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Triple positive solutions of fourth-order impulsive differential equations with integral boundary conditions

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Abstract

By using Leggett-Williams? fixed point theorem and Hölder?s inequality, the existence of three positive solutions for the fourth-order impulsive differential equations with integral boundary conditions $x^{(4)}(t) = \omega(t)f(t,x(t))$, $0 < t < 1$, $t \neq t_k$, $\Delta x|_{t=t_k} = I_k(t_k,x(t_k))$, $\Delta x'|_{t=t_k} = 0$, $k = 1,2,\ldots,m$, $x(0) = \int_0^1 g(s)x(s)\,ds$, $x'(1) = 0$, $x''(0) = \int_0^1 h(s)x''(s)\,ds$, $x'''(1) = 0$ is considered, where $\omega(t)$ is L^p-integrable. Our results cover a fourth-order boundary value problem without impulsive effects and are compared with some recent results.

Keywords: triple positive solutions; impulsive differential equations; integral boundary conditions; Leggett-Williams? fixed point theorem; Hölder?s inequality

1 Introduction

Impulsive differential equations occur in many applications. Various mathematical models, such as population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, *etc.*, can be expressed by differential equations with impulses. Therefore, the study of impulsive differential equations has gained prominence and it is a rapidly growing field; see [1–22] and the references therein. We note that the difficulties dealing with such problems are that theirs states are discontinuous. Therefore, the results of impulsive differential equations, especially for higher-order impulsive differential equations, are fewer in number than those of differential equations without impulses.

At the same time, owing to its importance in modeling the stationary states of the deflection of an elastic beam, fourth-order boundary value problems have attracted much attention from many authors; see, for example [23–53] and the references therein. In particular, we would like to mention some results of Yang [28], Anderson and Avery [31], and Zhang *et al.* [36]. In [28], Yang considered the following fourth-order two-point boundary value problem:

$$\begin{cases} x^{(4)}(t) = g(t)f(x(t)), & 0 \le t \le 1, \\ x(0) = x'(0) = x''(1) = x'''(1) = 0. \end{cases}$$

By using Krasnoselskii?s fixed point theorem, the author established some new estimates to the positive solutions to the above problem and obtained some sufficient conditions for the existence of at least one positive solution.

In [31], Anderson and Avery considered the following fourth-order four-point right focal boundary value problem:

$$\begin{cases} -x^{(4)}(t) = f(x(t)), & t \in [0,1], \\ x(0) = x'(q) = x''(r) = x'''(1) = 0, \end{cases}$$

where $0 < q < r < 1$ are two constants, $f : R \to R$ is continuous and $f(x) \geq 0$ for $x \geq 0$. By using the five functionals fixed point theorem, the authors gave sufficient conditions for the existence of three positive solutions of above problem.

Recently, Zhang *et al.* [36] studied the existence of positive solutions of the following fourth-order boundary value problem with integral boundary conditions:

$$\begin{cases} x^{(4)}(t) - \lambda f(t, x(t)) = \theta, & 0 < t < 1, \\ x(0) = x(1) = \int_0^1 g(s)x(s)\,ds, \\ x''(0) = x''(1) = \int_0^1 h(s)x(s)\,ds, \end{cases}$$

where θ is the zero element of E.

However, to the best of our knowledge, no paper has considered the existence results of triple positive solutions for fourth-order impulsive differential equations with integral boundary conditions till now; for example, see [54–58] and the references therein.

In this paper, we investigate the existence of three positive solutions for the following fourth-order impulsive differential equations with integral boundary conditions:

$$\begin{cases} x^{(4)}(t) = \omega(t)f(t, x(t)), & 0 < t < 1, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(t_k, x(t_k)), \\ \Delta x'|_{t=t_k} = 0, & k = 1, 2, \ldots, m, \\ x(0) = \int_0^1 g(s)x(s)\,ds, & x'(1) = 0, \\ x''(0) = \int_0^1 h(s)x''(s)\,ds, & x'''(1) = 0. \end{cases} \tag{1.1}$$

Here $\omega \in L^p[0,1]$ for some $1 \leq p \leq +\infty$, t_k $(k = 1, 2, \ldots, m)$ (where m is fixed positive integer) are fixed points with $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1$, $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, *i.e.* $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right-hand limit and left-hand limit of $x(t)$ at $t = t_k$, respectively. In addition, ω, f, I_k, g, and h satisfy

(H_1) $\omega \in L^p[0,1]$ for some $1 \leq p \leq +\infty$ and there exists $n > 0$ such that $\omega(t) \geq n$ a.e. on J;
(H_2) $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $I_k \in C([0,1] \times [0,+\infty), [0,+\infty))$;
(H_3) $g, h \in L^1[0,1]$ are nonnegative and $\mu \in [0,1)$, $\nu \in [0,1)$, where

$$\nu = \int_0^1 g(t)\,dt, \qquad \mu = \int_0^1 h(t)\,dt. \tag{1.2}$$

Remark 1.1 The idea of impulsive effect for problem (1.1) is from Ding and O?Regan [59].

Some special cases of problem (1.1) have been investigated. For example, Zhang and Ge [45] studied the existence and multiplicity of symmetric positive solutions for problem (1.1) with $I_k \equiv 0$ $(k = 1, 2, \ldots, m)$ and $\omega \in C(0,1)$, not $\omega \in L^p[0,1]$.

Motivated by the results mentioned above, in this paper we study the existence of three positive solutions for problem (1.1) by new technique (different from the proof of Theorems 3.1-3.4 of [45]) to overcome difficulties arising from the appearances of $I_k \neq 0$ ($k = 1, 2, \dots, m$) and $\omega(t)$ is L^p-integrable. The arguments are based upon a fixed point theorem due to Leggett and Williams which deals with fixed points of a cone-preserving operator defined on an ordered Banach space.

The rest of the paper is organized as follows: In Section 2, we provide some necessary background. In particular, we state some properties of the Green?s function associated with problem (1.1). In Section 3, the main results of problem (1.1) will be stated and proved.

2 Preliminaries

Let $J' = J \backslash \{t_1, t_2, \dots, t_m\}$, and

$$PC[0,1] = \{x : x \text{ is continuous at } t \neq t_k, x(t_k^-) = x(t_k) \text{ and } x(t_k^+) \text{ exists}, k = 1, 2, \dots, m\}.$$

Then $PC[0,1]$ is a real Banach space with norm

$$\|x\| = \max_{t \in J} |x(t)|.$$

Definition 2.1 (See [60]) Let E be a real Banach space over R. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$ and

(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2 The map β is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Definition 2.3 A function $x \in PC[0,1] \cap C^4(J')$ is called a solution of problem (1.1) if it satisfies (1.1).

We shall reduce problem (1.1) to an integral equation. With this goal, firstly by means of the transformation

$$x''(t) = -y(t), \tag{2.1}$$

we convert problem (1.1) into

$$\begin{cases} y''(t) + \omega(t) f(t, x(t)) = 0, & t \in J, \\ y(0) = \int_0^1 h(t) y(t)\,dt, & y'(1) = 0, \end{cases} \tag{2.2}$$

and

$$\begin{cases} -x''(t) = y(t), & t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(t_k, x(t_k)), & \\ \Delta x'|_{t=t_k} = 0, & k = 1, 2, \dots, m, \\ x(0) = \int_0^1 g(t) x(t)\,dt, & x'(1) = 0. \end{cases} \tag{2.3}$$

Lemma 2.1 *Assume that* (H_1)-(H_3) *hold. Then problem* (2.2) *has a unique solution* y *given by*

$$y(t) = \int_0^1 H(t,s)\omega(s)f\bigl(s,x(s)\bigr)\,ds, \tag{2.4}$$

where

$$H(t,s) = G(t,s) + \frac{1}{1-\mu}\int_0^1 G(s,\tau)h(\tau)\,d\tau, \tag{2.5}$$

$$G(t,s) = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1. \end{cases} \tag{2.6}$$

Proof The proof of Lemma 2.1 is similar to that of Lemma 2.1 in [61]. □

Write $e(t) = t$. Then from (2.5) and (2.6), we can prove that $H(t,s)$ and $G(t,s)$ have the following properties.

Proposition 2.1 *Let* $\delta \in (0,\frac{1}{2})$, $J_\delta = [\delta, 1-\delta]$. *If* $\mu \in [0,1)$, *then we have*

$$H(t,s) > 0, \qquad G(t,s) > 0, \quad \forall t,s \in (0,1), \tag{2.7}$$

$$H(t,s) \ge 0, \qquad G(t,s) \ge 0, \quad \forall t,s \in J, \tag{2.8}$$

$$e(t)e(s) \le G(t,s) \le G(t,t) = t = e(t) \le 1, \quad \forall t,s \in J, \tag{2.9}$$

$$\rho e(t)e(s) \le H(t,s) \le \gamma s = \gamma e(s) \le \gamma, \quad \forall t,s \in J, \tag{2.10}$$

$$G(t,s) \ge \delta G(s,s), \qquad H(t,s) \ge \delta H(s,s), \quad \forall t \in J_\delta, s \in J, \tag{2.11}$$

where

$$\gamma = \frac{1}{1-\mu}, \qquad \rho = 1 + \frac{\int_0^1 sh(s)\,ds}{1-\mu}. \tag{2.12}$$

Remark 2.1 From (2.5) and (2.11), we can obtain

$$H(t,s) \ge \delta s = \delta G(s,s), \quad \forall t \in J_\delta, s \in J.$$

Lemma 2.2 *If* (H_2) *and* (H_3) *hold, then problem* (2.3) *has a unique solution* x *and* x *can be expressed in the form*

$$x(t) = \int_0^1 H_1(t,s)y(s)\,ds + \sum_{k=1}^m H'_{1s}(t,t_k)I_k\bigl(t_k,x(t_k)\bigr), \tag{2.13}$$

where

$$H_1(t,s) = G(t,s) + \frac{1}{1-\nu}\int_0^1 G(s,\tau)g(\tau)\,d\tau, \tag{2.14}$$

$$H'_{1s}(t,s) = G'_s(t,s) + \frac{1}{1-\nu}\int_0^1 G'_s(\tau,s)g(\tau)\,d\tau, \tag{2.15}$$

$$G'_s(t,s) = \begin{cases} 0, & 0 \le t \le s \le 1, \\ 1, & 0 \le s \le t \le 1. \end{cases} \tag{2.16}$$

Proof The proof of Lemma 2.2 is similar to that of Lemma 2.6 in [53]. □

From (2.14)-(2.16), we can prove that $H_1(t,s)$, $H'_{1s}(t,s)$, and $G'_s(t,s)$ have the following properties.

Proposition 2.2 *If* $\nu \in [0,1)$, *then we have*

$$H_1(t,s) \ge 0, \quad \forall t,s \in J; \tag{2.17}$$

$$\rho_1 e(t)e(s) \le H_1(t,s) \le \gamma_1 s = \gamma_1 e(s) \le \gamma_1, \quad \forall t,s \in J, \tag{2.18}$$

$$H_1(t,s) \ge \delta H_1(s,s), \quad \forall t \in J_\delta, s \in J. \tag{2.19}$$

$$G'_s(t,s) \le 1, \qquad 0 \le H'_{1s}(t,s) \le \frac{1}{1-\nu}, \tag{2.20}$$

where

$$\gamma_1 = \frac{1}{1-\nu}, \qquad \rho_1 = 1 + \frac{\int_0^1 sg(s)\,ds}{1-\nu}. \tag{2.21}$$

Remark 2.2 From (2.14) and (2.19), we can obtain

$$H_1(t,s) \ge \delta s = \delta G(s,s), \quad \forall t \in J_\delta, s \in J.$$

Remark 2.3 From (2.20), one can prove that

$$0 < H'_{1s}(t,s)(1-\nu) \le 1, \quad \forall t \in J_\delta, s \in [0,1). \tag{2.22}$$

Suppose that x is a solution of problem (1.1). Then from Lemma 2.1 and Lemma 2.2, we have

$$x(t) = \int_0^1 \int_0^1 H_1(t,s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds + \sum_{k=1}^m H'_{1s}(t,t_k)I_k\bigl(t_k,x(t_k)\bigr).$$

Define a cone in $PC[0,1]$ by

$$K = \bigl\{x \in PC[0,1] : x \ge 0\bigr\}. \tag{2.23}$$

It is easy to see K is a closed convex cone of $PC[0,1]$.

Define an operator $T : K \to PC[0,1]$ by

$$(Tx)(t) = \int_0^1 \int_0^1 H_1(t,s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds + \sum_{k=1}^m H'_{1s}(t,t_k)I_k\bigl(t_k,x(t_k)\bigr). \tag{2.24}$$

From (2.24), we know that $x \in PC[0,1]$ is a solution of problem (1.1) if and only if x is a fixed point of operator T.

Lemma 2.3 *Suppose that* (H_1)-(H_3) *hold. Then* $T(K) \subset K$ *and* $T : K \to K$ *is completely continuous.*

Proof The proof of Lemma 2.3 is similar to that of Lemma 2.4 in [53]. □

Let $0 < a < b$ be given and let β be a nonnegative continuous concave functional on the cone K. Define the convex sets K_a, $K(\beta, a, b)$ by

$$K_a = \{x \in K : \|x\| < a\},$$
$$K(\beta, a, b) = \{x \in K : a \le \beta(x), \|x\| \le b\}.$$

Finally we state Leggett-Williams? fixed point theorem [52].

Lemma 2.4 *Let* K *be a cone in a real Banach space* E, $A : \bar{K}_a \to \bar{K}_a$ *be completely continuous and* β *be a nonnegative continuous concave functional on* K *with* $\beta(x) \le \|x\|$ *for all* $x \in K_a$. *Suppose there exist* $0 < d < a < b \le c$ *such that*

(i) $\{x \in K(\beta, a, b) : \beta(x) > a\} \neq \emptyset$ *and* $\beta(Ax) > a$ *for* $x \in K(\beta, a, b)$;

(ii) $\|Ax\| < d$ *for* $\|x\| \le d$;

(iii) $\beta(Ax) > a$ *for* $x \in K(\beta, a, c)$ *with* $\|Ax\| > b$.

Then A *has at least three positive solutions* x_1, x_2, x_3 *satisfying*

$$\|x_1\| < d, \qquad a < \beta(x_2), \qquad \|x_3\| > d \quad \text{and} \quad \beta(x_3) < a.$$

To obtain some of the norm inequalities in Theorem 3.1 and Corollary 3.1, we employ Hölder?s inequality.

Lemma 2.5 (Hölder) *Let* $f \in L^p[a, b]$ *with* $p > 1$, $g \in L^q[a, b]$ *with* $q > 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $fg \in L^1[a, b]$ *and*

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

Let $f \in L^1[a, b]$, $g \in L^\infty[a, b]$. *Then* $fg \in L^1[a, b]$ *and*

$$\|fg\|_1 \le \|f\|_1 \|g\|_\infty.$$

3 Existence of triple positive solutions to (1.1)

In this section, we apply Lemma 2.4 and Lemma 2.5 to establish the existence of triple positive solutions for problem (1.1). We consider the following three cases for $\omega \in L^p[0, 1]$: $p > 1$, $p = 1$, and $p = \infty$. Case $p > 1$ is treated in the following theorem.

For convenience, we introduce the following notation:

$$D = \gamma \gamma_1 \|e\|_q \|\omega\|_p, \qquad D_1 = \frac{m}{1 - \nu},$$
$$\delta_1 = \min_{t \in J_\delta, s \in (0,1)} H'_{1s}(t, s)(1 - \nu), \qquad \delta^* = \min\left\{\frac{\delta}{\gamma_1}, \delta_1\right\},$$
$$f^\infty = \limsup_{x \to \infty} \max_{t \in J} \frac{f(t, x)}{x}, \qquad I^\infty(k) = \limsup_{x \to \infty} \max_{t \in J} \frac{I_k(t, x)}{x}, \quad k = 1, 2, \ldots, m.$$

Theorem 3.1 *Assume that* (H_1)-(H_3) *hold. Furthermore, suppose that there exist constants* $0 < d < a < \frac{a}{\delta^*} \leq c$ *such that*

(H_4) $f^\infty < \frac{1}{2D}$, $I^\infty(k) < \frac{1}{2D_1}$, $k = 1, 2, \ldots, m$;
(H_5) $f(t,x) > \frac{3a}{\delta^2(1-2\delta)n}$ *for* $(t,x) \in J_\delta \times [a, \frac{a}{\delta^*}]$;
(H_6) $f(t,x) < \frac{d}{2D}$, $I_k(t,x) < \frac{d}{2D_1}$ *for* $(t,x) \in J \times [0,d]$, $k = 1, 2, \ldots, m$.

Then problem (1.1) *has at least three positive solutions* x_1, x_2, *and* x_3 *such that*

$$\|x_1\| < d, \qquad a < \beta(x_2), \qquad \text{and} \quad x_3 > d \quad \text{with } \beta(x_3) < a.$$

Proof By the definition of operator T and its properties, it suffices to show that the conditions of Lemma 2.4 hold with respect to T.

Let $\beta(x) = \min_{t\in J_\delta} x(t)$. Then $\beta(x)$ is a nonnegative continuous concave functional on the cone K satisfying $\beta(x) \leq \|x\|$ for all $x \in K$.

For convenience, we denote $b = \frac{a}{\delta^*}$.

Considering (H_4), there exist $0 < \sigma < \frac{1}{2D}$, $0 < \sigma_1 < \frac{1}{2D_1}$, and $l > 0$ such that

$$f(t,x) \leq \sigma x, \qquad I_k(t,x) \leq \sigma_1 x, \quad k = 1, 2, \ldots, m, \forall t \in J, x \geq l.$$

Let

$$\eta = \max_{0\leq x\leq l, t\in J} f(t,x), \qquad \eta_1 = \max_{0\leq x\leq l, t\in J} I_k(t,x), \quad k = 1, 2, \ldots, m.$$

Then

$$f(t,x) \leq \sigma x + \eta, \qquad I_k(t,x) \leq \sigma_1 x + \eta_1, \quad \forall t \in J, 0 \leq x \leq +\infty. \tag{3.1}$$

Set

$$c > \max\left\{\frac{2D\eta}{1-2D\sigma}, \frac{2D_1\eta_1}{1-2D_1\sigma_1}, \frac{a}{\delta^*}\right\}.$$

Then, for $x \in \bar{K}_c$, it follows from (2.19), (2.22), and (3.1) that

$$\begin{aligned}
(Tx)(t) &= \int_0^1 \int_0^1 H_1(t,s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds + \sum_{k=1}^m H'_{1s}(t,t_k)I_k\bigl(t_k,x(t_k)\bigr) \\
&\leq \int_0^1 \int_0^1 \gamma_1\gamma e(\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds + \frac{1}{1-\nu}\sum_{k=1}^m I_k\bigl(t_k,x(t_k)\bigr) \\
&\leq \int_0^1 \int_0^1 \gamma_1\gamma e(\tau)\omega(\tau)(\sigma x + \eta)\,d\tau\,ds + \frac{1}{1-\nu}\sum_{k=1}^m (\sigma_1 x + \eta_1) \\
&\leq \int_0^1 \int_0^1 \gamma_1\gamma e(\tau)\omega(\tau)\bigl(\sigma\|x\| + \eta\bigr)\,d\tau\,ds + \frac{1}{1-\nu}\sum_{k=1}^m \bigl(\sigma_1\|x\| + \eta_1\bigr) \\
&\leq \gamma_1\gamma(\sigma c + \eta)\int_0^1 e(\tau)\omega(\tau)\,d\tau + \frac{m}{1-\nu}(\sigma_1 c + \eta_1)
\end{aligned}$$

$$\begin{aligned}
&\leq \gamma_1\gamma(\sigma c+\eta)\|e\|_q\|\omega\|_p+\frac{m}{1-\nu}(\sigma_1 c+\eta_1)\\
&<\frac{c}{2}+\frac{c}{2}=c,
\end{aligned}$$

which shows that $Tx\in K_c$.

Hence, we have shown that if (H_4) holds, then T maps $\bar{K}_c$ into K_c.

Next, we verify that $\{x\in K(\beta,a,b):\beta(x)>a\}\neq\emptyset$ and $\beta(Tu)>a$ for all $x\in K(\beta,a,b)$.

Take $\varphi_0(t)\equiv\frac{\delta^*+1}{2\delta^*}a$, for $t\in J$. Then

$$\varphi_0\in\left\{x\;\middle|\;x\in K\left(\beta,a,\frac{a}{\delta^*}\right),\beta(x)>a\right\}.$$

This shows that

$$\left\{x\in K(\beta,a,b):\beta(x)>a\right\}\neq\emptyset.$$

Therefore, it follows from (H_5) that

$$\begin{aligned}
\beta(Tx)&=\min_{t\in J_\delta}(Tx)(t)\\
&=\min_{t\in J_\delta}\int_0^1\int_0^1 H_1(t,s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds+\sum_{k=1}^m H'_{1s}(t,t_k)I_k\bigl(t_k,x(t_k)\bigr)\\
&\geq\min_{t\in J_\delta}\int_0^1\int_0^1 H_1(t,s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds\\
&\geq\delta\int_0^1\int_0^1 e(s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds\\
&\geq\delta\int_0^1\int_\delta^{1-\delta} e(s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds\\
&\geq\delta^2\int_0^1 s^2\,ds\int_\delta^{1-\delta}\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\\
&>\frac{1}{3}\delta^2 n(1-2\delta)\frac{3a}{\delta^2(1-2\delta)n}\\
&=a.
\end{aligned}$$

If $x\in\bar{K}_d$, then it follows from (H_6) that

$$\begin{aligned}
(Tx)(t)&=\int_0^1\int_0^1 H_1(t,s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds+\sum_{k=1}^m H'_{1s}(t,t_k)I_k\bigl(t_k,x(t_k)\bigr)\\
&\leq\int_0^1\int_0^1\gamma_1\gamma e(\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds+\frac{1}{1-\nu}\sum_{k=1}^m I_k\bigl(t_k,x(t_k)\bigr)\\
&<\int_0^1\int_0^1\gamma_1\gamma e(\tau)\omega(\tau)\frac{d}{2D}\,d\tau\,ds+\frac{1}{1-\nu}\sum_{k=1}^m\frac{d}{2D_1}\\
&=d.
\end{aligned}$$

Finally, we assert that if $x \in K(\beta, a, c)$ and $\|Tx\| > b$, then $\beta(Tx) > a$.

Suppose $x \in K(\beta, a, c)$ and $\|Tx\| > b$, then it follows from (2.18), (2.20), and (2.23) that

$$\begin{aligned}
\beta(Tx) &= \min_{t\in J_\delta}(Tx)(t) \\
&= \min_{t\in J_\delta}\left[\int_0^1\int_0^1 H_1(t,s)H(s,\tau)\omega(\tau)f\big(\tau,x(\tau)\big)\,d\tau\,ds + \sum_{k=1}^m H'_{1s}(t,t_k)I_k\big(t_k,x(t_k)\big)\right] \\
&\geq \delta\int_0^1\int_0^1 e(s)H(s,\tau)\omega(\tau)f\big(\tau,x(\tau)\big)\,d\tau\,ds \\
&\quad + \min_{t\in J_\delta}\sum_{k=1}^m H'_{1s}(t,t_k)(1-\nu)\frac{1}{1-\nu}I_k\big(t_k,x(t_k)\big) \\
&\geq \frac{\delta}{\gamma_1}\int_0^1\int_0^1 \gamma_1 e(s)H(s,\tau)\omega(\tau)f\big(\tau,x(\tau)\big)\,d\tau\,ds + \delta_1\sum_{k=1}^m\frac{1}{1-\nu}I_k\big(t_k,x(t_k)\big) \\
&\geq \min\left\{\frac{\delta}{\gamma_1},\delta_1\right\}\left[\int_0^1\int_0^1 \gamma_1 e(s)H(s,\tau)\omega(\tau)f\big(\tau,x(\tau)\big)\,d\tau\,ds + \sum_{k=1}^m\frac{1}{1-\nu}I_k\big(t_k,x(t_k)\big)\right] \\
&\geq \delta^*\|Tx\| \\
&> a.
\end{aligned}$$

To sum up, the hypotheses of Lemma 2.5 hold. Therefore, an application of Lemma 2.5 implies problem (1.1) has at least three positive solutions x_1, x_2, and x_3 such that

$$\|x_1\| < d, \qquad a < \beta(x_2), \quad \text{and} \quad x_3 > d \quad \text{with } \beta(x_3) < a. \qquad \square$$

The following theorem deals with the case $p = \infty$.

Corollary 3.1 *Assume that* (H_1)-(H_6) *hold. Then problem* (1.1) *has at least three positive solutions* x_1, x_2, *and* x_3 *such that*

$$\|x_1\| < d, \qquad a < \beta(x_2), \quad \textit{and} \quad x_3 > d \quad \textit{with } \beta(x_3) < a.$$

Proof Let $\|e\|_1\|\omega\|_\infty$ replace $\|e\|_p\|\omega\|_q$ and repeat the argument above. □

Finally we consider the case of $p = 1$. Let

$(H_4)'$ $f^\infty < \frac{1}{D'}$, $I^\infty(k) < \frac{1}{D_1}$, $k = 1, 2, \ldots, m$;
$(H_6)'$ $f(t,x) \leq \frac{d}{2D'}$, $I_k(t,x) \leq \frac{d}{2D_1}$ $(k = 1, 2, \ldots, m)$ for $(t,x) \in J \times [0,d]$,

where

$$D' = \gamma\gamma_1\|\omega\|_1.$$

Corollary 3.2 *Assume that* (H_1)-(H_3), $(H_4)'$, (H_5), *and* $(H_6)'$ *hold. Then problem* (1.1) *has at least three positive solutions* x_1, x_2, *and* x_3 *such that*

$$\|x_1\| < d, \qquad a < \beta(x_2), \quad \textit{and} \quad x_3 > d \quad \textit{with } \beta(x_3) < a.$$

Proof Set

$$c' > \max\left\{\frac{2D'\eta}{1-2D'\sigma'}, \frac{2D_1\eta_1}{1-2D_1\sigma_1}, \frac{a}{\delta^*}\right\},$$

where $0 < \sigma' < \frac{1}{2D'}$. Then, for $x \in \bar{K}_{c'}$, it follows from (2.19), (2.22), and (3.1) that

$$\begin{aligned}
(Tx)(t) &= \int_0^1 \int_0^1 H_1(t,s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds + \sum_{k=1}^m H'_{1s}(t,t_k)I_k\bigl(t_k,x(t_k)\bigr) \\
&\le \int_0^1 \int_0^1 \gamma_1\gamma e(\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds + \frac{1}{1-\nu}\sum_{k=1}^m I_k\bigl(t_k,x(t_k)\bigr) \\
&\le \int_0^1 \int_0^1 \gamma_1\gamma e(\tau)\omega(\tau)(\sigma x+\eta)\,d\tau\,ds + \frac{1}{1-\nu}\sum_{k=1}^m (\sigma_1 x+\eta_1) \\
&\le \int_0^1 \int_0^1 \gamma_1\gamma e(\tau)\omega(\tau)\bigl(\sigma\|x\|+\eta\bigr)\,d\tau\,ds + \frac{1}{1-\nu}\sum_{k=1}^m \bigl(\sigma_1\|x\|+\eta_1\bigr) \\
&\le \gamma_1\gamma\bigl(\sigma c'+\eta\bigr)\int_0^1 \omega(\tau)\,d\tau + \frac{m}{1-\nu}\bigl(\sigma_1 c'+\eta_1\bigr) \\
&\le \gamma_1\gamma\bigl(\sigma c'+\eta\bigr)\|\omega\|_1 + \frac{m}{1-\nu}\bigl(\sigma_1 c'+\eta_1\bigr) \\
&< \frac{c'}{2} + \frac{c'}{2} = c',
\end{aligned}$$

which shows that $Tx \in K_{c'}$.

Hence, we have shown that if $(\mathrm{H}_4)'$ holds, then T maps $\bar{K}_{c'}$ into $K_{c'}$.

If $x \in \bar{K}_d$, then it follows from $(\mathrm{H}_6)'$ that

$$\begin{aligned}
(Tx)(t) &= \int_0^1 \int_0^1 H_1(t,s)H(s,\tau)\omega(\tau)f\bigl(\tau,x(\tau)\bigr)\,d\tau\,ds + \sum_{k=1}^m H'_{1s}(t,t_k)I_k\bigl(t_k,x(t_k)\bigr) \\
&\le \int_0^1 \int_0^1 \gamma_1\gamma e(\tau)\omega(\tau)\frac{d}{2D'}\,d\tau\,ds + \frac{1}{1-\nu}\sum_{k=1}^m \frac{d}{2D_1} \\
&\le \gamma_1\gamma\frac{d}{2D'}\int_0^1 \omega(\tau)\,d\tau + \frac{1}{1-\nu}\sum_{k=1}^m \frac{d}{2D_1} \\
&= d.
\end{aligned}$$

Similar to the proof of Theorem 3.1, one can find the results of Corollary 3.2. □

We remark that the condition (H_6) in Theorem 3.1 can be replaced by the following condition:

$(\mathrm{H}_6)''$ $f_0^d \le \frac{1}{2D}$, $I_0^d(k) \le \frac{1}{2D_1}$, $k = 1,2,\ldots,m$, where

$$f_0^d = \max\left\{\max_{t\in J}\frac{f(t,x)}{d} : x \in [0,d]\right\}, \qquad I_0^d(k) = \max\left\{\max_{t\in J}\frac{I_k(t,x)}{d} : x \in [0,d]\right\}.$$

$(\mathrm{H}_6)'''$ $f^0 \le \frac{1}{2D}$, $I^0(k) \le \frac{1}{2D_1}$, $k = 1,2,\ldots,m$.

Corollary 3.3 *If the condition* (H_6) *in Theorem* 3.1 *is replaced by* $(\mathrm{H}_6)''$ *or* $(\mathrm{H}_6)'''$, *respectively, then the conclusion of Theorem* 3.1 *also holds*.

Proof It follows from the proof of Theorem 3.1 that Corollary 3.3 holds. □

Remark 3.1 Comparing with Zhang and Ge [45], the main features of this paper are as follows.

(i) Triple positive solutions are available.

(ii) $I_k \not\equiv 0$ $(k = 1, 2, \dots, m)$ is considered.

(iii) $\omega(t)$ is L^p-integrable, not only $\omega(t) \in C(0,1)$ for $t \in J$.

4 Example

To illustrate how our main results can be used in practice, we present an example.

Example 4.1 Let $\delta = \frac{1}{4}$, $m = 1$, $t_1 = \frac{1}{2}$, $p = 1$. It follows from $p = 1$ that $q = \infty$. Consider the following boundary value problem:

$$\begin{cases} x^{(4)}(t) = \omega(t)f(t,x(t)), & 0 < t < 1, t \neq \frac{1}{2}, \\ \Delta x|_{t=\frac{1}{2}} = I_1(\frac{1}{2}, x(\frac{1}{2})), \\ \Delta x'|_{t=\frac{1}{2}} = 0, \\ x(0) = \int_0^1 g(s)x(s)\,ds, & x'(1) = 0, \\ x''(0) = \int_0^1 h(s)x''(s)\,ds, & x'''(1) = 0, \end{cases} \tag{4.1}$$

where $\omega(t) = 2t + 3 \in L^1[0,1]$, $g(t) = h(t) = t$, $I_1(t,x) = \frac{tx}{20\delta}$,

$$f(t,x) = \begin{cases} \frac{d}{48}, & t \in J, x \in [0,d], \\ \frac{d}{48} \times \frac{a-x}{a-d} + 64a\frac{x-d}{a-d}, & t \in J, x \in [d,a], \\ 64a, & t \in J, x \in [a, \frac{a}{\delta^*}], \\ 64a + t\sqrt{x - \frac{a}{\delta^*}}, & t \in J, x \in [\frac{a}{\delta^*}, \infty). \end{cases}$$

Thus it is easy to see by calculating that $\omega(t) \geq n = 3$ for a.e. $t \in J$, and

$$\mu = \int_0^1 h(t)\,dt = \frac{1}{2}, \qquad \nu = \int_0^1 g(t)\,dt = \frac{1}{2},$$
$$\gamma = \frac{1}{1-\mu} = 2, \qquad \gamma_1 = \frac{1}{1-\nu} = 2, \qquad \delta_1 = \frac{3}{4}, \qquad \delta^* = \frac{1}{8}.$$

Therefore, it follows from the definitions ω, f, I_1, g, and h that (H_1)-(H_3) hold.

On the other hand, it follows from $\omega(t) = 2t + 3$ and $e(t) = t$ that

$$\|\omega\|_1 = \int_0^1 (2t+3)\,dt = 4, \qquad \|e\|_q = \|e\|_\infty = \lim_{q\to\infty}\left(\int_0^1 t^q\,dt\right)^{\frac{1}{q}} = \lim_{q\to\infty}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} = 1.$$

Thus, we have

$$D = \gamma\gamma_1\|\omega\|_1\|e\|_\infty = 16, \qquad D_1 = \frac{m}{1-\mu} = 2, \qquad \frac{1}{2D} = \frac{1}{32}, \qquad \frac{1}{2D_1} = \frac{1}{4}.$$

Choosing $0 < d < a < 8a \leq c$, we have

$$f^{\infty} = 0 < \frac{1}{32} = \frac{1}{2D}, \qquad I^{\infty}(1) = \frac{1}{5} < \frac{1}{4} = \frac{1}{2D_1},$$
$$f(t,x) = 64a > 32a = \frac{3a}{\delta^2(1-2\delta)n}, \quad \forall (t,x) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [a, 8a],$$
$$f(t,x) = \frac{d}{48} < \frac{d}{32} = \frac{1}{2D}, \qquad I_1(t,x) \leq \frac{d}{5} < \frac{d}{4} = \frac{d}{2D_1}, \quad \forall (t,x) \in J \times [0,d],$$

which shows that (H_4)-(H_6) hold.

By Corollary 3.2, problem (4.1) has at least three positive solutions x_1, x_2, and x_3 such that

$$\|x_1\| < d, \qquad a < \beta(x_2), \quad \text{and} \quad x_3 > d \quad \text{with } \beta(x_3) < a.$$

Remark 4.1 In Example 4.1, we consider the norm of $L^{\infty}[0,1]$, which is different from that used in [28, 31, 36, 45].

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
All results belong to YZ and XZ. All authors read and approved the final manuscript.

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