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Multiplicity of solutions of perturbed Schrödinger equation with electromagnetic fields and critical nonlinearity in $\mathbb{R}^N$

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Abstract

In this paper, we deal with the existence and multiplicity of solutions for perturbed Schrödinger equation with electromagnetic fields and critical nonlinearity in $\mathbb{R}^N$: $-\varepsilon^2\Delta_A u(x) + V(x)u(x) = |u|^{2^*-2}u + h(x,|u|^2)u$ for all $x \in \mathbb{R}^N$, where $\nabla_A u(x) := (\nabla + iA(x))u$, $V(x)$ is a nonnegative potential. By using Lions' second concentration compactness principle and concentration compactness principle at infinity to prove that the $(PS)_c$ condition holds locally and by variational method, we show that this equation has at least one solution provided that $\varepsilon < \mathcal{E}$, for any $m \in \mathbb{N}$, it has m pairs of solutions if $\varepsilon < \mathcal{E}_m$, where $\mathcal{E}$ and $\mathcal{E}_m$ are sufficiently small positive numbers.

MSC: 35J60; 35B33

Keywords: perturbed Schrödinger equation; critical nonlinearity; magnetic fields; variational methods

1 Introduction

In this paper, we are concerned with the existence of nontrivial solutions to the following perturbed Schrödinger equation with electromagnetic fields and critical nonlinearity in $\mathbb{R}^N$:

$$-\varepsilon^2\Delta_A u(x) + V(x)u(x) = |u|^{2^*-2}u + h(x,|u|^2)u \quad \text{for } x \in \mathbb{R}^N, \tag{1.1}$$

where $\nabla_A u(x) := (\nabla + iA(x))u$. Here, i is the imaginary unit, $2^* := 2N/(N-2)$ denotes the Sobolev critical exponent and $N \geq 3$, $V(x)$ and $h(x,u)$ are functions satisfying some conditions.

This paper is motivated by some works concerning the nonlinear Schrödinger equation

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\big(\nabla + iA(x)\big)^2\psi + W(x)\psi \\ &\quad - K(x)|\psi|^{2^*-2}\psi - h\big(x,|\psi|^2\big)\psi \quad \text{for } x \in \mathbb{R}^N, \end{aligned} \tag{1.2}$$

where $\hbar$ is Plank constant, $A(x) = (A_1(x), A_2(x), \ldots, A_N(x)) : \mathbb{R}^N \to \mathbb{R}^N$ is a real vector (magnetic) potential with magnetic field $B = \operatorname{curl} A$ and $W(x) : \mathbb{R}^N \to \mathbb{R}^N$ is a scalar electric potential.

In physics, we are interested in standing wave solutions, *i.e.*, solutions of type (1.2) when $\hbar$ is sufficiently small, when E is a real number and $u(x)$ is a complex-value function which satisfies

$$-\bigl(\nabla + iA(x)\bigr)^2 u(x) + \lambda V(x)u(x) = \lambda K(x)|u|^{2^*-2}u + \lambda h\bigl(x,|u|^2\bigr)u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $\lambda^{-1} = \frac{\hbar^2}{2m}$ and $V(x) = W(x) - E$. The transition from quantum mechanics to classical mechanics can be formally performed by letting $\hbar \to 0$. Thus the existence of solutions for $\hbar$ small, semi-classical solutions, has important physical interest.

It is well known that the linear Schrödinger equation is a basic tool of quantum mechanics, and it provides a description of the dynamics of a particle in a non-relativistic setting. The nonlinear Schrödinger equation arises in different physical theories, *e.g.*, the description of Bose-Einstein condensates and nonlinear optics, see [1] and the references cited there. Both the linear and the nonlinear Schrödinger equations have been widely considered in the literature. The main purpose of this paper is to study the existence and multiplicity of solutions of perturbed Schrödinger equations with electromagnetic fields and critical nonlinearity (1.1).

Problem (1.3) with $A(x) \equiv 0$ has an extensive literature. Different approaches have been taken to attack this problem under various hypotheses on the potential and the nonlinearity. See, for example, [2–13] and the references therein. Observe that in all these papers the nonlinearities are assumed to be subcritical. In [11], using a Lyapunov-Schmidt reduction, Floer and Weinstein established the existence of single and multiple spike solutions. Their method and results were later generalized by Oh [12] to the higher-dimensional case. Kang and Wei [14] established the existence of positive solutions with any prescribed number of spikes, clustering around a given local maximum point of the potential function. In accordance with the Sobolev critical nonlinearities, there have been many papers devoted to studying the existence of solutions to elliptic boundary-valued problems on bounded domains after the pioneering work by Brezis and Nirenberg [15]. Ding and Lin [9] first studied the existence of semi-classical solutions to the problem on the whole space with critical nonlinearities and established the existence of positive solutions as well as of those that change sign exactly once. They also obtained multiplicity of solutions when the nonlinearity is odd.

When $A(x) \not\equiv 0$, there are also many works dealing with the magnetic case. The first one seems to be [16] where the existence of standing waves was obtained for $\hbar > 0$ fixed and for special classes of magnetic fields. If A and W are periodic functions, the existence of various types of solutions for fixed $\hbar > 0$ was proved in [17] by applying minimax arguments. Concerning semi-classical bound states, it was proved in [18] that for $\hbar > 0$ small and admits a least energy solution which concentrates near the global minimum of W. A multiplicity result for solutions was obtained in [4] by using a topological argument. There it was also proved that the magnetic potential A only contributes to the phase factor of the solitary solutions for $\hbar > 0$ sufficiently small. In [19] single-bump bound states were obtained by using perturbation methods. These concentrate near a non-degenerate critical point of W as $\hbar \to 0$. For the critical growth case, Wang [20] studied the electromagnetic Schrödinger equations

$$-\bigl(\nabla + iA(x)\bigr)^2 u(x) + \lambda V(x)u(x) = K(x)|u|^{2^*-2}u \quad \text{for } x \in \mathbb{R}^N. \tag{1.4}$$

By using the linking theorem twice to the corresponding functional, they established the existence results. Chabrowski and Szulkin [21] considered problem (1.4) under assumption that $V(x)$ changes sign; by using a min-max type argument based on a topological linking, they obtained a solution in the Sobolev space which was defined in the paper. Assume $K(x) \equiv 1$, Han [22] studied problem (1.4) and established the existence of nontrivial solutions in the critical case by means of variational method. For more results, we refer the reader to [20, 23–27] and the references therein.

In the present paper, we consider the existence of solutions for problem (1.1) under the condition $\inf_{x\in\mathbb{R}^N} V(x) = 0$ and critical nonlinearity. It seems that Byeon and Wang [1] were the first to study energy level and the asymptotic behavior of positive solutions to Schrödinger equations under the condition $\inf_{x\in\mathbb{R}^N} V(x) = 0$. In [25], Cao and Tang extended the results of Byeon and Wang [1]. However, to the best knowledge, it seems that there are few works on the existence of solutions to be the problems on $\mathbb{R}^N$ involving critical nonlinearities with electromagnetic fields. We mainly follow the idea of [9, 10]. Let us point out that although the idea was used before for other problems, the adaptation of the procedure to our problem is not trivial at all. Because of the appearance of electromagnetic potential $A(x)$, we must consider our problem for complex-valued functions, and so we need more delicate estimates. Furthermore, we use Lions' second concentration compactness principle and concentration compactness principle at infinity to prove that the $(PS)_c$ condition holds, which is different from methods used in [9, 10].

2 Main results

Let $\lambda = \varepsilon^{-2}$. Equation (1.1) reads then as

$$-\big(\nabla + iA(x)\big)^2 u(x) + \lambda V(x)u(x) = \lambda |u|^{2^*-2}u + \lambda h\big(x, |u|^2\big)u \quad \text{for } x \in \mathbb{R}^N. \tag{2.1}$$

We make the following assumptions on $A(x)$, $V(x)$ and h throughout this paper:

(V) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$; $V(x_0) = \min_{x\in\mathbb{R}^N} V = 0$, and there is $b > 0$ such that the set $V^b = \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure;

(A) $A_j(x) \in C(\mathbb{R}^N, \mathbb{R})$ $(j = 1, 2, \dots, N)$ and $A(x_0) = 0$;

(H) (h_1) $h \in C(\mathbb{R}^N \times [0, +\infty), \mathbb{R})$ and $h(x, t) = o(1)$ uniformly in x as $t \to 0$;

(h_2) there are $C_0 > 0$ and $q \in (2, 2^*)$ such that $|h(x, t)| \le C_0(1 + t^{\frac{q-2}{2}})$;

(h_3) there are $a_0 > 0$, $p > 2$ and $\mu > 2$ such that $H(x, t) \ge a_0 t^{\frac{p}{2}}$ and $\frac{\mu}{2} H(x, t) \le h(x, t)t$ for all (x, t), where $H(x, t) = \int_0^t h(x, s)\, ds$.

Set

$$\nabla_A u = \big(\nabla + iA(x)\big)u$$

and

$$H_A^1\big(\mathbb{R}^N\big) = \big\{u \in L^2\big(\mathbb{R}^N\big) : \nabla_A u \in L^2\big(\mathbb{R}^N\big)\big\}.$$

Hence $H_A^1(\mathbb{R}^N)$ is the Hilbert space under the scalar product

$$\langle u, v\rangle = \mathrm{Re} \int_{\mathbb{R}^N} \big(\big(\nabla u + iA(x)u\big)\overline{\big(\nabla v + iA(x)v\big)} + u\bar{v}\big)\, dx,$$

the norm induced by the product $(\cdot,\cdot)$ is

$$\begin{aligned}\|u\|_{H^1_A(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N}\left(|\nabla_A u|^2+|u|^2\right)dx\right)^{\frac{1}{2}}\\ &= \left(\int_{\mathbb{R}^N}\left(\left|\nabla u+iA(x)u\right|^2+|u|^2\right)dx\right)^{\frac{1}{2}}\\ &= \left(\int_{\mathbb{R}^N}\left(|\nabla u|^2+\left(\left|iA(x)\right|^2+1\right)|u|^2\right)dx-2\operatorname{Re}\int_{\mathbb{R}^N}iA(x)\overline{u}\nabla u\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Let

$$E:=\left\{u\in H^1_A\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}\lambda V(x)|u|^2\,dx<\infty\right\},$$

which is a Hilbert space equipped with the norm

$$\|u\|^2_\lambda=\int_{\mathbb{R}^N}\left(|\nabla_A u|^2+\lambda V(x)|u|^2\right)dx.$$

Remark 2.1 We have the following diamagnetic inequality (see [16] for example):

$$\left|\nabla_A u(x)\right|\geq\left|\nabla\left|u(x)\right|\right|\quad\text{for } u\in H^1_A\left(\mathbb{R}^N\right).$$

Indeed, since A is real-valued,

$$\left|\nabla|u|(x)\right|=\left|\operatorname{Re}\left(\nabla u\frac{\overline{u}}{|u|}\right)\right|=\left|\operatorname{Re}(\nabla u+iAu)\frac{\overline{u}}{|u|}\right|\leq|\nabla u+iAu|$$

(the bar denotes complex conjugation) this fact means that if $u\in H^1_A(\mathbb{R}^N)$, then $|u|\in H^1(\mathbb{R}^N)$, and therefore $u\in L^p(\mathbb{R}^N)$ for any $p\in[2,2^*]$.

Remark 2.2 The spaces $H^1_A(\mathbb{R}^N)$ and the spaces $H^1(\mathbb{R}^N)$ are not comparable; more precisely, in general $H^1_A(\mathbb{R}^N)\not\subseteq H^1(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)\not\subseteq H^1_A(\mathbb{R}^N)$. However, it is proved by Arioli and Szulkin [17] that if K is a bounded domain with regular boundary, then $H^1_A(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ are equivalent, where $H^1_A(K)=\{u\in L^2(K):\nabla u\in L^2(K)\}$ with the norm $\|u\|_{H^1_A(K)}=(\int_K(|\nabla_A u|^2+|u|^2)\,dx)^{\frac{1}{2}}$.

Let

$$E_\lambda:=\left\{u\in H^1_A\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}V(x)|u|^2\,dx<\infty\right\}$$

with the norms

$$\|u\|^2_\lambda=\int_{\mathbb{R}^N}\left(|\nabla_A u|^2+\lambda V(x)|u|^2\right)dx.$$

Thus, it is easy to see that the norm $\|\cdot\|_E$ is equivalent to the one $\|\cdot\|_\lambda$ for each $\lambda>0$. From Remark 2.2, for each $s\in[2,2^*]$, there is $c_s>0$ (independent of λ) such that if $\lambda>1$, then

$$\left(\int_{\mathbb{R}^N}|u|^s\right)^{\frac{1}{s}} \le c_s\left(\int_{\mathbb{R}^N}\left|\nabla|u|\right|^2\right)^{\frac{1}{2}} \le c_s\left(\int_{\mathbb{R}^N}|\nabla_A u|^2\right)^{\frac{1}{2}} \le c_s\|u\|_\lambda. \tag{2.2}$$

Consider the functional

$$\begin{aligned} J_\lambda(u) &:= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla_A u|^2 + \lambda V(x)|u|^2\right)dx - \frac{\lambda}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\,dx \\ &\quad - \frac{\lambda}{2}\int_{\mathbb{R}^N} H\left(x,|u|^2\right)dx \\ &= \frac{1}{2}\|u\|_\lambda^2 - \frac{\lambda}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\,dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} H\left(x,|u|^2\right)dx. \end{aligned}$$

Under the assumptions [28], $J_\lambda \in C^1(E_\lambda, \mathbb{R})$ and its critical points are solutions of (2.1).

Theorem 2.1 *Let* (V), (A) *and* (H) *be satisfied*. *Thus*:

(1) *For any* $\sigma > 0$, *there is* $\Lambda_\sigma > 0$ *such that problem* (2.1) *has at least one solution* u_λ *for each* $\lambda \ge \Lambda_\sigma$ *satisfying* $0 < J_\lambda(u_\lambda) \le \sigma\lambda^{1-\frac{N}{2}}$.

(2) *Assume additionally that* $h(x,t)$ *is odd in* t; *for any* $m \in N$ *and* $\sigma > 0$, *there is* $\Lambda_{m\sigma} > 0$ *such that problem* (2.1) *has at least* m *pairs of solutions* u_λ *with* $0 < J_\lambda(u_\lambda) \le \sigma\lambda^{1-\frac{N}{2}}$ *whenever* $\lambda \ge \Lambda_{m\sigma}$.

Remark 2.3 We should point out that Theorem 2.3 is different from the previous results of [9, 10] in three main directions:

(i) $A(x) \not\equiv 0$. There exist many functions $h(x,t)$ satisfying condition (H), for example, $h(x,t) = P(x)|t|^{p-2}t$, where $P(x)$ is a positive and bounded function.

(ii) Other potentials $V(x)$ guaranteeing compactness of the embedding from $E_\lambda \hookrightarrow L^p(\mathbb{R}^N)$ can also be used in this paper. For example, (1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\liminf_{|x|\to\infty} V(x) > V(x_0) = \min_{x\in\mathbb{R}^N} V = 0$; (2) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ with periodic function (or bounded function) and $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$.

(iii) We use Lions' second concentration compactness principle and concentration compactness principle at infinity to prove that the (PS) condition holds, which is different from methods used in [9, 10].

3 $(PS)_c$ Condition

Recall that we say that a sequence (u_n) is a (PS) sequence at level c ($(PS)_c$-sequence, for short) if $\Phi_\lambda(u_n) \to c$ and $\Phi'_\lambda(u_n) \to 0$. Φ_λ is said to satisfy the $(PS)_c$ condition if any $(PS)_c$-sequence contains a convergent subsequence.

Lemma 3.1 *Let* (V), (A) *and* (H) *be satisfied*. *Then there exists constant* $M(c)$ *which is independent of* $\lambda \ge 0$ *such that* $c \ge 0$ *and*

$$\limsup_{n\to\infty}\|u_n\|_\lambda^2 \le M(c).$$

Proof Let $\{u_n\}$ be a sequence in E such that

$$c + o(1) = J(u_n) = \frac{1}{2}\|u_n\|_\lambda^2 - \frac{\lambda}{2^*}\int_{\mathbb{R}^N}|u_n|^{2^*}\,dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} H\left(x,|u_n|^2\right)dx, \tag{3.1}$$

$$\begin{aligned} o(1)\|u_n\| &= \langle J'(u_n), v\rangle \\ &= \operatorname{Re}\biggl\{\int_{\mathbb{R}^N} \bigl(\nabla_A u_n \cdot \overline{\nabla_A v} + \lambda V(x) u_n \bar{v}\bigr)\,dx \\ &\quad - \lambda \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n \bar{v}\,dx - \lambda \int_{\mathbb{R}^N} h\bigl(x, |u_n|^2\bigr) u_n \bar{v}\,dx\biggr\}. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2) we have

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\mu} J'_\lambda(u_n) u_n &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \bigl(|\nabla_A u_n|^2 + \lambda V(x)|u_n|^2\bigr)\,dx \\ &\quad + \left(\frac{1}{\mu} - \frac{1}{2^*}\right)\lambda \int_{\mathbb{R}^N} |u_n|^{2^*}\,dx \\ &\quad + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu} h\bigl(x, |u_n|^2\bigr)|u_n|^2 - \frac{1}{2} H\bigl(x, |u_n|^2\bigr)\right)dx. \end{aligned} \tag{3.3}$$

On the other hand, condition (h_3) implies that

$$\frac{1}{\mu} h\bigl(x, |u_n|^2\bigr)|u_n|^2 - \frac{1}{2} H\bigl(x, |u_n|^2\bigr) \geq 0.$$

Thus, it follows from (3.3) that

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_\lambda^2 \leq c + o(1) + \varepsilon_n \|u_n\|_\lambda,$$

hence for n large enough, we have

$$\|u_n\|_\lambda^2 \leq \frac{2\mu}{\mu - 2} c.$$

Thus $\|u_n\|_\lambda$ is bounded as $n \to \infty$. Taking the limit in (3.3) shows that $c \geq 0$. This completes the proof of Lemma 3.1. □

The main result in this section is the following compactness result.

Lemma 3.2 *Suppose that* (V), (A) *and* (H) *hold. For any* $\lambda \geq 1$, J_λ *satisfies the* $(PS)_c$ *condition, for all* $c \in (0, \alpha_0 \lambda^{1-\frac{N}{2}})$, *where* $\alpha_0 = (\frac{1}{\mu} - \frac{1}{2^*}) S^{\frac{N}{2}}$; *that is, any* $(PS)_c$*-sequence* $(u_n) \subset E_\lambda$ *has a strongly convergent subsequence in* E_λ.

Proof Let $\{u_n\}$ be a $(PS)_c$ sequence, by Lemma 3.1, $\{u_n\}$ is bounded in $H_A^1(\mathbb{R}^N)$. Hence, by diamagnetic inequality, $\{|u_n|\}$ is bounded in $H_A^1(\mathbb{R}^N)$. Then, for some subsequence, there is $u \in H_A^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H_A^1(\mathbb{R}^N)$. We claim that

$$\int_{\mathbb{R}^N} |u_n|^{2^*}\,dx \to \int_{\mathbb{R}^N} |u|^{2^*}\,dx. \tag{3.4}$$

In order to prove this claim, we suppose that

$$\bigl|\nabla |u_n|\bigr|^2 \rightharpoonup \bigl|\nabla |u|\bigr|^2 + \sigma \quad \text{and} \quad |u_n|^{2^*} \rightharpoonup |u|^{2^*} + \nu \quad \bigl(\text{weak}^* \text{ sense of measures}\bigr).$$

Using the concentration compactness principle due to Lions (*cf.* [29, Lemma 1.2]), we obtain a countable index set I, sequences $\{x_i\} \subset \mathbb{R}^N$, $\{\sigma_i\}, \{\nu_i\} \subset (0, \infty)$ such that

$$\nu = \sum_{i\in I} \delta_{x_i} \nu_i, \qquad \sigma \geq \sum_{i\in I} \delta_{x_i} \sigma_i \quad \text{and} \quad \sigma_i \geq S \nu_i^{2/2^*} \tag{3.5}$$

for all $i \in I$, where δ_{x_i} are Dirac measures at x_i and σ_i, ν_i are constants.

Now, let x_i be a singular point of the measures σ and ν. We define a function $\phi(x) \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\phi(x) = 1$ in $B(x_i, \varepsilon)$, $\phi(x) = 0$ in $\mathbb{R}^N \setminus B(x_i, 2\varepsilon)$ and $|\nabla\phi| \leq 2/\varepsilon$ in $\mathbb{R}^N$. Since $\{u_n\phi\}$ is bounded in $H_A^1(\mathbb{R}^N)$ and ϕ takes values in $\mathbb{R}$, a direct calculation shows that

$$\langle J'_\lambda(u_n), u_n\phi\rangle \to 0$$

and

$$\overline{\nabla_A(u_n\phi)} = i\overline{u}_n \nabla\phi + \phi\overline{\nabla_A u_n}.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla_A u_n|^2 \phi\, dx + \int_{\mathbb{R}^N} \lambda V(x)|u_n|^2 \phi\, dx \\
&\quad = -\mathrm{Re}\left(\int_{\mathbb{R}^N} i\overline{u}_n \nabla_A u_n \nabla\phi\, dx\right) + \lambda \int_{\mathbb{R}^N} h\big(x, |u_n|^2\big)|u_n|^2 \phi\, dx \\
&\qquad + \lambda \int_{\mathbb{R}^N} |u_n|^{2^*} \phi\, dx + o_n(1).
\end{aligned} \tag{3.6}$$

By Hölder's inequality, it is not difficult to prove that

$$\limsup_{n\to\infty} \left|\mathrm{Re}\left(\int_{\mathbb{R}^N} i\overline{u}_n \nabla_A u_n \nabla\phi\, dx\right)\right| = 0.$$

In this way, it follows that

$$\int_{\mathbb{R}^N} |\nabla_A u_n|^2 \phi\, dx \leq \lambda \int_{\mathbb{R}^N} h\big(x, |u_n|^2\big)|u_n|^2 \phi\, dx + \lambda \int_{\mathbb{R}^N} |u_n|^{2^*} \phi\, dx + o_n(1).$$

Consequently, using the fact that $u_n \to u$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $1 \leq s < 2^*$ and ϕ has compact support, we can let $n \to \infty$ in the last inequality to obtain

$$\int_{\mathbb{R}^N} \phi\, d\sigma \leq \lambda \int_{\mathbb{R}^N} \phi\, d\nu.$$

Letting $\varepsilon \to 0$, we obtain $\sigma_i \leq \lambda\nu_i$. Combining this with (3.5), we obtain $\nu_i \geq \lambda^{-1} S \nu_i^{\frac{2}{2^*}}$. This result implies that

$$\text{(I)} \quad \nu_i = 0 \quad \text{or} \quad \text{(II)} \quad \nu_i \geq \big(\lambda^{-1}S\big)^{\frac{N}{2}}.$$

To obtain the possible concentration of mass at infinity, we will use the concentration compactness principle at infinity [30]. Similarly, we define a cut-off function $\phi_R \in$

$C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\phi_R(x) = 0$ on $|x| < R$ and $\phi_R(x) = 1$ on $|x| > R+1$. Note that $\{u_n\phi_R\}$ is bounded in $H_A^1(\mathbb{R}^N)$ and ϕ takes values in $\mathbb{R}$. A direct calculation shows that $\langle J_\lambda'(u_n), u_n\phi_R\rangle \to 0$, this fact implies that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla_A u_n|^2\phi_R\,dx + \int_{\mathbb{R}^N} \lambda V(x)|u_n|^2\phi_R\,dx \\
&\quad = -\mathrm{Re}\left(\int_{\mathbb{R}^N} i\overline{u}_n\nabla_A u_n\nabla\phi_R\,dx\right) + \lambda\int_{\mathbb{R}^N} h\big(x,|u_n|^2\big)|u_n|^2\phi_R\,dx \\
&\qquad + \lambda\int_{\mathbb{R}^N} |u_n|^{2^*}\phi_R\,dx + o_n(1).
\end{aligned} \tag{3.7}$$

It is easy to prove that

$$-\lim_{R\to\infty}\lim_{n\to\infty}\mathrm{Re}\left(\int_{\mathbb{R}^N} i\overline{u}_n\nabla_A u_n\nabla\phi_R\,dx\right) = 0.$$

Letting $R\to\infty$, we obtain $\sigma_\infty \le \lambda\nu_\infty$. Thus $\nu_\infty \ge \lambda^{-1}S\nu_\infty^{\frac{2}{2^*}}$. This result implies that

$$\text{(III)}\quad \nu_\infty = 0 \quad\text{or}\quad \text{(IV)}\quad \nu_\infty \ge \big(\lambda^{-1}S\big)^{\frac{N}{2}}.$$

Next, we claim that (II) and (IV) cannot occur. If case (IV) holds for some $i\in I$, then by condition (H) we have

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle\right) \\
&\ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\big(|\nabla_A u_n|^2 + \lambda V(x)|u_n|^2\big)\,dx + \left(\frac{1}{\mu}-\frac{1}{2^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{2^*}\,dx \\
&\quad + \lambda\int_{\mathbb{R}^N}\left(\frac{1}{\mu}h\big(x,|u_n|^2\big)|u_n|^2 - \frac{1}{2}H\big(x,|u_n|^2\big)\right)dx \\
&\ge \left(\frac{1}{\mu}-\frac{1}{2^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{2^*}\,dx \ge \left(\frac{1}{\mu}-\frac{1}{2^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{2^*}\phi_R\,dx \\
&= \left(\frac{1}{\mu}-\frac{1}{2^*}\right)\lambda\nu_\infty \ge \alpha_0\lambda^{1-\frac{N}{2}} \quad \text{as } R\to\infty,
\end{aligned}$$

where $\alpha_0 = (\frac{1}{\mu}-\frac{1}{2^*})S^{\frac{N}{2}}$. This is impossible. Consequently, $\nu_i = 0$ for all $i\in I$. Similarly, if case (II) holds for some $i\in I$, then by condition (H) we have

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle\right) \\
&\ge \left(\frac{1}{\mu}-\frac{1}{2^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{2^*}\,dx \ge \left(\frac{1}{\mu}-\frac{1}{2^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{2^*}\phi\,dx \\
&= \left(\frac{1}{\mu}-\frac{1}{2^*}\right)\lambda\nu \ge \alpha_0\lambda^{1-\frac{N}{2}} \quad \text{as } \varepsilon\to 0,
\end{aligned}$$

which leads to a contradiction. Thus, we must have that (II) cannot occur for each i. Thus limit (3.4) holds.

Thus, from the Brezis-Lieb lemma [31], we have

$$\begin{aligned} o(1)\|u_n\|_\lambda &= \langle J_\lambda'(u_n), u_n\rangle = \|u_n\|_\lambda^2 - \lambda \int_{\mathbb{R}^N} |u_n|^{2^*}\,dx - \lambda \int_{\mathbb{R}^N} H\big(x, |u_n|^2\big)\,dx \\ &= \|u_n - u\|_\lambda^2 + \|u\|_\lambda^2 - \lambda \int_{\mathbb{R}^N} |u|^{2^*}\,dx - \lambda \int_{\mathbb{R}^N} H\big(x, |u|^2\big)\,dx \\ &= \|u_n - u\|_\lambda^2 + o(1)\|u\|_\lambda, \end{aligned}$$

here we use $J_\lambda'(u) = 0$. Thus we prove that $\{u_n\}$ strongly converges to u in $H_A^1(\mathbb{R}^N)$. This completes the proof of Lemma 3.2. □

4 Proof of Theorem 2.1

In the following, we always consider $\lambda \geq 1$. By assumptions (V), (A) and (H), one can see that $J_\lambda(u)$ has mountain pass geometry.

Lemma 4.1 *Assume that* (V), (A) *and* (H) *hold. There exist* $\alpha_\lambda, \rho_\lambda > 0$ *such that* $J_\lambda(u) > 0$ *if* $u \in B_{\rho_\lambda} \setminus \{0\}$ *and* $J_\lambda(u) \geq \alpha_\lambda$ *if* $u \in \partial B_{\rho_\lambda}$, *where* $B_{\rho_\lambda} = \{u \in E : \|u\|_\lambda \leq \rho_\lambda\}$.

Proof By condition (H), for $\delta \leq (4\lambda c_s)^{-1}$, there is $C_\delta > 0$ such that

$$\frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*}\,dx + \frac{1}{2} \int_{\mathbb{R}^N} H\big(x, |u|^2\big)\,dx \leq \delta |u|_2^2 + C_\delta |u|_{2^*}^{2^*}.$$

So, from (A) and (V) it follows that

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{2}\|u\|_\lambda^2 - \lambda\delta|u|_2^2 - \lambda C_\delta |u|_{2^*}^{2^*} \\ &\geq \frac{1}{4}\|u\|_\lambda^2 - \lambda C_\delta |u|_{2^*}^{2^*}. \end{aligned}$$

By (2.2) and $2^* > 2$, we know that the conclusion of Lemma 4.1 holds. This completes the proof of Lemma 4.1. □

Lemma 4.2 *Under the assumption of Lemma* 4.1, *for any finite-dimensional subspace* $F \subset E_\lambda$,

$$J_\lambda(u) \to -\infty \quad \textit{as } u \in F, \|u\| \to \infty.$$

Proof Using conditions (A), (V) and (H), we can get

$$J_\lambda(u) \leq \frac{1}{2}\|u\|_\lambda^2 - \lambda a_0 |u|_p^p$$

for all $u \in E$ since all norms in a finite-dimensional space are equivalent and $p > 2$. This completes the proof of Lemma 4.2. □

Since $J_\lambda(u)$ does not satisfy the $(PS)_c$ condition for all $c > 0$, in the following we will find a special finite-dimensional subspace by which we construct sufficiently small minimax levels.

The assumption (V) implies that there is $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Without loss of generality we assume from now on that $x_0 = 0$.

Observe that by (h_3)

$$\frac{\lambda}{2^*}\int_{\mathbb{R}^N} K(x)|u|^{2^*}\,dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} H\bigl(x,|u|^2\bigr)\,dx \ge a_0\lambda\int_{\mathbb{R}^N}|u|^p\,dx.$$

Define the function $I_\lambda \in C^1(E_\lambda,\mathbb{R})$ by

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N}\bigl(|\nabla_A u|^2 + \lambda V(x)|u|^2\bigr)\,dx - a_0\lambda\int_{\mathbb{R}^N}|u|^p\,dx.$$

Then $J_\lambda(u) \le I_\lambda(u)$ for all $u \in E$, and it suffices to construct small minimax levels for I_λ.

Note that

$$\inf\left\{\int_{\mathbb{R}^N}|\nabla\phi|^2\,dx : \phi \in C_0^\infty\bigl(\mathbb{R}^N\bigr), |\phi|_2 = 1\right\} = 0.$$

For any $\delta > 0$, one can choose $\phi_\delta \in C_0^\infty(\mathbb{R}^N)$ with $|\phi_\delta|_2 = 1$ and $\operatorname{supp}\phi_\delta \subset B_{r_\delta}(0)$ so that $|\nabla\phi_\delta|_2^2 < \delta$. Set

$$f_\lambda = \phi_\delta\bigl(\lambda^{\frac{1}{2}}x\bigr), \tag{4.1}$$

then

$$\operatorname{supp} f_\lambda \subset B_{\lambda^{-\frac{1}{2}}r_\delta}(0).$$

Thus, for $t \ge 0$, we have

$$\begin{aligned} I_\lambda(tf_\lambda) &= \frac{t^2}{2}\int_{\mathbb{R}^N}\bigl(|\nabla_A f_\lambda|^2 + \lambda V(x)|f_\lambda|^2\bigr)\,dx - t^p a_0\lambda\int_{\mathbb{R}^N}|f_\lambda|^p\,dx \\ &= \lambda^{1-\frac{N}{2}}\left(\frac{t^2}{2}\int_{\mathbb{R}^N}\bigl(|\nabla_A\phi_\delta|^2 + V\bigl(\lambda^{-\frac{1}{2}}x\bigr)|\phi_\delta|^2\bigr)\,dx - t^p a_0\int_{\mathbb{R}^N}|\phi_\delta|^p\,dx\right) \\ &= \lambda^{1-\frac{N}{2}}\Psi_\lambda(t\phi_\delta), \end{aligned}$$

where $\Psi_\lambda \in C^1(E_\lambda,\mathbb{R})$ defined by

$$\Psi_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N}\bigl(|\nabla_A u|^2 + V\bigl(\lambda^{-\frac{1}{2}}x\bigr)|u|^2\bigr)\,dx - a_0\int_{\mathbb{R}^N}|u|^p\,dx.$$

Obviously,

$$\max_{t\ge 0}\Psi_\lambda(t\phi_\delta) = \frac{p-2}{2p(pa_0)^{\frac{2}{p-2}}}\left[\int_{\mathbb{R}^N}\bigl(|\nabla_A\phi_\delta|^2 + V\bigl(\lambda^{-\frac{1}{2}}x\bigr)|\phi_\delta|^2\bigr)\,dx\right]^{\frac{p}{p-2}}.$$

On the one hand, since $V(0) = 0$ and note that $\operatorname{supp}\phi_\delta \subset B_{r_\delta}(0)$, there is $\Lambda_{\delta_1} > 0$ such that

$$V\bigl(\lambda^{-\frac{1}{2}}x\bigr) \le \frac{\delta}{|\phi_\delta|_2^2}\quad \text{for all } |x| \le r_\delta \text{ and } \lambda \ge \Lambda_{\delta_1}.$$

On the other hand, by Hölder's inequality, we have

$$\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^2\,dx \le \int_{\mathbb{R}^N} \bigl(2|\nabla\phi_\delta|^2 + 2\bigl|A\bigl(\lambda^{-\frac{1}{2}}x\bigr)\phi_\delta\bigr|^2\bigr)\,dx. \tag{4.2}$$

Since $A(x)$ is continuous on $\mathbb{R}^N$ and $A(0) = 0$, there exists $\Lambda_{\delta_2} > 0$ such that

$$\bigl|A\bigl(\lambda^{-\frac{1}{2}}x\bigr)\bigr| \le \sqrt{\frac{\delta}{|\phi_\delta|_2^2}} \quad \text{for all } |x| \le r_\delta \text{ and } \lambda \ge \Lambda_{\delta_2}. \tag{4.3}$$

Without loss of generality, we take $\Lambda_\delta := \{\Lambda_{\delta_1}, \Lambda_{\delta_2}\}$. So, by (4.2) and (4.3) we can get

$$\max_{t\ge 0} \Psi_\lambda(t\phi_\delta) \le \frac{p-2}{2p(pa_0)^{\frac{2}{p-2}}}(5\delta)^{\frac{p}{p-2}}. \tag{4.4}$$

Therefore, for all $\lambda \ge \Lambda_\delta$,

$$\max_{t\ge 0} J_\lambda(t\phi_\delta) \le \frac{p-2}{2p(pa_0)^{\frac{2}{p-2}}}(5\delta)^{\frac{p}{p-2}}\lambda^{1-\frac{N}{2}}. \tag{4.5}$$

Thus we have the following lemma.

Lemma 4.3 *Under the assumption of Lemma* 4.1, *for any* $\sigma > 0$, *there exists* $\Lambda_\sigma > 0$ *such that for each* $\lambda \ge \Lambda_\sigma$, *there is* $\hat{f}_\lambda \in E_\lambda$ *with* $\|\hat{f}_\lambda\| > \rho_\lambda$, $J_\lambda(\hat{f}_\lambda) \le 0$ *and*

$$\max_{t\in[0,1]} J_\lambda(t\hat{f}_\lambda) \le \sigma\lambda^{1-\frac{N}{2}}. \tag{4.6}$$

Proof Choose $\delta > 0$ so small that

$$\frac{p-2}{2p(pa_0)^{\frac{2}{p-2}}}(5\delta)^{\frac{p}{p-2}} \le \sigma,$$

and let $f_\lambda \in E$ be the function defined by (4.1). Taking $\Lambda_\sigma = \Lambda_\delta$. Let $\hat{t}_\lambda > 0$ be such that $\hat{t}_\lambda\|f_\lambda\|_\lambda > \rho_\lambda$ and $J_\lambda(tf_\lambda) \le 0$ for all $t \ge \hat{t}_\lambda$. By (4.3), let $\hat{f}_\lambda = \hat{t}_\lambda f_\lambda$; we know that the conclusion of Lemma 4.3 holds. □

For any $m^* \in N$, one can choose m^* functions $\phi_\delta^i \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi_\delta^i \cap \operatorname{supp}\phi_\delta^k = \emptyset$, $i \ne k$, $|\phi_\delta^i|_p = 1$ and $|\nabla\phi_\delta^i|_2^2 < \delta$. Let $r_\delta^{m^*} > 0$ be such that $\operatorname{supp}\phi_\delta^i \subset B_{r_\delta}^i(0)$ for $i = 1, 2, \ldots, m^*$. Set

$$f_\lambda^i(x) = \phi_\delta^i\bigl(\lambda^{\frac{1}{2}}x\bigr) \quad \text{for } j = 1, 2, \ldots, m^*$$

and

$$H_{\lambda\delta}^{m^*} = \operatorname{span}\bigl\{f_\lambda^1, f_\lambda^2, \ldots, f_\lambda^{m^*}\bigr\}.$$

Observe that for each $u = \sum_{i=1}^{m^*} c_i f_\lambda^i \in H_{\lambda\delta}^{m^*}$,

$$\int_{\mathbb{R}^N} |\nabla_A u|^2\,dx = \sum_{i=1}^{m^*} |c_i|^2 \int_{\mathbb{R}^N} \bigl|\nabla_A f_\lambda^i\bigr|^2\,dx,$$

$$\int_{\mathbb{R}^N} V(x)|u|^2\,dx = \sum_{i=1}^{m^*} |c_i|^2 \int_{\mathbb{R}^N} V(x)\big|f_\lambda^i\big|^2\,dx,$$

$$\frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,dx = \frac{1}{2^*}\sum_{i=1}^{m^*} |c_i|^{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,dx$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N} H\big(x,|u|^2\big)\,dx = \frac{1}{2}\sum_{i=1}^{m^*}\int_{\mathbb{R}^N} H\big(x,\big|c_i f_\lambda^i\big|^2\big)\,dx.$$

Thus

$$J_\lambda(u) = \sum_{i=1}^{m^*} J_\lambda\big(c_i f_\lambda^i\big),$$

and as before

$$J_\lambda\big(c_i f_\lambda^i\big) \le \lambda^{1-\frac{N}{2}}\,\Psi\big(|c_i| f_\lambda^i\big).$$

Set

$$\beta_\delta := \max\big\{\big|\phi_\delta^i\big|_2^2 : j = 1,2,\ldots,m^*\big\}$$

and choose $\Lambda_{m^*\delta} > 0$ so that

$$V\big(\lambda^{1-\frac{N}{2}}x\big) \le \frac{\delta}{\beta_\delta} \quad \text{for all } |x| \le r_\delta^{m^*} \text{ and } \lambda \ge \Lambda_{m^*\delta}.$$

As before, we can obtain the following:

$$\max_{u\in H_{\lambda\delta}^{m^*}} J_\lambda(u) \le \frac{m^*(p-2)}{2p(pa_0)^{\frac{2}{p-2}}}(5\delta)^{\frac{p}{p-2}}\lambda^{1-\frac{N}{2}} \tag{4.7}$$

for all $\lambda \ge \Lambda_{m^*\delta}$.

Using this estimate we have the following.

Lemma 4.4 *Under the assumption of Lemma* 4.1, *for any* $m^* \in \mathbb{N}$ *and* $\sigma > 0$, *there exists* $\Lambda_{m^*\sigma} > 0$ *such that for each* $\lambda \ge \Lambda_{m^*\sigma}$, *there exists an* m^**-dimensional subspace* $F_{\lambda m^*}$ *satisfying*

$$\max_{u\in F_{\lambda\delta}} J_\lambda(u) \le \sigma\lambda^{1-\frac{N}{2}}.$$

Proof Choose $\delta > 0$ so small that

$$\frac{m^*(p-2)}{2p(pa_0)^{\frac{2}{p-2}}}(5\delta)^{\frac{p}{p-2}} \le \sigma$$

and take $F_{\lambda\delta} = H_{\lambda\delta}^{m^*}$. By (4.5), we know that the conclusion of Lemma 4.4 holds. □

We now establish the existence and multiplicity results.

Proof of Theorem 2.1 Using Lemma 4.3, we choose $\Lambda_\sigma > 0$ and define for $\lambda \geq \Lambda_\sigma$ the minimax value

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(t\hat{f}_\lambda),$$

where

$$\Gamma_\lambda := \{\gamma \in C([0,1],E) : \gamma(0) = 0 \text{ and } \gamma(1) = \hat{f}_\lambda\}.$$

By Lemma 4.1, we have $\alpha_\lambda \leq c_\lambda \leq \sigma\lambda^{1-\frac{N}{2}}$. In virtue of Lemma 3.2, we know that J_λ satisfies the $(PS)_{c_\lambda}$ condition, there is $u_\lambda \in E$ such that $J'_\lambda(u_\lambda) = 0$ and $J_\lambda(u_\lambda) = c_\lambda$, hence the existence is proved.

Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ for each $Z \in \Sigma$. Let gen(Z) be the Krasnoselski genus and

$$i(Z) := \min_{h \in \Gamma_{m^*}} \operatorname{gen}(h(Z) \cap \partial B_{\rho_\lambda}),$$

where Γ_{m^*} is the set of all odd homeomorphisms $h \in C(E,E)$ and ρ_λ is the number from Lemma 4.1. Then i is a version of Benci's pseudo-index [32]. Let

$$c_{\lambda i} := \inf_{i(Z) \geq i} \sup_{u \in Z} J_\lambda(u), \quad 1 \leq i \leq m^*.$$

Since $J_\lambda(u) \geq \alpha_\lambda$ for all $u \in \partial B^+_{\rho\lambda}$ and since $i(F_{\lambda m^*}) = \dim F_{\lambda m^*} = m^*$,

$$\alpha_\lambda \leq c_{\lambda 1} \leq \cdots \leq c_{\lambda m^*} \leq \sup_{u \in H_{\lambda m^*}} J_\lambda(u) \leq \sigma\lambda^{1-\frac{N}{2}}.$$

It follows from Lemma 3.2 that J_λ satisfies the $(PS)_{c_\lambda}$ condition at all levels c_i. By the usual critical point theory, all c_i are critical levels and J_λ has at least m^* pairs of nontrivial critical points. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SL carried out the theoretical studies, participated in the sequence alignment and drafted the manuscript. YS participated in the design of the study and performed the statistical analysis. All authors read and approved the final manuscript.

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