem (1.8) by the Galerkin method. Let $\{w_j\}$ be the system of base functions of $H_0^2(\Omega) \cap L^m(\Omega)$.

Now suppose that the approximate weak solutions of the problem (1.8) can be written

$$u_l(x,t) = \sum_{j=1}^{l} d_l^j(t) w_j(x), \quad l = 1,2,\ldots. \tag{2.7}$$

According to the Galerkin method, these coefficients $d_l^j(t)$ need to satisfy the following initial value problem of the nonlinear ordinary differential equations:

$$\begin{cases} (u_{ltt}, w_j) + (\nabla u_l, \nabla w_j) + (\nabla u_{lt}, \nabla w_j) + (\triangle u_l, \triangle w_j) + (\nabla u_{ltt}, \nabla w_j) \\ \quad + (a|u_{lt}|^{m-2}u_{lt}, w_j) = (b|u_l|^{p-2}u_l, w_j), \quad x \in \Omega, t > 0, \\ u_l(x,0) = \sum_{j=1}^{l} d_l^j(0) w_j(x) \longrightarrow u_0(x), \quad \text{in } H_0^2(\Omega) \cap L^p(\Omega), \\ u_{lt}(x,0) = \sum_{j=1}^{l} d_l^j(0)' w_j(x) \longrightarrow u_1(x), \quad \text{in } H_0^1(\Omega) \cap L^m(\Omega). \end{cases} \tag{2.8}$$

Under some assumption for the nonlinear terms and a priori estimates in Section 3, we prove that the initial value problem (2.8) of the nonlinear ordinary differential equations has global solutions in the interval $[0,T]$. Furthermore, we show that the solutions of the problem (1.8) can be approximated by the functions $u_l(x,t)$. □

3 The existence of global weak solutions

In this section, we establish the existence of global weak solutions for the problem (1.8).

Theorem 3.1 *Suppose that* $a, b > 0$, $2 < p \leq m < \infty$, $u_0(x) \in H_0^2(\Omega) \cap L^p(\Omega)$, $u_1(x) \in H_0^1(\Omega) \cap L^m(\Omega)$, *Then for any* $T > 0$, *the problem* (1.8) *admits a global weak solution* $u \in L^\infty(0, T; H_0^2(\Omega) \cap L^p(\Omega))$, $u_t \in L^\infty(0, T; H_0^1(\Omega)) \cap L^m(Q_T)$.

Proof Multiplying (2.8) by $d_l^j(t)'$ and summing for $j = 1, \ldots, l$, we have

$$\frac{d}{dt}\left[\frac{1}{2}\|u_{lt}\|_2^2 + \frac{1}{2}\|\nabla u_l\|_2^2 + \frac{1}{2}\|\triangle u_l\|_2^2 + \frac{1}{2}\|\nabla u_{lt}\|_2^2 - \frac{b}{p}\|u_l\|_p^p\right] + \|\nabla u_{lt}\|_2^2 + a\|u_{lt}\|_m^m = 0. \tag{3.1}$$

Integrating (3.1) with respect to t from 0 to t, we find

$$\begin{aligned} &\frac{1}{2}\|u_{lt}\|_2^2 + \frac{1}{2}\|\nabla u_l\|_2^2 + \frac{1}{2}\|\triangle u_l\|_2^2 + \frac{1}{2}\|\nabla u_{lt}\|_2^2 \\ &\quad - \frac{b}{p}\|u_l\|_p^p + \int_0^t \|\nabla u_{lt}\|_2^2\,d\tau + a\int_0^t \|u_{lt}\|_m^m\,d\tau \\ &\quad = \frac{1}{2}\|u_{lt}(0)\|_2^2 + \frac{1}{2}\|\nabla u_l(0)\|_2^2 + \frac{1}{2}\|\triangle u_l(0)\|_2^2 + \frac{1}{2}\|\nabla u_{lt}(0)\|_2^2 - \frac{b}{p}\|u_l(0)\|_p^p, \end{aligned}$$

since

$$u_l(t) = u_l(0) + \int_0^t u_{lt}\,d\tau.$$

By the Minkowski inequality, we have

$$\begin{aligned} &\|u_l(t)\|_p \leq \|u_l(0)\|_p + \int_0^t \|u_{lt}\|_p\,d\tau, \\ &\|u_l(t)\|_p^p \leq 2^p\left[\|u_l(0)\|_p^p + \left(\int_0^t \|u_{lt}\|_p\,d\tau\right)^p\right]. \end{aligned} \tag{3.2}$$

Taking $q = \frac{p}{p-1}$ and considering the Hölder inequality, we get

$$\left(\int_0^t \|u_{lt}\|_p\,d\tau\right)^p \leq \left(\int_0^t 1^q\,d\tau\right)^{\frac{p}{q}} \int_0^t \|u_{lt}\|_p^p\,d\tau = t^{p-1}\int_0^t \|u_{lt}\|_p^p\,d\tau. \tag{3.3}$$

Using the Young inequality, we see that

$$t^{p-1}|u_{lt}|^p \leq \varepsilon\frac{p}{m}|u_{lt}|^m + c(\varepsilon)\frac{m-p}{m}t^\beta, \quad m \geq p, \tag{3.4}$$

where $\beta = (p-1)\frac{m}{m-p}$, $c(\varepsilon) = \varepsilon^{\frac{p}{p-m}}$. Hence, we have

$$t^{p-1}\int_0^t \|u_{lt}\|_p^p\,d\tau \leq \varepsilon\frac{p}{m}\|u_{lt}\|^m + c(\varepsilon)\frac{m-p}{m}t^{\beta+1}|\Omega|. \tag{3.5}$$

From (3.2)-(3.5), we get

$$\begin{aligned}\frac{b}{p}\|u_l(t)\|_p^p &\le \frac{b}{p}2^p\left[\|u_l(0)\|_p^p + \left(\int_0^t \|u_{lt}\|_p\,d\tau\right)^p\right]\\ &= \frac{b}{p}2^p\|u_l(0)\|_p^p + \frac{b}{p}2^p\left(\int_0^t \|u_{lt}\|_p\,d\tau\right)^p\\ &\le \frac{b}{p}2^p\|u_l(0)\|_p^p + \varepsilon\frac{b}{m}2^p\|u_{lt}\|^m + c(\varepsilon)\frac{m-p}{pm}b2^p t^{\beta+1}|\Omega|. \end{aligned}\tag{3.6}$$

Choosing ε such that $\varepsilon\frac{b}{m}2^p = \frac{a}{2}$, we see that

$$\begin{aligned}&\frac{1}{2}\|u_{lt}\|_{H^1}^2 + \frac{1}{2}\|\nabla u_l\|_2^2 + \frac{1}{2}\|\triangle u_l\|_2^2 + \int_0^t \|\nabla u_{lt}\|_2^2\,d\tau + \frac{a}{2}\int_0^t \|u_{lt}\|_m^m\,d\tau\\ &\quad\le \frac{1}{2}\|u_{lt}(0)\|_{H^1}^2 + \frac{1}{2}\|\nabla u_l(0)\|_2^2 + \frac{1}{2}\|\triangle u_l(0)\|_2^2\\ &\qquad - \frac{(2^p-1)b}{p}\|u_l(0)\|_p^p + c(\varepsilon)\frac{m-p}{pm}b2^p t^{\beta+1}|\Omega|\\ &\quad\le M(T). \end{aligned}\tag{3.7}$$

Thus, we find that

$$\|u_{lt}\|_{H^1}^2,\qquad \|\nabla u_l\|_2^2,\qquad \|\triangle u_l\|_2^2,\qquad \int_0^t \|\nabla u_{lt}\|_2^2\,d\tau,\qquad \int_0^t \|u_{lt}\|_m^m\,d\tau$$

are finite and can be controlled by a constant depending on T. Together with (3.6), we find that

$$\|u_l\|_p^p \le M(T),\quad 0\le t\le T.$$

Hence, $u_l \in L^\infty(0,T;H_0^2(\Omega)\cap L^p(\Omega))$, $u_{lt}\in L^\infty(0,T;H_0^1(\Omega))\cap L^m(Q_T)$. We see that

$$\left(a|u_{lt}|^{m-2}u_{lt},u_{lt}\right) = a\|u_{lt}\|_m^m,\qquad \left(b|u_l|^{p-2}u_l,u_l\right) = b\|u\|_p^p,$$

so we obtain $a|u_{lt}|^{m-2}u_{lt}\in L^{m'}(Q_T)$, $b|u_l|^{p-2}u_l \in L^\infty(0,T;L^{p'}(\Omega))$.

Thus, we can show that there exist a subsequence $\{u_\nu\}$ from the sequences $\{u_l\}$ and limit functions u, ξ, η, such that

$$\begin{aligned}&u_\nu \longrightarrow u \quad \text{in } L^\infty\big(0,T;H_0^2(\Omega)\cap L^p(\Omega)\big) \text{ weakly star}, \nu\longrightarrow\infty,\\ &u_\nu \longrightarrow u \quad \text{a.e. in } Q=\Omega\times[0,T), \nu\longrightarrow\infty,\\ &u_{\nu t}\longrightarrow u_t \quad \text{in } L^\infty\big(0,T;H_0^1(\Omega)\big) \text{ weakly star}, \nu\longrightarrow\infty,\\ &u_{\nu t}\longrightarrow u_t \quad \text{in } L^m(Q_T) \text{ weakly}, \nu\longrightarrow\infty,\\ &a|u_{\nu t}|^{m-2}u_{\nu t}\longrightarrow \xi \quad \text{in } L^{m'}(Q_T) \text{ weakly}, \nu\longrightarrow\infty,\\ &b|u_\nu|^{p-2}u_\nu \longrightarrow \eta \quad \text{in } L^\infty\big(0,T;L^{p'}(\Omega)\big) \text{ weakly star}, \nu\longrightarrow\infty. \end{aligned}$$

Making use of the Lions lemma in [27], it follows that $b|u_l|^{p-2}u_l \longrightarrow b|u|^{p-2}u = \eta$.

Integrating (2.8) with respect to t, we get

$$\begin{aligned}&(u_{lt}, w_j) + (\nabla u_{lt}, \nabla w_j) + \int_0^t (\nabla u_l, \nabla w_j)\,d\tau \\ &\quad + (\nabla u_l, \nabla w_j) + \int_0^t (\triangle u_l, \triangle w_j)\,d\tau + a\int_0^t \big(|u_{lt}|^{m-2}u_{lt}, w_j\big)\,d\tau \\ &\quad = b\int_0^t \big(|u_l|^{p-2}u_l, w_j\big)\,d\tau + \big(u_{lt}(0), w_j\big) + \big(\nabla u_{lt}(0), \nabla w_j\big) + \big(\nabla u_l(0), \nabla w_j\big).\end{aligned} \tag{3.8}$$

Taking $l = \nu \to \infty$ in (3.8), we have

$$\begin{aligned}&(u_t, w_j) + (\nabla u_t, \nabla w_j) + \int_0^t (\nabla u, \nabla w_j)\,d\tau \\ &\quad + (\nabla u, \nabla w_j) + \int_0^t (\triangle u, \triangle w_j)\,d\tau + \int_0^t (\xi, w_j)\,d\tau \\ &\quad = b\int_0^t \big(|u|^{p-2}u, w_j\big)\,d\tau + (u_1, w_j) + (\nabla u_1, \nabla w_j) + (\nabla u_0, \nabla w_j),\end{aligned} \tag{3.9}$$

and consequently, differentiating (3.9) with respect to t, we deduce

$$\begin{aligned}&(u_{tt}, w_j) + (\nabla u, \nabla w_j) + (\nabla u_t, \nabla w_j) + (\triangle u, \triangle w_j) \\ &\quad + (\nabla u_{tt}, \nabla w_j) + (\xi, w_j) = \big(b|u|^{p-2}u, w_j\big).\end{aligned} \tag{3.10}$$

Considering that the basis $\{w_j(x)\}_{j=1}^{\infty}$ is dense in $H_0^2(\Omega) \cap L^m(\Omega)$, we choose a function $v \in H_0^2(\Omega) \cap L^m(\Omega)$ having the form $v = \sum_{j=1}^{\infty} d_j w_j(x)$, where $\{d_j\}_1^{\infty}$ are given functions. Multiplying (3.10) by d_j, summing for $j = 1, \ldots,$ then we deduce

$$\begin{aligned}&(u_{tt}, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) + (\triangle u, \triangle v) + (\nabla u_{tt}, \nabla v) + (\xi, v) \\ &\quad = \big(b|u|^{p-2}u, v\big), \quad \forall v \in H_0^2(\Omega) \cap L^m(\Omega).\end{aligned}$$

Next, we need to prove that

$$\xi = a|u_t|^{m-2}u_t. \tag{3.11}$$

In fact, from (2.8) we obtain

$$\begin{aligned}&\frac{1}{2}\|u_{lt}\|_{H^1}^2 + \frac{1}{2}\|\nabla u_l\|_2^2 + \frac{1}{2}\|\triangle u_l\|_2^2 + \int_0^t \|\nabla u_{lt}\|_2^2\,d\tau + \int_0^t \big(\gamma(u_{lt}), u_{lt}\big)\,d\tau \\ &\quad = \frac{1}{2}\big\|u_{lt}(0)\big\|_{H^1}^2 + \frac{1}{2}\big\|\nabla u_l(0)\big\|_2^2 + \frac{1}{2}\big\|\triangle u_l(0)\big\|_2^2 + \int_0^t \big(b|u_l|^{p-2}u_l, u_{lt}\big)\,d\tau,\end{aligned}$$

where $\gamma(u_{lt}) = a|u_{lt}|^{m-2}u_{lt}$. From what has been discussed above, taking $l \longrightarrow \infty$, we have

$$\begin{aligned}&\frac{1}{2}\|u_t\|_{H^1}^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\triangle u\|_2^2 + \int_0^t \|\nabla u_t\|_2^2\,d\tau + \liminf \int_0^t \big(\gamma(u_{lt}), u_{lt}\big)\,d\tau \\ &\quad \le \frac{1}{2}\|u_1\|_{H^1}^2 + \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|\triangle u_0\|_2^2 + \int_0^t \big(b|u|^{p-2}u, u_t\big)\,d\tau.\end{aligned} \tag{3.12}$$

Combining (3.12) and Lemma 2.2, we deduce

$$\int_0^t (g, u_t)\,d\tau \le \int_0^t \big(b|u|^{p-2}u, u_t\big)\,d\tau - \liminf \int_0^t \big(\gamma(u_{lt}), u_{lt}\big)\,d\tau. \tag{3.13}$$

Noting that $m' = \frac{m}{m-1} < \frac{p}{p-1} = p'$, it follows that $b|u|^{p-2}u \in L^\infty(0,T;L^{p'}(\Omega)) \subset L^{m'}(Q_T)$. From Lemma 2.2, we know that the function $g \in L^2(0,T;H^{-1}(\Omega)) + L^{m'}(Q_T)$. Hence, we can choose $g = b|u|^{p-2}u - \xi$ such that

$$\liminf \int_0^t \big(\gamma(u_{lt}), u_{lt}\big)\,d\tau \le \int_0^t (\xi, u_t)\,d\tau. \tag{3.14}$$

Considering $u_{vt} \to u_t$ weakly in $L^m(Q_T)$ and $a|u_{vt}|^{m-2}u_{vt} \to \xi$ weakly in $L^{m'}(Q_T)$, $\forall\varphi \in L^m(Q_T)$, we have

$$\lim \int_0^t \big(\gamma(u_{lt}), -\varphi\big)\,d\tau = \int_0^t (\xi, -\varphi)\,d\tau, \tag{3.15}$$

$$\lim \int_0^t \big(-\gamma(\varphi), u_{lt} - \varphi\big)\,d\tau = \int_0^t \big(-\gamma(\varphi), u_t - \varphi\big)\,d\tau. \tag{3.16}$$

By the combination of (3.14), (3.15), and (3.16), it follows that

$$\liminf \int_0^t \big(\gamma(u_{lt}) - \gamma(\varphi), u_{lt} - \varphi\big)\,d\tau \le \int_0^t \big(\xi - \gamma(\varphi), u_t - \varphi\big)\,d\tau. \tag{3.17}$$

Utilizing the monotonicity of the function $\gamma(s) = a|s|^{m-2}s$, it means that

$$\big(\gamma(u_{lt}) - \gamma(\varphi), u_{lt} - \varphi\big) = a(m-1)\int_\Omega |u_{lt} + \theta\varphi|^{m-2}|u_{lt} - \varphi|^2\,dx \ge 0, \tag{3.18}$$

where $0 < \theta < 1$. Thus, from (3.17) and (3.18), we have

$$\int_0^t \big(\xi - \gamma(\varphi), u_t - \varphi\big)\,d\tau \ge 0.$$

Taking $t \longrightarrow T$, we obtain

$$\int_0^T \big(\xi - \gamma(\varphi), u_t - \varphi\big)\,d\tau \ge 0. \tag{3.19}$$

In order to prove (3.11) from (3.19), we use the semi-continuity of the function $\gamma(s)$ $(s \in R)$. Let $\varphi = u_t - \lambda w_t$, $\lambda > 0$, and $\forall w_t \in L^\infty(0,T;H_0^1(\Omega)) \cap L^m(Q_T)$, then

$$\lambda \int_0^T \big(\xi - \gamma(u_t - \lambda w_t), w_t\big)\,d\tau \ge 0,$$

and

$$\int_0^T \big(\xi - \gamma(u_t - \lambda w_t), w_t\big)\,d\tau \ge 0.$$

Passing to the limits as $\lambda \longrightarrow 0$, we obtain

$$\int_0^T \big(\xi - \gamma(u_t), w_t\big)\, d\tau \geq 0, \quad \forall w_t \in L^\infty\big(0,T;H_0^1(\Omega)\big) \cap L^m(Q_T). \tag{3.20}$$

In a similar way, let $\varphi = u_t - \lambda w_t$, $\lambda < 0$, and $\forall w_t \in L^\infty(0,T;H_0^1(\Omega)) \cap L^m(Q_T)$, then

$$\int_0^T \big(\xi - \gamma(u_t), w_t\big)\, d\tau \leq 0, \quad \forall w_t \in L^\infty\big(0,T;H_0^1(\Omega)\big) \cap L^m(Q_T). \tag{3.21}$$

From (3.20), (3.21), we see that $\xi = a|u_t|^{m-2}u_t$. Thus, the theorem is completed. □

4 The asymptotic behavior of global weak solutions

In this section, we consider the asymptotic behavior of global weak solutions for the problem (1.8).

To obtain the results of this section, we now define some functionals as follows:

$$\begin{aligned} I\big(u(t)\big) &= \|\nabla u\|_2^2 + \|\triangle u\|_2^2 - b\|u\|_p^p, \\ J\big(u(t)\big) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\triangle u\|_2^2 - \frac{b}{p}\|u\|_p^p, \end{aligned}$$

and

$$\begin{aligned} E(t) = E\big(u(t)\big) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + J\big(u(t)\big) \\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\triangle u\|_2^2 - \frac{b}{p}\|u\|_p^p. \end{aligned}$$

Next, let us introduce the set

$$W = \big\{u \in H_0^2(\Omega) | I\big(u(t)\big) > 0\big\} \cup \{0\}.$$

Lemma 4.1 *Let $(u_0, u_1) \in W \times H_0^1(\Omega)$ be given. Assume that $a, b > 0$ and*

$$2 < p \leq m < \infty \quad \text{if } n = 1,2 \quad \text{and} \quad 2 < p \leq m \leq \frac{2n}{n-2} \quad \text{if } n \geq 3, \tag{4.1}$$

$$\alpha = bC_{*p}^p\left[\frac{2p}{p-2}E(0)\right]^{\frac{p-2}{2}} < 1, \tag{4.2}$$

where C_{*p} is the Sobolev constant of the space $H_0^1(\Omega)$ into $L^p(\Omega)$. Then for any $t \geq 0$, the global weak solutions of the problem (1.8) *satisfy $u(t) \in W$.*

Proof Since $I(u_0) > 0$ and the time continuity of $I(u)$, then there exists t_1 such that $I(u(t)) \geq 0$ for all $t \in [0, t_1]$. Thus, we see that

$$\begin{aligned} J\big(u(t)\big) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\triangle u\|_2^2 - \frac{b}{p}\|u\|_p^p \\ &= \frac{1}{2}\big[\|\nabla u\|_2^2 + \|\triangle u\|_2^2\big] - \frac{1}{p}\big[\|\nabla u\|_2^2 + \|\triangle u\|_2^2\big] + \frac{1}{p}I\big(u(t)\big) \\ &\geq \frac{p-2}{2p}\big[\|\nabla u\|_2^2 + \|\triangle u\|_2^2\big], \end{aligned} \tag{4.3}$$

for any $t \in [0, t_1]$. Hence, we obtain

$$\|\nabla u\|_2^2 + \|\triangle u\|_2^2 \leq \frac{2p}{p-2} J(t) \leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2} E(0), \tag{4.4}$$

for any $t \in [0, t_1]$. If $\|\nabla u\|_2^2 = 0$, we have $u = 0$ (since $u \in H_0^2(\Omega)$). Hence, we have from the definition of the set W: $u \in W$. If $\|\nabla u\|_2^2 \neq 0$, by the Sobolev inequality and (4.2), (4.4), we have

$$\begin{aligned} b\|u\|_p^p &\leq bC_{*p}^p \|\nabla u\|_2^p = bC_{*p}^p \|\nabla u\|_2^{p-2} \|\nabla u\|_2^2 \\ &\leq bC_{*p}^p \left[\frac{2p}{p-2} E(0)\right]^{\frac{p-2}{2}} \|\nabla u\|_2^2 = \alpha \|\nabla u\|_2^2 < \|\nabla u\|_2^2, \end{aligned} \tag{4.5}$$

for any $t \in [0, t_1]$. Hence, we obtain $\|\nabla u\|_2^2 - b\|u\|_p^p > 0$, $\forall t \in [0, t_1]$. This shows that $u(t) \in W$, $\forall t \in [0, t_1]$. We see that

$$\lim_{t \longrightarrow t_1} bC_{*p}^p \left[\frac{2p}{p-2} E(t)\right]^{\frac{p-2}{2}} \leq \alpha < 1, \tag{4.6}$$

so the above argument may be repeated, and the solution can thus be extended to the time $t_1 \leq t < t_2$. Continuing in this way, the assertion of the lemma is proved. □

Lemma 4.2 *Assume that* (4.1) *and* (4.2) *hold. Then the global weak solutions of the problem* (1.8) *satisfy*

$$\|u(t)\|_m^m \leq LE(t), \tag{4.7}$$

for some constant L depending on m, p and $E(0)$ *only.*

Proof Using the Sobolev inequality and (4.4), we have

$$\begin{aligned} \|u(t)\|_m^m &\leq C_{*m}^m \|\nabla u\|_2^m = C_{*m}^m \|\nabla u\|_2^{m-2} \|\nabla u\|_2^2 \\ &\leq C_{*m}^m \left[\frac{2p}{p-2} E(0)\right]^{\frac{m-2}{2}} \frac{2p}{p-2} E(t) \leq LE(t), \end{aligned} \tag{4.8}$$

where $L = C_{*m}^m [\frac{2p}{p-2} E(0)]^{\frac{m-2}{2}} \frac{2p}{p-2}$, and C_{*m} is the Sobolev constant of the space $H_0^1(\Omega)$ into $L^m(\Omega)$, and the proof of the lemma is completed. □

Theorem 4.1 *Assume that* $a, b > 0$, *and the conditions* (4.1), (4.2) *hold. Let* $(u_0, u_1) \in W \times H_0^1(\Omega)$ *be given. Then for the global weak solutions of the problem* (1.8), *there exist positive constants M and k such that*

$$E(t) \leq Me^{-kt}, \quad \forall t \geq 0. \tag{4.9}$$

Proof From Lemma 4.1, we know that, for any $t \geq 0$, the global weak solutions of the problem (1.8) satisfy $u(t) \in W$. Now defining

$$F(t) = E(t) + \varepsilon \int_\Omega u u_t \, dx + \varepsilon \int_\Omega \nabla u \nabla u_t \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla u|^2 \, dx, \tag{4.10}$$

we can show that for ε small enough, there exist two positive constants C_1 and C_2 such that

$$C_1E(t) \le F(t) \le C_2E(t). \tag{4.11}$$

In fact,

$$\begin{aligned} F(t) &\le E(t) + \frac{\varepsilon}{2}\|u_t\|_2^2 + \frac{\varepsilon}{2}\|u\|_2^2 + \frac{\varepsilon}{2}\|\nabla u_t\|_2^2 + \varepsilon\|\nabla u\|_2^2 \\ &\le (1+\varepsilon)E(t) + \varepsilon C(\Omega)\|\nabla u\|_2^2 \\ &\le (1+\varepsilon)E(t) + \varepsilon\frac{2p}{p-2}E(t) \\ &\le C_2E(t) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} F(t) &\ge E(t) - \frac{\varepsilon}{4\gamma}\|u_t\|_2^2 - \varepsilon\gamma\|u\|_2^2 + \frac{\varepsilon}{2}\|\nabla u\|_2^2 - \frac{\varepsilon}{4\gamma}\|\nabla u_t\|_2^2 - \varepsilon\gamma\|\nabla u\|_2^2 \\ &\ge E(t) - \frac{\varepsilon}{4\gamma}\bigl[\|u_t\|_2^2 + \|\nabla u_t\|_2^2\bigr] + \varepsilon\left(\frac{1}{2} - \gamma - \gamma C_{*2}\right)\|\nabla u\|_2^2, \end{aligned} \tag{4.13}$$

where C_{*2} is the Sobolev constant of the space $H_0^1(\Omega)$ into $L^2(\Omega)$.

By choosing γ small enough, we have

$$\begin{aligned} F(t) &\ge E(t) - \frac{\varepsilon}{4\gamma}\bigl[\|u_t\|_2^2 + \|\nabla u_t\|_2^2\bigr] \\ &\ge J\bigl(u(t)\bigr) + \left(\frac{1}{2} - \frac{\varepsilon}{4\gamma}\right)\|u_t\|_2^2 + \left(\frac{1}{2} - \frac{\varepsilon}{4\gamma}\right)\|\nabla u_t\|_2^2. \end{aligned} \tag{4.14}$$

Once γ is chosen, we take ε so small that

$$F(t) \ge J\bigl(u(t)\bigr) + \frac{C_1}{2}\|u_t\|_2^2 + \frac{C_1}{2}\|\nabla u_t\|_2^2 \ge C_1E(t), \tag{4.15}$$

where $\frac{C_1}{2} \le \frac{1}{2} - \frac{\varepsilon}{4\gamma}$. Now differentiating (4.10) and utilizing (1.8), Lemma 2.1, and the Poincaré inequality, we have

$$\begin{aligned} F'(t) &= E'(t) + \varepsilon\int_\Omega uu_{tt}\,dx + \varepsilon\|u_t\|_2^2 + \varepsilon\int_\Omega \nabla u\nabla u_{tt}\,dx + \varepsilon\|\nabla u_t\|_2^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|\nabla u\|_2^2 \\ &= -\bigl(a\|u_t\|_m^m + \|\nabla u_t\|_2^2\bigr) + \varepsilon\bigl[\|u_t\|_2^2 + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 - \|\triangle u\|_2^2\bigr] \\ &\quad - a\varepsilon\int_\Omega |u_t|^{m-2}u_tu\,dx + \varepsilon b\|u\|_p^p \\ &\le -a\|u_t\|_m^m - [1 - C_{*2}\varepsilon - \varepsilon]\|\nabla u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 - \varepsilon\|\triangle u\|_2^2 \\ &\quad - a\varepsilon\int_\Omega |u_t|^{m-2}u_tu\,dx + \varepsilon b\|u\|_p^p. \end{aligned} \tag{4.16}$$

Using the energy functionals and the Sobolev inequality, we have

$$\begin{aligned} b\|u\|_p^p &= \lambda b\|u\|_p^p + (1-\lambda)b\|u\|_p^p \\ &\le \lambda\left[\frac{p}{2}\|u_t\|_2^2 + \frac{p}{2}\|\nabla u_t\|_2^2 + \frac{p}{2}\|\nabla u\|_2^2 + \frac{p}{2}\|\triangle u\|_2^2\right] \\ &\quad - p\lambda E(t) + (1-\lambda)\alpha\|\nabla u\|_2^2, \end{aligned} \tag{4.17}$$

where $\alpha = bC_{*p}^p[\frac{2p}{p-2}E(0)]^{\frac{p-2}{2}} < 1$.

Utilizing Lemma 4.2 and inserting (4.17) into (4.16), we have

$$\begin{aligned} F'(t) &\le -a\|u_t\|_m^m - [1 - C_{*2}\varepsilon - \varepsilon]\|\nabla u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 - \varepsilon\|\triangle u\|_2^2 \\ &\quad + \varepsilon\lambda\left[\frac{p}{2}\|u_t\|_2^2 + \frac{p}{2}\|\nabla u_t\|_2^2 + \frac{p}{2}\|\nabla u\|_2^2 + \frac{p}{2}\|\triangle u\|_2^2\right] \\ &\quad - p\varepsilon\lambda E(t) + (1-\lambda)\alpha\|\nabla u\|_2^2\,dx + a\varepsilon\delta\int_\Omega |u|^m\,dx + C(\delta)a\varepsilon\int_\Omega |u_t|^m\,dx \\ &\le -a\|u_t\|_m^m - [1 - C_{*2}\varepsilon - \varepsilon]\|\nabla u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 - \varepsilon\|\triangle u\|_2^2 \\ &\quad + \varepsilon\lambda\left[\frac{p}{2}\|u_t\|_2^2 + \frac{p}{2}\|\nabla u_t\|_2^2 + \frac{p}{2}\|\nabla u\|_2^2 + \frac{p}{2}\|\triangle u\|_2^2\right] - p\varepsilon\lambda E(t) \\ &\quad + (1-\lambda)\alpha\|\nabla u\|_2^2\,dx + a\varepsilon\delta LE(t) + a\varepsilon C(\delta)\bigl[\|u_t\|_m^m + \|\nabla u_t\|_2^2\bigr] \\ &\le -a\bigl[1 - \varepsilon C(\delta)\bigr]\|u_t\|_m^m - \varepsilon[\lambda p - a\delta L]E(t) \\ &\quad + \varepsilon\left[(1-\lambda)\alpha + \frac{p}{2}\lambda - 1\right]\|\nabla u\|_2^2 + \varepsilon\left[\frac{p}{2}\lambda - 1\right]\|\triangle u\|_2^2 \\ &\quad - \left\{1 - \varepsilon\left[C_{*2} + 1 + C(\delta)a + \lambda\frac{p}{2}C_{*2} + \lambda\frac{p}{2}\right]\right\}\|\nabla u_t\|_2^2, \end{aligned} \tag{4.18}$$

where L is the constant of Lemma 4.2, δ is any positive constant, and $C(\delta)$ is a constant depending on δ, m only.

Thus, we see that

$$\begin{aligned} F'(t) &\le -a\bigl[1 - \varepsilon C(\delta)\bigr]\|u_t\|_m^m - \varepsilon[\lambda p - a\delta L]E(t) \\ &\quad + \varepsilon\left[(1-\lambda)\alpha + \frac{p}{2}\lambda - 1\right]\|\triangle u\|_2^2 + \varepsilon\left[(1-\lambda)\alpha + \frac{p}{2}\lambda - 1\right]\|\nabla u\|_2^2 \\ &\quad - \left\{1 - \varepsilon\left[C_{*2} + 1 + C(\delta)a + \lambda\frac{p}{2}C_{*2} + \lambda\frac{p}{2}\right]\right\}\|\nabla u_t\|_2^2 \\ &\le -\left\{1 - \varepsilon\left[C_{*2} + 1 + C(\delta)a + \lambda\frac{p}{2}C_{*2} + \lambda\frac{p}{2}\right]\right\}\|\nabla u_t\|_2^2 - a\bigl[1 - \varepsilon C(\delta)\bigr]\|u_t\|_m^m \\ &\quad - \varepsilon[\lambda p - a\delta L]E(t) + \varepsilon\left[\frac{p-2}{2}\lambda - (1-\alpha)(1-\lambda)\right]\|\triangle u\|_2^2 \\ &\quad + \varepsilon\left[\frac{p-2}{2}\lambda - (1-\alpha)(1-\lambda)\right]\|\nabla u\|_2^2. \end{aligned} \tag{4.19}$$

By choosing λ close to zero such that $\frac{p-2}{2}\lambda - (1-\alpha)(1-\lambda) \leq 0$, then (4.19) becomes

$$F'(t) \leq -\left\{1 - \varepsilon\left[C_{*2} + 1 + C(\delta)a + \lambda\frac{p}{2}C_{*2} + \lambda\frac{p}{2}\right]\right\}\|\nabla u_t\|_2^2 - \varepsilon(\lambda p - a\delta L)E(t) - a\big[1 - \varepsilon C(\delta)\big]\|u_t\|_m^m. \tag{4.20}$$

Once δ is chosen such that $\lambda p - a\delta L > 0$, we can take ε so small that

$$1 - \varepsilon\left[C_{*2} + 1 + C(\delta)a + \lambda\frac{p}{2}C_{*2} + \lambda\frac{p}{2}\right] \geq 0, \qquad 1 - \varepsilon C(\delta) \geq 0.$$

Thus, we see that

$$F'(t) \leq -\varepsilon(\lambda p - a\delta L)E(t) \leq -\frac{\varepsilon(\lambda p - a\delta L)}{C_2}F(t). \tag{4.21}$$

By the Gronwall inequality, we see that

$$F(t) \leq F(0)e^{-kt}, \quad \forall t \geq 0, \tag{4.22}$$

where $k = \frac{\varepsilon(\lambda p - a\delta L)}{C_2}$. Combining with (4.11), we obtain

$$C_1E(t) \leq F(t) \leq F(0)e^{-kt}, \quad \forall t \geq 0,$$

and

$$E(t) \leq Me^{-kt}, \quad \forall t \geq 0, \tag{4.23}$$

where $M = \frac{F(0)}{C_1}$. Thus, the proof of the theorem is completed. □

Remark 4.1 From the (2.3), (4.4), (4.5), (4.9), we easily obtain

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 + \|\triangle u\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_p^p \leq Ce^{-kt}, \tag{4.24}$$

for any $t \geq 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally and read and approved the final manuscript.

Acknowledgements
This work is supported by the NSF of China (11401122, 40890153) and the Scientific Program (2008B080701042) of Guangdong Province.

Received: 12 September 2014 Accepted: 14 January 2015 Published online: 11 February 2015

References

1. Sattinger, DH: On global solution of nonlinear hyperbolic equations. Arch. Ration. Mech. Anal. **30**, 148-172 (1968)
2. Payne, LE, Sattinger, DH: Saddle points and instability on nonlinear hyperbolic equations. Isr. J. Math. **22**, 273-303 (1975)
3. Liu, YC: On potential wells and vacuum isolating of solutions for semilinear wave equation. J. Differ. Equ. **1992**, 155-169 (2003)
4. Liu, YC: On potential wells and applications to semilinear hyperbolic equations and parabolic equations. Nonlinear Anal. **64**, 2665-2678 (2006)
5. Lasiecka, I, Tataru, D: Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. Differ. Integral Equ. **6**(3), 507-533 (1993)
6. Vitillaro, E: A potential well method for the wave equation with nonlinear source and boundary damping terms. Glasg. Math. J. **44**, 375-395 (2002)
7. Cavalcanti, MM, Cavalcanti, VND, Martinez, P: Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term. J. Differ. Equ. **203**(1), 119-158 (2004)
8. Georgiev, V, Todorova, G: Existence of a solution of the wave equation with nonlinear damping and source terms. J. Differ. Equ. **109**, 295-308 (1994)
9. Ikehata, R: Some remarks on the wave equations with nonlinear damping and source terms. Nonlinear Anal. TMA **27**, 1165-1175 (1996)
10. Vitillaro, E: Global nonexistence theorems for a class of evolution equations with dissipation. Arch. Ration. Mech. Anal. **149**, 155-182 (1999)
11. Messaoudi, SA: Blow up in a nonlinear damped wave equation. Math. Nachr. **231**, 105-111 (2001)
12. Messaoudi, SA: Global nonexistence in a nonlinearly damped wave equation. Appl. Anal. **80**, 269-277 (2001)
13. Messaoudi, SA: Exponential decay of solutions of a nonlinearly damped wave equation. Nonlinear Differ. Equ. Appl. **12**, 391-399 (2005)
14. Serrin, J, Todorova, G, Vitillaro, E: Existence for a nonlinear wave equation with nonlinear damping and source terms. Differ. Integral Equ. **16**(1), 13-50 (2003)
15. Hayes, M, Saccomandi, G: Finite amplitude transverse waves in special incompressible viscoelastic solids. J. Elast. **59**, 213-225 (2000)
16. Shang, YD: Initial-boundary value problem for a class of fourth order nonlinear evolution equations. Math. Appl. **13**(1), 7-11 (2000)
17. Villiaggio, P: Mathematical Models for Elastic Structures. Cambridge University Press, Cambridge (1997)
18. Shang, YD: Initial boundary value problem of equation $u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u)$. Acta Math. Appl. Sin. **23**, 385-393 (2000)
19. Zhang, HW, Hu, QY: Existence of global weak solution and stability of a class nonlinear evolution equation. Acta Math. Sci. **24A**, 329-336 (2004)
20. Xie, YQ, Zhong, CK: The existence of global attractors for a class nonlinear evolution equation. J. Math. Anal. Appl. **336**, 54-69 (2007)
21. Xu, RZ, Zhao, XR, Shen, JH: Asymptotic behavior of solution for fourth order wave equation with dispersive and dissipative terms. Appl. Math. Mech. -Engl. Ed. **29**, 259-262 (2008)
22. Xie, YQ, Zhong, CK: Asymptotic behavior of a class of nonlinear evolution equations. Nonlinear Anal. **71**(11), 5095-5105 (2009)
23. Carvalho, AN, Cholewa, JW: Local well posedness, asymptotic behavior and asymptotic bootstrapping for a class of semilinear evolution equations of the second order in time. Trans. Am. Math. Soc. **361**(5), 2567-2586 (2009)
24. Sun, CY, Yang, L, Duan, JQ: Asymptotic behavior for a semilinear second order evolution equation. Trans. Am. Math. Soc. **363**(11), 6085-6109 (2011)
25. Chen, CS, Wang, H, Zhu, SL: Global attractor and decay estimates of solutions to a class of nonlinear evolution equations. Math. Methods Appl. Sci. **34**(5), 497-508 (2011)
26. Xu, RZ, Yang, YB: Finite time blow up for the nonlinear fourth-order dispersive-dissipative wave equation at high energy level. Int. J. Math. **23**, 1250060 (10 pages) (2012)
27. Lions, JL: Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaires. Dunod Gauthier-Villars, Paris (1969)
28. Evance, LC: Partial Differential Equations. Am. Math. Soc., Providence (1988)
29. Messaoudi, SA: Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation. J. Math. Anal. Appl. **320**, 902-915 (2006)