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About Dirichlet boundary value problem for the heat equation in the infinite angular domain

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Abstract

In this paper it is established that in an infinite angular domain for Dirichlet problem of the heat conduction equation the unique (up to a constant factor) non-trivial solution exists, which does not belong to the class of summable functions with the found weight. It is shown that for the adjoint boundary value problem the unique (up to a constant factor) non-trivial solution exists, which belongs to the class of essentially bounded functions with the weight found in the work. It is proved that the operator of a boundary value problem of heat conductivity in an infinite angular domain in a class of growing functions is Noetherian with an index which is equal to minus one.

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1 Introduction

Different kinds of processes of mass and heat transfer lead to solving boundary value problems for parabolic equations in a domain with a moving in time boundary (non-cylindrical domain). These processes are the most important factor that affects, for example, the reliability of various contact systems. Due to the increased speed-in-action of the electrical contacts, that is, because of the short duration of the process, it is experimentally impossible to determine accurately the temperature field of the contact system and the dynamics of its change in time. Therefore, the study of boundary value problems of heat conduction in domains with moving boundary and the degeneracy at the initial time is actual. Consideration of a wide range of issues of mathematical physics [1, 2], in particular, the solving of boundary value problems in the heat equation degenerating domains leads to the need to study the singular integral equations of Volterra type when the norm of a integral operator is equal to unit. These problems have a direct connection with the theory of loaded equations [3, 4]. It turned out that these issues have a close connection with the problem of establishing the classes of uniqueness from [5–8], which have been further developed in [9–16] and other works.

2 On classes of uniqueness

Let us give a brief overview of some works on uniqueness classes for parabolic equations. In the domain $Q = R^n \times (0, T)$ for the boundary value problem

$$u_t(x,t) - \Delta u(x,t) = 0, \qquad u(x,t)|_{t=0} = 0$$

the following classes of uniqueness are established: $u(x,t) \leq C \cdot \exp\{k|x|^2(\ln|x|)^\alpha\}$, $\alpha \in [0,1]$ (Holmgren [5]). $u(x,t) \leq C \cdot \exp\{a_T|x|^2\}$ (Tikhonov [6]). $u(x,t) \leq C \cdot \exp\{|x|h(|x|)\}$, $\int_1^\infty \frac{dr}{h(r)} = \infty$ (Täcklind [7]). For the boundary value problem in $Q = R^n \times (0,T)$:

$$u_t(x,t) + Au(x,t) = 0, \qquad u|_{t=0} = 0,$$

where A is the linear elliptic operator of orders $2p$, the following classes of uniqueness are established:

$u(x,t) \leq C \cdot \exp\{k|x|^{\frac{2p}{2p-1}}\}$ (Ladyzhenskaya [9] for one equation with coefficients depending only on t).

$u(x,t) \leq C \cdot \exp\{k|x|^{\frac{2p}{2p-1}}\}$ (Oleinik [10] for systems of parabolic equations with coefficients depending on x and t; [11] for the Cauchy-Neumann problem in an unbounded domain, arbitrarily 'tapering' at infinity).

We also note works of Oleinik and Radkevich [12], Gagnidze [13], Kozhevnikova [14–16], and others, devoted to the establishment of classes of uniqueness for parabolic equations and systems.

3 VP Mihajlov's example on the existence of non-trivial solution for the homogeneous Dirichlet problem in the degenerate domain

Let $Q \subset R^2$ be the domain bounded by a closed curve Γ: $x^2 = -2t\ln t$, passing through the origin of coordinates and the point $\{x = 0, t = 1\}$ and symmetrical with respect to the axis $0t$. The boundary value problem [8]

$$u_t(x,t) - u_{xx}(x,t) = 0, \qquad u|_\Gamma = 0,$$

has a non-trivial solution

$$u(x,t) = t^{-1/2}\exp\left\{-\frac{x^2}{4t}\right\} - 1 \in L_2(Q).$$

We note that although $u_t, u_{xx} \notin L_2(Q)$, the following inclusions hold:

$$t^2u_t(x,t), t^2u_{xx}(x,t) \in L_2(Q).$$

4 Statement of the boundary value problem *L*

In the domain $G = \{(x;t) : -\infty < t < 0, 0 < x < -t\}$ it is required to find a solution to the heat conduction equation

$$u_t(x,t) = a^2u_{xx}(x,t), \tag{1}$$

satisfying the boundary conditions

$$\lim_{t\to-\infty} u(x,t) = 0, \qquad u(x,t)|_{x=0} = 0, \qquad u(x,t)|_{x=-t} = 0, \tag{2}$$

where $u(x,t)$ has to belong to the class

$$\gamma(x,t)\cdot u(x,t)\in L_1(G), \tag{3}$$

when

$$\gamma(x,t)=\max\left[-\frac{\sqrt{-t}}{t+x}\exp\left\{\frac{x^2}{4a^2t}\right\};1+\exp\left\{-\frac{t+x}{a^2}\right\}\right]\geq 2,\quad \{x,t\}\in G.$$

It is required to show that problem (1)-(2) has not in the class (3) non-trivial solutions.
We note that the functions

$$u_1(x,t)=x^2+2a^2t,\qquad u_2(x,t)=(-t)^{-1/2}\exp\left\{-\frac{x^2}{4a^2t}\right\},$$
$$u_3(x,t)=\exp\{x+a^2t\},\quad (x,t)\in G,$$

satisfying (1), do not belong to (3).

Boundary value problems of the form (1)-(2) arise in the mathematical modeling of thermophysical processes in high-current electric arc of the disconnecting device. Tool for describing the physics of the processes in the arc is the heat equation, in which the influence of the heat sources in the arc and the effect of the contraction of the axial section of the arc in the cathode region to a contact spot are taken into account. The diameter of the contact spot is several orders smaller than the diameter of the developed section of the arc column and this fact gives the chance to consider the contact spot as a mathematical point. The solution domain changes over time according to the law which is defined by the conditions of the bridging contact. At the fixed terminal time the contacts close and the solution domain of the problem degenerate. From a mathematical point of view, problematic nature of the problem consists in the presence of a moving boundary and degeneracy of the solution domain at the fixed terminal time. It should be emphasized that the boundary value problems for parabolic equations in a domain with a moving boundary are fundamentally different from the classical problems. Due to the size of the domain depending on the time for this type of problems methods of separating the variables and integral transformations not be applied, as remaining within the classical methods of mathematical physics you cannot conform the solution of the heat equation to the motion of the border line of the heat transport domain.

In addition, the finding of a nonzero solution of the homogeneous problem (1)-(2) allows one to define precisely the uniqueness classes.

Let us consider the boundary value problem (1)-(2).

5 Reduction of problem *L* (1)-(2) to an integral equation

We are looking for a solution of boundary problem (1)-(2) as the sum of the heat potentials of the double layer [17]

$$u(x,t)=\frac{1}{4a^3\sqrt{\pi}}\int_{-\infty}^{t}\frac{x}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\nu(\tau)\,d\tau$$
$$+\frac{1}{4a^3\sqrt{\pi}}\int_{-\infty}^{t}\frac{-x-\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{(x+\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau. \tag{4}$$

Using conditions (2) and the properties of heat potentials, we have the following system of integral equations for the unknown densities $\nu(t)$ and $\varphi(t)$ [17, p.480]:

$$\begin{cases} 0 = \frac{\nu(t)}{2a^2} + \frac{1}{4a^3\sqrt{\pi}}\int_{-\infty}^t \frac{-\tau}{(t-\tau)^{\frac{3}{2}}}\exp\{-\frac{\tau^2}{4a^2(t-\tau)}\}\varphi(\tau)\,d\tau, \\ 0 = -\frac{\varphi(t)}{2a^2} + \frac{1}{4a^3\sqrt{\pi}}\int_{-\infty}^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\{-\frac{t-\tau}{4a^2}\}\varphi(\tau)\,d\tau \\ \quad + \frac{1}{4a^3\sqrt{\pi}}\int_{-\infty}^t \frac{-t}{(t-\tau)^{\frac{3}{2}}}\exp\{-\frac{t^2}{4a^2(t-\tau)}\}\nu(\tau)\,d\tau. \end{cases} \tag{5}$$

Excluding from system (5) $\nu(t)$, we find

$$\nu(t) = -\frac{1}{2a\sqrt{\pi}}\int_{-\infty}^t \frac{-\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau, \tag{6}$$

$$\begin{aligned} 0 = {}& -\varphi(t) + \frac{1}{2a\sqrt{\pi}}\int_{-\infty}^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)\,d\tau \\ & - \frac{1}{4a^2\pi}\int_{-\infty}^t\int_{-\infty}^\tau \frac{t\tau_1}{[(t-\tau)(\tau-\tau_1)]^{\frac{3}{2}}} \\ & \times \exp\left\{-\frac{t^2}{4a^2(t-\tau)} - \frac{\tau_1^2}{4a^2(\tau-\tau_1)}\right\}\varphi(\tau_1)\,d\tau_1\,d\tau. \end{aligned} \tag{7}$$

We introduce the following notation:

$$J(t) = \frac{1}{4a^2\pi}\int_{-\infty}^t t\tau_1\varphi(\tau_1)I(t,\tau_1)\,d\tau_1, \tag{8}$$

where

$$\begin{aligned} I(t,\tau_1) &= \int_{\tau_1}^t \frac{1}{(t-\tau)^{\frac{3}{2}}(\tau-\tau_1)^{\frac{3}{2}}}\exp\left\{-\frac{t^2}{4a^2(t-\tau)} - \frac{\tau_1^2}{4a^2(\tau-\tau_1)}\right\}d\tau \\ &= 2a\sqrt{\pi}\frac{t+\tau_1}{t\tau_1(t-\tau_1)^{\frac{3}{2}}}\exp\left\{-\frac{(t+\tau_1)^2}{4a^2(t-\tau_1)}\right\}. \end{aligned}$$

Here we have used substitution $z = \sqrt{\frac{t-\tau}{\tau-\tau_1}}$ in integral $I(t,\tau_1)$ and the known equality [18, p.321, 3.325]

$$\int_0^\infty \exp\left\{-\mu z^2 - \frac{\eta}{z^2}\right\}dz = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\mu}}\exp\{-2\sqrt{\mu\eta}\}.$$

Substituting the value of the integral $I(t,\tau_1)$ into (8) we have

$$J(t) = \frac{1}{2a\sqrt{\pi}}\int_{-\infty}^t \frac{t+\tau_1}{(t-\tau_1)^{\frac{3}{2}}}\exp\left\{-\frac{(t+\tau_1)^2}{4a^2(t-\tau_1)}\right\}\varphi(\tau_1)\,d\tau_1. \tag{9}$$

In view of (8)-(9), we rewrite (7) in the form

$$\varphi(t) - \int_{-\infty}^t K(t,\tau)\varphi(\tau)\,d\tau = 0, \quad -\infty < t < 0, \tag{10}$$

where

$$K(t,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ -\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)} \right) + \frac{1}{(t-\tau)^{\frac{1}{2}}} \exp\left(-\frac{t-\tau}{4a^2} \right) \right\}.$$

We note that the kernel $K(t,\tau)$ has the following properties:

(1) $K(t,\tau) \geq 0$ and is continuous on $-\infty < \tau \leq t < 0$;
(2) $\lim_{t\to t_0+0} \int_{t_0}^{t} K(t,\tau)\,d\tau = 0$;
(3) $\lim_{t_0\to -\infty} \int_{t_0}^{t} K(t,\tau)\,d\tau = 1$, $-\infty < t_0 < t < 0$.

Indeed, properties (2) and (3) follow from the following equality:

$$\int_{t_0}^{t} K(t,\tau)\,d\tau = \exp\left\{ -\frac{2t}{a^2} \right\} \cdot \operatorname{erfc}\left(-\frac{t+t_0}{2a\sqrt{t-t_0}} \right) + \operatorname{erf}\left(\frac{\sqrt{t-t_0}}{2a} \right),$$

where t_0, t satisfy a condition $-\infty < t_0 < t < 0$.

To prove (3) we note that the first summand of this equality tends to zero when $t_0 \to -\infty$, since for large $-t_0$ for the exponent we have the inequality:

$$0 < -\frac{2t}{a^2} < \frac{(t+t_0)^2}{4a^2(t-t_0)}.$$

Here we have used the asymptotic formula [17, p.718, Formula 3].

6 Investigation of the integral equation (10)

In the homogeneous integral equation (10) we transform its kernel. Using the relations

$$t+\tau = 2t-(t-\tau), \qquad \frac{(t+\tau)^2}{4a^2(t-\tau)} = \frac{t\tau}{a^2(t-\tau)} + \frac{t-\tau}{4a^2},$$

we obtain

$$K(t,\tau) = k(t,\tau)\exp\left\{ -\frac{t-\tau}{4a^2} \right\}, \tag{11}$$

where

$$\begin{aligned} k(t,\tau) = {}& -\frac{t}{a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} \exp\left\{ -\frac{t\tau}{a^2(t-\tau)} \right\} \\ & + \frac{1}{2a\sqrt{\pi}(t-\tau)^{\frac{1}{2}}} \left(1 + \exp\left\{ -\frac{t\tau}{a^2(t-\tau)} \right\} \right). \end{aligned} \tag{12}$$

It is well known that in order to solve (10) it is sufficient to find a solution of the following equation:

$$\tilde{\varphi}(t) - \int_{-\infty}^{t} k(t,\tau)\tilde{\varphi}(\tau)\,d\tau = 0, \quad \text{where } \tilde{\varphi}(t) = \exp\bigl\{ t/\bigl(4a^2\bigr) \bigr\}\varphi(t), -\infty < t < 0. \tag{13}$$

To solve (13) we introduce the following substitutions:

$$t = -1/y, \qquad \tau = -1/x, \qquad d\tau = dx/\bigl(x^2\bigr), \qquad -\infty < \tau < t < 0, \qquad 0 < x < y < +\infty.$$

As a result, (13) takes the form

$$\tilde{\varphi}(-1/y) + \frac{1}{a\sqrt{\pi}} \int_0^y \frac{-y^{3/2}x^{3/2}}{y(y-x)^{3/2}x^2} \cdot \frac{x-y+y}{x} \cdot \exp\left\{-\frac{1}{a^2(y-x)}\right\} \tilde{\varphi}(-1/x)\,dx$$
$$- \frac{1}{2a\sqrt{\pi}} \int_0^y \frac{y^{1/2}x^{1/2}}{(y-x)^{1/2}x^2} \left[1 + \exp\left\{-\frac{1}{a^2(y-x)}\right\}\right] \tilde{\varphi}(-1/x)\,dx = 0, \quad 0 < y < \infty.$$

From the latter equation using the next substitution of the required function,

$$\psi(y) = y^{-3/2}\tilde{\varphi}(-1/y),$$

we obtain

$$y \cdot \psi(y) - \frac{1}{2a\sqrt{\pi}} \int_0^y \frac{1}{(y-x)^{1/2}} \left[1 - \exp\left\{-\frac{1}{a^2(y-x)}\right\}\right] \psi(x)\,dx$$
$$- y \cdot \frac{1}{a\sqrt{\pi}} \int_0^y \frac{1}{(y-x)^{3/2}} \exp\left\{-\frac{1}{a^2(y-x)}\right\} \psi(x)\,dx = 0, \quad 0 < y < \infty. \tag{14}$$

Applying to (14) the Laplace transform, we obtain

$$-\frac{d\Psi(p)}{dp} - \frac{1}{2a\sqrt{p}}\left(1 - \exp\left(-\frac{2\sqrt{p}}{a}\right)\right)\Psi(p) + \frac{d}{dp}\left\{\exp\left(-\frac{2\sqrt{p}}{a}\right)\Psi(p)\right\} = 0,$$

that is, we have

$$\frac{d\Psi(p)}{dp} + \frac{1}{2a\sqrt{p}} \cdot \frac{\operatorname{ch}\frac{\sqrt{p}}{a}}{\operatorname{sh}\frac{\sqrt{p}}{a}}\Psi(p) = 0. \tag{15}$$

The general solution of differential equation (15) is determined by the following formula:

$$\Psi(p) = \frac{C}{\sinh\frac{\sqrt{p}}{a}}, \quad C = \text{const}. \tag{16}$$

To find the original of this function, we rewrite it in the form of a series

$$\Psi(p) = 2C\sum_{n=0}^{\infty} \exp\left\{-\frac{(2n+1)\sqrt{p}}{a}\right\}. \tag{17}$$

Applying the inverse Laplace transform to (17) we have

$$\psi(y) = \frac{C}{a\sqrt{\pi}} \cdot \frac{1}{y^{3/2}} \sum_{n=0}^{\infty}(2n+1)\exp\left\{-\frac{(2n+1)^2}{4a^2y}\right\}, \quad 0 < y < \infty. \tag{18}$$

By inverse substitutions $t = -\frac{1}{y}$, $\tau = -\frac{1}{x}$ and recalling that $\psi(y) = \frac{1}{y^{3/2}}\tilde{\varphi}(-\frac{1}{y})$ equality (18) takes the form

$$\tilde{\varphi}(t) = \frac{C}{a\sqrt{\pi}} \sum_{n=0}^{\infty}(2n+1)\exp\left\{\frac{(2n+1)^2}{4a^2}t\right\}, \quad -\infty < t < 0, \tag{19}$$

that is, for (10) a non-trivial solution is given by

$$\varphi(t) = \frac{C}{a\sqrt{\pi}} \sum_{n=0}^{\infty} (2n+1) \exp\left\{ \frac{n(n+1)}{a^2} t \right\}, \quad -\infty < t < 0. \tag{20}$$

Thus, (19) defines a solution of the homogeneous integral equation (13) and (20) defines a solution of the homogeneous integral equation (10), respectively.

However, the solution $u_{\mathrm{hom}}(x,t)$ (4) of problem (1)-(2) defined by the function $\varphi(t)$ (20) does not belong to class (3), defined by the inclusion of

$$\gamma(x,t) \cdot u(x,t) \in L_1(G),$$

when

$$\gamma(x,t) = \max\left[-\frac{\sqrt{-t}}{t+x} \exp\left\{ \frac{x^2}{4a^2 t} \right\}; 1 + \exp\left\{ \frac{t+x}{a^2} \right\} \right], \quad \{x,t\} \in G,$$

that is

$$\gamma(x,t) \cdot u_{\mathrm{hom}}(x,t) \notin L_1(G).$$

For violation of condition (3), it is sufficient to show it for the solution $u_{\mathrm{hom}}(x,t)$ (4), corresponding to the first term of the sum (20) which is constant. Violation of condition (3) really takes place, the homogeneous boundary value problem L (1)-(2) in class (3) has only the trivial solution.

Thus we have established the following.

Proposition 1 *For the boundary value problem* L (1)-(2) *in class* (3)

$$\dim\{\mathrm{Ker}\{L\}\} = 0.$$

7 Statement of the adjoint boundary value problem L^*

In the domain $G = \{(x;t) : -\infty < t < 0, 0 < x < -t\}$ we consider the following problem: it is required to find a non-trivial solution to the heat conduction equation

$$-u_t^*(x,t) = a^2 u_{xx}^*(x,t), \tag{21}$$

satisfying the boundary conditions:

$$u^*(x,t)|_{x=0} = 0, \qquad u^*(x,t)|_{x=-t} = 0, \tag{22}$$

where $u^*(x,t)$ has to belong to the class

$$[\gamma(x,t)]^{-1} \cdot u^*(x,t) \in L_\infty(G) \tag{23}$$

and

$$\gamma(x,t) = \max\left[-\frac{\sqrt{-t}}{t+x} \exp\left\{ \frac{x^2}{4a^2 t} \right\}; 1 + \exp\left\{ -\frac{t+x}{a^2} \right\} \right] \geq 2, \quad \{x,t\} \in G.$$

It is required to show that problem (21)-(22) has in class (23) only one non-trivial solution.

We note that the functions

$$u_1^*(x,t) = x^2 - 2a^2 t, \qquad u_2^*(x,t) = (-t)^{-1/2}\exp\left\{\frac{x^2}{4a^2 t}\right\},$$
$$u_3^*(x,t) = \exp\{x - a^2 t\}, \quad (x,t) \in G,$$

satisfying (21), belong to (23).

8 Reduction of problem L^* (21)-(22) to an integral equation

We are looking for a solution of boundary problem (21)-(22) as the sum of the heat potentials of the double layer [17]

$$\begin{aligned} u^*(x,t) &= \frac{1}{4a^3\sqrt{\pi}}\int_t^0 \frac{x}{(\tau-t)^{\frac{3}{2}}}\exp\left\{-\frac{x^2}{4a^2(\tau-t)}\right\}\nu^*(\tau)\,d\tau \\ &\quad + \frac{1}{4a^3\sqrt{\pi}}\int_t^0 \frac{-x-\tau}{(\tau-t)^{\frac{3}{2}}}\exp\left\{-\frac{(x+\tau)^2}{4a^2(\tau-t)}\right\}\varphi^*(\tau)\,d\tau. \end{aligned} \tag{24}$$

Using conditions (22) and the properties of heat potentials, we have the following system of integral equations for the unknown densities $\nu^*(t)$ and $\varphi^*(t)$ [17, p.480]:

$$\begin{cases} 0 = \frac{\nu^*(t)}{2a^2} + \frac{1}{4a^3\sqrt{\pi}}\int_t^0 \frac{-\tau}{(\tau-t)^{\frac{3}{2}}}\exp\{-\frac{\tau^2}{4a^2(\tau-t)}\}\varphi^*(\tau)\,d\tau, \\ 0 = \frac{\varphi^*(t)}{2a^2} - \frac{1}{4a^3\sqrt{\pi}}\int_t^0 \frac{1}{(\tau-t)^{\frac{1}{2}}}\exp\{-\frac{\tau-t}{4a^2}\}\varphi^*(\tau)\,d\tau \\ \quad + \frac{1}{4a^3\sqrt{\pi}}\int_t^0 \frac{-t}{(t-\tau)^{\frac{3}{2}}}\exp\{-\frac{t^2}{4a^2(\tau-t)}\}\nu^*(\tau)\,d\tau. \end{cases} \tag{25}$$

Excluding from the system (25) $\nu^*(t)$, we find

$$\nu^*(t) = -\frac{1}{2a\sqrt{\pi}}\int_t^0 \frac{-\tau}{(\tau-t)^{\frac{3}{2}}}\exp\left\{-\frac{\tau^2}{4a^2(\tau-t)}\right\}\varphi^*(\tau)\,d\tau, \tag{26}$$

$$\varphi^*(t) - \frac{1}{2a\sqrt{\pi}}\int_t^0 \frac{1}{(\tau-t)^{\frac{1}{2}}}\exp\left\{-\frac{\tau-t}{4a^2}\right\}\varphi^*(\tau)\,d\tau - J(t) = 0, \tag{27}$$

where

$$J(t) = \frac{1}{4a^2\pi}\int_t^0 t\tau_1\varphi^*(\tau_1)I(t,\tau_1)\,d\tau_1, \tag{28}$$

and

$$\begin{aligned} I(t,\tau_1) &= \int_t^{\tau_1} \frac{1}{[(\tau-t)(\tau_1-\tau)]^{\frac{3}{2}}}\exp\left\{-\frac{t^2}{4a^2(\tau-t)} - \frac{\tau_1^2}{4a^2(\tau_1-\tau)}\right\}d\tau \\ &= -2a\sqrt{\pi}\frac{\tau_1+t}{t\tau_1(\tau_1-t)^{\frac{3}{2}}}\exp\left\{-\frac{(\tau_1+t)^2}{4a^2(\tau_1-t)}\right\}. \end{aligned}$$

Here we have used the substitution $z = \sqrt{\frac{\tau - t}{\tau_1 - \tau}}$ in the integral $I(t, \tau_1)$ and the known equality [18, p.321, 3.325]

$$\int_0^\infty \exp\left\{-\mu z^2 - \frac{\eta}{z^2}\right\} dz = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\mu}} \exp\{-2\sqrt{\mu\eta}\}.$$

Substituting the value of the integral $I(t, \tau_1)$ into (28), we have

$$J(t) = -\frac{1}{2a\sqrt{\pi}} \int_t^0 \frac{\tau_1 + t}{(\tau_1 - t)^{\frac{3}{2}}} \exp\left\{-\frac{(\tau_1 + t)^2}{4a^2(\tau_1 - \tau)}\right\} \varphi^*(\tau_1)\, d\tau_1. \tag{29}$$

In view of (28)-(29), we rewrite (27) in the form

$$\varphi^*(t) - \int_t^0 K^*(t, \tau)\varphi^*(\tau)\, d\tau = 0, \tag{30}$$

where

$$K^*(t, \tau) = \frac{1}{2a\sqrt{\pi}}\left\{-\frac{\tau + t}{(\tau - t)^{\frac{3}{2}}} \exp\left(-\frac{(\tau + t)^2}{4a^2(\tau - t)}\right) + \frac{1}{(\tau - t)^{\frac{1}{2}}} \exp\left(-\frac{\tau - t}{4a^2}\right)\right\}.$$

We note that the kernel $K^*(t, \tau)$ has the following properties:

(1) $K^*(t, \tau) \geq 0$ and is continuous on $-\infty < t < \tau < 0$;

(2) $\lim_{t \to t_0 - 0} \int_t^{t_0} K^*(t, \tau)\, d\tau = 0$, $t_0 \leq \varepsilon < 0$;

(3) $\lim_{t \to -0} \int_t^0 K^*(t, \tau)\, d\tau = 1$.

Indeed, taking into account that $x = \sqrt{\tau - t}$ we have

$$\begin{aligned}
\int_t^{t_0} K^*(t, \tau)\, d\tau &= -\exp\left\{\frac{2t}{a^2}\right\} \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t_0 - t}} \exp\left\{\left(\frac{x}{2a} - \frac{t}{ax}\right)^2\right\} d\left(\frac{x}{2a} - \frac{t}{ax}\right) \\
&\quad + \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t_0 - t}} \exp\left\{-\frac{x^2}{4a^2}\right\} d\left(\frac{x}{2a}\right) \\
&= \exp\left\{\frac{2t}{a^2}\right\} \operatorname{erfc}\left(\frac{t_0 - 3t}{2a\sqrt{t_0 - t}}\right) + \operatorname{erf}\left(\frac{\sqrt{t_0 - t}}{2a}\right).
\end{aligned}$$

From the last equality, the validity of properties (2), (3) of the function $K^*(t, \tau)$ follows.

9 Investigation of the integral equation (30)

An important feature of (30) given by property (3) of the kernel $K^*(t, \tau)$ is expressed by the fact that the corresponding non-homogeneous equation cannot be solved by the method of successive approximations.

We note that boundary value problems for spectrally loaded parabolic equation are reduced to analogous integral equations when the load line moves according to the law $x = t$ [19–21].

In the homogeneous integral equation (30) we transform its kernel. Using the relations

$$\tau + t = (\tau - t) + 2t, \qquad \frac{(\tau + t)^2}{4a^2(\tau - t)} = \frac{t\tau}{a^2(\tau - t)} + \frac{\tau - t}{4a^2},$$

we obtain

$$K^*(t,\tau) = k^*(t,\tau)\exp\left\{-\frac{\tau - t}{4a^2}\right\}, \tag{31}$$

where

$$\begin{aligned} k^*(t,\tau) = &-\frac{t}{a\sqrt{\pi}(\tau - t)^{\frac{3}{2}}}\exp\left\{-\frac{t\tau}{a^2(\tau - t)}\right\} \\ &+ \frac{1}{2a\sqrt{\pi}(\tau - t)^{\frac{1}{2}}}\left(1 - \exp\left\{-\frac{t\tau}{a^2(\tau - t)}\right\}\right). \end{aligned} \tag{32}$$

It is well known that to solve (30) it is sufficient to find a solution of the following equation:

$$\psi^*(t) - \int_t^0 k^*(t,\tau)\psi^*(\tau)\,d\tau = 0, \quad \text{where } \psi^*(t) = \exp\{-t/(4a^2)\}\varphi^*(t). \tag{33}$$

10 Solving the characteristic equation

To study the integral equation (33) we allocate its characteristic part, namely

$$\psi^*(t) - \int_t^0 k_0^*(t,\tau)\psi^*(\tau)\,d\tau = f_1^*(t), \tag{34}$$

where

$$k_0^*(t,\tau) = -\frac{t}{a\sqrt{\pi}(\tau - t)^{\frac{3}{2}}}\exp\left\{-\frac{t\tau}{a^2(\tau - t)}\right\}, \qquad f_1^*(t) = \int_t^0 k_1^*(t,\tau)\psi^*(\tau)\,d\tau, \tag{35}$$

and

$$k_1^*(t,\tau) = \frac{1}{2a\sqrt{\pi}(\tau - t)^{\frac{1}{2}}}\left(1 - \exp\left\{-\frac{t\tau}{a^2(\tau - t)}\right\}\right).$$

Equation (34) is the characteristic equation for (33) since

$$\lim_{t\to -0}\int_t^0 k_0^*(t,\tau)\,d\tau = 1; \qquad \lim_{t\to -0}\int_t^0 k_1^*(t,\tau)\,d\tau = 0.$$

Assuming that the right side of (34) is known, we will find its solution, that is, the solution to the characteristic equation.

Similarly, as in [22, 23], (34) will be reduced to an equation with a difference kernel. To this end, performing in (34) the substitutions

$$\begin{aligned} &t = -1/y, \qquad \tau = -1/x, \qquad d\tau = dx/(x^2), \qquad -\infty < t < \tau < 0, \qquad 0 < y < x < \infty, \\ &\tilde{\psi}^*(y) = y^{-1/2}\psi^*(-1/y), \qquad f_2^*(y) = y^{-1/2}f_1^*(-1/y), \end{aligned} \tag{36}$$

we have

$$\psi^*(-1/y) + \frac{1}{a\sqrt{\pi}}\int_y^\infty \frac{-x^{3/2}y^{3/2}}{y(x-y)^{3/2}x^2}\exp\left\{-\frac{1}{a^2(x-y)}\right\}\psi^*(-1/x)\,dx = f_1^*(-1/y),$$

or, further,

$$\tilde{\psi}^*(y) - \frac{1}{a\sqrt{\pi}} \int_y^\infty \frac{1}{(x-y)^{3/2}} \exp\left\{-\frac{1}{a^2(x-y)}\right\} \tilde{\psi}^*(x)\,dx = f_2^*(y), \quad y > 0. \tag{37}$$

The homogeneous equation corresponding to (37) has the unique solution $\tilde{\psi}^*(y) = C_1$ ($C_1 = \text{const}$); a solution of the non-homogeneous equation (37) is a function

$$\tilde{\psi}^*(y) = f_2^*(y) + \int_y^\infty r_-(y-x) f_2^*(x)\,dx + C_1 \quad (C_1 = \text{const}), \tag{38}$$

where (see [24, p.86, according to the (1.2.56)] for $\lambda = 1$)

$$r_-(-\theta) = \frac{1}{a\sqrt{\pi}\theta^{\frac{3}{2}}} \sum_{n=1}^\infty n \cdot \exp\left\{-\frac{n^2}{a^2\theta}\right\}, \quad \theta > 0.$$

After substitutions inverse to (36) we obtain the solution of the non-homogeneous equation (34):

$$\psi^*(t) = f_1^*(t) + \int_t^0 r(t,\tau) f_1^*(\tau)\,d\tau + \frac{C_1}{\sqrt{-t}}, \tag{39}$$

where

$$r(t,\tau) = -\frac{t}{a\sqrt{\pi}(\tau - t)^{\frac{3}{2}}} \sum_{n=1}^\infty n \cdot \exp\left\{-n^2 \frac{t\tau}{a^2(\tau - t)}\right\}. \tag{40}$$

11 Reducing (33) to the Abelian equation

Using (39)-(40) for the solution of the characteristic equation (34) and taking into account (35) for the function $f_1^*(t)$, we obtain

$$\begin{aligned} \psi^*(t) &= \int_t^0 k_1^*(t,\tau)\psi^*(\tau)\,d\tau + \int_t^0 r(t,\tau)\left(\int_\tau^0 k_1^*(\tau,\tau_1)\psi^*(\tau_1)\,d\tau_1\right) d\tau + \frac{C_1}{\sqrt{-t}} \\ &= \int_t^0 \left\{k_1^*(t,\tau) + \int_t^\tau r(t,\tau_1) k_1^*(\tau_1,\tau)\,d\tau_1\right\} \psi^*(\tau)\,d\tau + \frac{C_1}{\sqrt{-t}}, \quad -\infty < t < 0. \end{aligned} \tag{41}$$

We now calculate the inner integral in (41). We have

$$\begin{aligned} J(t,\tau) &= \int_t^\tau r(t,\tau_1) k_1^*(\tau_1,\tau)\,d\tau_1 \\ &= -\frac{t}{2a^2\pi} \sum_{n=1}^\infty n \cdot \int_t^\tau \frac{1}{(\tau_1 - t)^{\frac{3}{2}}\sqrt{\tau - \tau_1}} \exp\left\{-n^2 \frac{t\tau_1}{a^2(\tau_1 - t)}\right\} \\ &\quad \times \left(1 - \exp\left\{-\frac{\tau_1 \tau}{a^2(\tau - \tau_1)}\right\}\right) d\tau_1 = -\frac{t}{2a^2\pi} \sum_{n=1}^\infty n \cdot I_n(t,\tau), \end{aligned} \tag{42}$$

where

$$I_n(t,\tau) = \int_t^\tau \frac{1}{(\tau_1 - t)^{\frac{3}{2}}\sqrt{\tau - \tau_1}} \exp\left\{-n^2 \frac{t\tau_1}{a^2(\tau_1 - t)}\right\} \times \left(1 - \exp\left\{-\frac{\tau_1\tau}{a^2(\tau - \tau_1)}\right\}\right) d\tau_1 = I_n^{(1)}(t,\tau) - I_n^{(2)}(t,\tau),$$

and

$$I_n^{(1)}(t,\tau) = \int_t^\tau \frac{1}{(\tau_1 - t)^{\frac{3}{2}}(\tau - \tau_1)^{\frac{1}{2}}} \exp\left\{-\frac{n^2 t\tau_1}{a^2(\tau_1 - t)}\right\} d\tau_1,$$
$$I_n^{(2)}(t,\tau) = \int_t^\tau \frac{1}{(\tau_1 - t)^{\frac{3}{2}}(\tau - \tau_1)^{\frac{1}{2}}} \exp\left\{-\left(\frac{n^2 t\tau_1}{a^2(\tau_1 - t)} + \frac{\tau_1\tau}{a^2(\tau - \tau_1)}\right)\right\} d\tau_1.$$

We calculate the integrals $I_n^{(1)}(t;\tau)$ and $I_n^{(2)}(t;\tau)$. Making the substitution $z = \sqrt{\frac{\tau-\tau_1}{\tau_1-t}}$ we find

$$I_n^{(1)}(t;\tau) = -\frac{a\sqrt{\pi}}{nt\sqrt{\tau - t}} \exp\left\{-\frac{n^2 t\tau}{a^2(\tau - t)}\right\},$$
$$I_n^{(2)}(t;\tau) = -\frac{2}{\tau - t} \exp\left\{-\frac{(n^2+1)t\tau}{a^2(\tau - t)}\right\} \int_0^\infty \exp\left\{-\frac{n^2t^2z^2}{a^2(\tau - t)} - \frac{\tau^2}{a^2(\tau - t)z^2}\right\} dz.$$

For the integral $I_n^{(2)}(t;\tau)$ using the equality [18, p.321, 3.325]

$$\int_0^\infty \exp\left\{-\mu z^2 - \frac{\eta}{z^2}\right\} dz = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\mu}} \exp\{-2\sqrt{\mu\eta}\},$$

we obtain

$$I_n^{(2)}(t;\tau) = -\frac{a\sqrt{\pi}}{nt\sqrt{\tau - t}} \exp\left\{-\frac{(n+1)^2 t\tau}{a^2(\tau - t)}\right\}.$$

So, we have

$$I_n(t;\tau) = I_n^{(1)}(t;\tau) - I_n^{(2)}(t;\tau) = -\frac{a\sqrt{\pi}}{nt\sqrt{\tau - t}}\left(\exp\left\{-\frac{n^2 t\tau}{a^2(\tau - t)}\right\} - \exp\left\{-\frac{(n+1)^2 t\tau}{a^2(\tau - t)}\right\}\right).$$

Substituting the value $I_n(t;\tau)$ in (42), we obtain

$$J(t,\tau) = \frac{1}{2a\sqrt{\pi(\tau - t)}} \sum_{n=1}^\infty \left(\exp\left\{-\frac{n^2 t\tau}{a^2(\tau - t)}\right\} - \exp\left\{-\frac{(n+1)^2 t\tau}{a^2(\tau - t)}\right\}\right) = \frac{1}{2a\sqrt{\pi(\tau - t)}} \exp\left\{-\frac{t\tau}{a^2(\tau - t)}\right\}.$$

Thus, (41) takes the form

$$\psi^*(t) - \frac{1}{2a\sqrt{\pi}} \int_t^0 \frac{\psi^*(\tau)}{\sqrt{\tau - t}} d\tau = \frac{C_1}{\sqrt{-t}}, \quad -\infty < t < 0. \tag{43}$$

Thus, the integral equation (33) is reduced to (43) which is a non-homogeneous Abelian integral equation of the second kind.

12 Solving the Abelian equation (43)

We will find a solution $\psi^*(t)$ of the Abelian equation (43), corresponding homogeneous equation (33) (for simplicity, we assume a constant C_1 equal to unity).

The solution of the Abelian equation (43) is determined by the formula [25]

$$\psi^*(t) = \frac{1}{\sqrt{-t}} + \frac{\sqrt{\pi}}{2a}\exp\left\{-\frac{t}{4a^2}\right\}\left[1+\operatorname{erf}\left(\frac{\sqrt{-t}}{2a}\right)\right], \quad -\infty < t < 0. \tag{44}$$

Direct verification [24] shows that the function (44) is indeed a solution to (43).

We note that after the multiplication of (44) by $\exp\{t/(4a^2)\}$ (*i.e.* with the substitution after (33)), we obtain the solution $\varphi^*(t)$ of the homogeneous equation corresponding to the initial equation (30)

$$\varphi^*(t) = \frac{1}{\sqrt{-t}}\exp\left\{\frac{t}{4a^2}\right\} + \frac{\sqrt{\pi}}{2a}\left[1+\operatorname{erf}\left(\frac{\sqrt{-t}}{2a}\right)\right]. \tag{45}$$

We prove (44) using a representation of the solution of the Abelian equation (43) by using the convolution of the fundamental solution in the form of the Mittag-Leffler function [22, p.33, (1.90)]

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha > 0,$$

with the right part of (43) with $C_1 = 1$ and $\alpha = 1/2$, by the following Hille-Tamarkin formula [23, p.93]:

$$\begin{aligned}
\psi^*(t) &= -\frac{d}{dt}\left(\int_t^0 \sum_{k=0}^{\infty}\frac{1}{\Gamma(\frac{k}{2}+1)}\left(\frac{\sqrt{\tau-t}}{2a}\right)^k \frac{d\tau}{\sqrt{-\tau}}\right)\\
&= \frac{1}{\sqrt{-t}} + \sum_{k=1}^{\infty}\frac{k}{2(2a)^k\cdot\frac{k}{2}\Gamma(\frac{k}{2})}\int_t^0 \frac{(\tau-t)^{\frac{k}{2}-1}}{\sqrt{-\tau}}\,d\tau\\
&= \frac{1}{\sqrt{-t}} + \frac{1}{a}\sum_{k=0}^{\infty}\frac{1}{\Gamma(\frac{k-1}{2}+1)}\left(\frac{-t}{4a^2}\right)^{\frac{k}{2}}\int_0^{\pi/2}\cos^k\theta d\theta.
\end{aligned} \tag{46}$$

Here we use the replacement $\tau = t\sin^2\theta$. To reduce solution (46) to (44) we use the following formula [18, p.383, 3.621.3, 3.621.4]:

$$\int_0^{\pi/2}\cos^k\theta\,d\theta = \begin{cases}\frac{(k-1)!!}{k!!}\cdot\frac{\pi}{2}, & \text{if } k \text{ is an even number,}\\ \frac{(k-1)!!}{k!!}, & \text{if } k \text{ is an odd number.}\end{cases} \tag{47}$$

In view of (47) from (46) we have

$$\begin{aligned}
\psi^*(t) &= \frac{1}{\sqrt{-t}} + \frac{\sqrt{\pi}}{2a} + \frac{1}{a}\sum_{k=1}^{\infty}\left[\frac{(2k-2)!!}{\Gamma(k)(2k-1)!!}\left(\frac{-t}{4a^2}\right)^{k-1/2}\right.\\
&\quad\left. + \frac{(2k-1)!!}{\Gamma(k+1/2)(2k)!!}\left(\frac{-t}{4a^2}\right)^{k}\cdot\frac{\pi}{2}\right]\\
&= \frac{1}{\sqrt{-t}} + \frac{\sqrt{\pi}}{2a} + \frac{1}{a}\sum_{k=1}^{\infty}\left[\frac{2^{k-1}(2k)!!}{(2k-1)!!(2k)!!}\left(\frac{-t}{4a^2}\right)^{k-1/2}\right.\\
&\quad\left. + \frac{\pi}{2}\cdot\frac{2^k}{(2k-1)!!\sqrt{\pi}}\cdot\frac{(2k-1)!!}{(2k)!!}\left(\frac{-t}{4a^2}\right)^{k}\right]\\
&= \sum_{k=0}^{\infty}\left[\frac{2^{2k}\cdot k!}{2a(2k)!}\cdot\left(\frac{-t}{4a^2}\right)^{k-1/2} + \frac{\sqrt{\pi}}{2a}\cdot\frac{1}{k!}\left(\frac{-t}{4a^2}\right)^{k}\right].
\end{aligned} \tag{48}$$

It remains to transform the first sum in (48):

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{2^{2k}k!}{2a(2k)!}\left(\frac{-t}{4a^2}\right)^{k-1/2} &= \frac{1}{\sqrt{-t}} + \sum_{k=1}^{\infty}\frac{2^{2k}k!}{2a(2k)!}\left(\frac{-t}{4a^2}\right)^{k-1/2} 7\\
&= \frac{1}{\sqrt{-t}} + \frac{1}{a}\sum_{k=0}^{\infty}\frac{2^{2k+1}(k+1)!}{(2k+2)!}\left(\frac{-t}{4a^2}\right)^{k+1/2}\\
&= \frac{1}{\sqrt{-t}} + \frac{1}{a}\sum_{k=0}^{\infty}\frac{2^{k}(2k+2)!!}{(2k+2)!!(2k+1)!!}\left(\frac{-t}{4a^2}\right)^{k+1/2}\\
&= \frac{1}{\sqrt{-t}} + \frac{1}{a}\sum_{k=0}^{\infty}\frac{2^{k}}{(2k+1)!!}\left(\frac{-t}{4a^2}\right)^{k+1/2}\\
&= \frac{1}{\sqrt{-t}} + \frac{\sqrt{\pi}}{2a}\exp\left\{\frac{-t}{4a^2}\right\}\operatorname{erf}\left(\frac{\sqrt{-t}}{2a}\right),
\end{aligned}$$

since we have the decomposition [18, p.320 3.321 (1)]

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\exp\{-z^2\}\sum_{k=0}^{\infty}\frac{2^k}{(2k+1)!!}\cdot z^{2k+1}.$$

The second sum of (48) is equal to

$$\frac{\sqrt{\pi}}{2a}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-t}{4a^2}\right)^{k} = \frac{\sqrt{\pi}}{2a}\exp\left(\frac{-t}{4a^2}\right).$$

Thus, the solution presented using the convolution from (46) coincides with the solution (44).

13 Solving the initial boundary value problem (21)-(22)

Now we can find the solution of the homogeneous boundary value problem for the heat conductivity equation in an infinite angular domain,

$$u_t^*(x,t) + a^2 u_{xx}^*(x,t) = 0, \qquad u^*(x,t)|_{x=0} = 0, \qquad u^*(x,t)|_{x=-t} = 0,$$

which has a nonzero solution $u^*(x,t)$, defined by the formula

$$u^*(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int_t^0 \frac{x}{(\tau - t)^{\frac{3}{2}}} \exp\left\{-\frac{x^2}{4a^2(\tau - t)}\right\} \nu^*(\tau)\, d\tau + \frac{1}{4a^3\sqrt{\pi}} \int_t^0 \frac{-x-\tau}{(\tau - t)^{\frac{3}{2}}} \exp\left\{-\frac{(x+\tau)^2}{4a^2(\tau - t)}\right\} \varphi^*(\tau)\, d\tau, \tag{49}$$

where

$$\nu^*(t) = -\frac{1}{2a\sqrt{\pi}} \int_t^0 \frac{-\tau}{(\tau - t)^{\frac{3}{2}}} \exp\left\{-\frac{\tau^2}{4a^2(\tau - t)}\right\} \varphi^*(\tau)\, d\tau,$$

and the function $\varphi^*(t)$ is determined according to (45).

14 Estimate of non-trivial solution (49)

To find the class of the non-trivial solution $u(x,t)$ (49) we find the accurate estimate on its order of growth

$$u^*(x,t) = u_1^*(x,t) + u_2^*(x,t), \tag{50}$$

where

$$\varphi^*(t) = \frac{1}{\sqrt{-t}} \exp\left\{\frac{t}{4a^2}\right\} + \frac{\sqrt{\pi}}{2a}\left[\operatorname{erf}\left(\frac{\sqrt{-t}}{2a}\right) + 1\right] = \varphi_1^*(t) + \varphi_2^*(t).$$

We estimate the second summand of (50). We have for $\varphi_1^*(t)$

$$\begin{aligned} u_{21}^*(x,t) &= \frac{1}{4a^3\sqrt{\pi}} \int_t^0 \frac{-x-\tau}{(\tau - t)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4a^2(\tau - t)}\right\} \frac{1}{\sqrt{-\tau}} \exp\left\{\frac{\tau}{4a^2}\right\} d\tau \\ &= \frac{1}{4a^3\sqrt{\pi}} \exp\left\{\frac{x^2}{4a^2 t}\right\} \left[\int_0^\infty \frac{-x-t}{-t} \cdot \frac{1}{\sqrt{y}} \exp\{-\alpha^2 y\}\, dy \right. \\ &\quad \left. - \int_0^\infty \frac{1}{y^{1/2}(1+y)} \exp\{-\alpha^2 y\}\, dy\right] \\ &\le \left|u_2^{*(1)}(x,t)\right| + \left|u_2^{*(2)}(x,t)\right|, \quad \alpha = \frac{-x-t}{2a\sqrt{-t}}, \end{aligned}$$

where we use the substitution $y = \frac{-\tau}{\tau - t}$. Further,

$$\begin{aligned} u_2^{*(1)}(x,t) &= \frac{1}{2a^2\sqrt{-t}} \exp\left\{\frac{x^2}{4a^2 t}\right\}, \\ -u_2^{*(2)}(x,t) &= \frac{1}{2a^3\sqrt{\pi}} \exp\left\{\frac{x^2}{4a^2 t}\right\} \int_0^\infty \frac{1}{1+z^2} \exp\{-\alpha^2 z^2\}\, dz \\ &\le \frac{1}{2a^3\sqrt{\pi}} \exp\left\{\frac{x^2}{4a^2 t}\right\} \int_0^\infty \exp\{-\alpha^2 z^2\}\, dz \\ &= \frac{1}{2a^2} \cdot \frac{\sqrt{-t}}{-t-x} \exp\left\{\frac{x^2}{4a^2 t}\right\}. \end{aligned}$$

Since $\varphi_2^*(\tau) \le \frac{\sqrt{\pi}}{a}$, $-\infty < t < \tau < 0$ we have for the function $\varphi_2^*(t)$

$$\begin{aligned} u_{22}^*(x,t) &= \frac{1}{4a^3\sqrt{\pi}} \int_t^0 \frac{-x-\tau}{(\tau-t)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4a^2(\tau-t)}\right\} \frac{\sqrt{\pi}}{2a}\left[\operatorname{erf}\left(\frac{\sqrt{-\tau}}{2a}\right)+1\right] d\tau \\ &\le \frac{1}{4a^4} \int_t^0 \frac{-x-\tau}{(\tau-t)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4a^2(\tau-t)}\right\} d\tau = \frac{\sqrt{\pi}}{a} \cdot \overline{u}_{22}^*(x,t), \\ \overline{u}_{22}^*(x,t) &= \frac{1}{4a^3\sqrt{\pi}} \int_t^0 \frac{-x-\tau}{(\tau-t)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4a^2(\tau-t)}\right\} d\tau \\ &= \frac{1}{a^2\sqrt{\pi}} \exp\left\{\frac{-t-x}{a^2}\right\} \int_{\frac{1}{\sqrt{-t}}}^{\infty} \exp\left\{-\left(\frac{-t-x}{2a}y + \frac{1}{2ay}\right)^2\right\} d\left(\frac{-t-x}{2a}y + \frac{1}{2ay}\right) \\ &= \frac{1}{2a^2} \exp\left\{\frac{-t-x}{a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{-t}}{a} - \frac{x}{2a\sqrt{-t}}\right), \end{aligned}$$

where we use the substitution $y = \frac{1}{\sqrt{\tau-t}}$. Hence it follows

$$\begin{aligned} \overline{u}_{22}^*(x,t)|_{x=0} &= \frac{1}{2a^2} \exp\left\{\frac{-t}{a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{-t}}{a}\right), \\ \overline{u}_{22}^*(x,t)|_{x=-t} &= \frac{1}{2a^2} \operatorname{erfc}\left(\frac{\sqrt{-t}}{2a}\right). \end{aligned}$$

Thus we have

$$\begin{aligned} &\left|u_2^{*(1)}(x,t)\right| \le C_3 \frac{1}{\sqrt{-t}} \exp\left\{\frac{x^2}{4a^2t}\right\}, \qquad \left|u_2^{*(2)}(x,t)\right| \le C_4 \frac{\sqrt{-t}}{-t-x} \exp\left\{\frac{x^2}{4a^2t}\right\}, \\ &\left|u_{22}^*(x,t)\right| \le C_5 \exp\left\{\frac{-t-x}{a^2}\right\}. \end{aligned} \tag{51}$$

The first two inequalities can be replaced by the following one:

$$\left|u_{21}^*(x,t)\right| \le C_6 \frac{\sqrt{-t}}{-t-x} \cdot \exp\left\{\frac{x^2}{4a^2t}\right\}, \quad \text{since } \frac{1}{\sqrt{-t}} < \frac{\sqrt{-t}}{-t-x}, \forall -t > x > 0. \tag{52}$$

Further, for the first summand in the function $u^*(x,t)$ (50) we have

$$u_1^*(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int_t^0 \frac{x}{(\tau-t)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(\tau-t)}\right\} \nu^*(\tau)\,d\tau.$$

We express the function $\nu^*(t)$ as the sum

$$\nu^*(t) = \nu_1^*(t) + \nu_2^*(t),$$

where

$$\begin{aligned} \nu_1^*(t) &= -\frac{1}{2a\sqrt{\pi}} \int_t^0 \frac{-\tau}{(\tau-t)^{\frac{3}{2}}} \exp\left\{-\frac{\tau^2}{4a^2(\tau-t)}\right\} \varphi_1^*(\tau)\,d\tau, \\ \nu_2^*(t) &= -\frac{1}{2a\sqrt{\pi}} \int_t^0 \frac{-\tau}{(\tau-t)^{\frac{3}{2}}} \exp\left\{-\frac{\tau^2}{4a^2(\tau-t)}\right\} \varphi_2^*(\tau)\,d\tau. \end{aligned}$$

We have

$$\begin{aligned}
-\nu_1^*(t) &= \frac{1}{2a\sqrt{\pi}} \int_t^0 \frac{\sqrt{-\tau}}{(\tau - t)^{3/2}} \exp\left\{-\frac{t\tau}{4a^2(\tau - t)}\right\} d\tau \\
&= \frac{1}{\sqrt{-\pi t}} \int_0^\infty \frac{y}{1+y} \exp\left\{\frac{ty}{4a^2}\right\} d\left(\frac{\sqrt{-ty}}{2a}\right) = \frac{1}{\sqrt{-\pi t}} \int_0^\infty \exp\left\{\frac{ty}{4a^2}\right\} d\left(\frac{\sqrt{-ty}}{2a}\right) \\
&\quad - \frac{1}{\sqrt{-\pi t}} \int_0^\infty \frac{1}{1+z^2} \exp\left\{\frac{tz^2}{4a^2}\right\} d\left(\frac{\sqrt{-t}z}{2a}\right) \le \frac{C_7}{\sqrt{-t}},
\end{aligned}$$

where we use the substitution $y = \frac{-\tau}{\tau - t}$. Further, since $\varphi_2^*(\tau) \le \frac{\sqrt{\pi}}{a}$, we have

$$\begin{aligned}
-\nu_2^*(t) &= \frac{1}{2a\sqrt{\pi}} \int_t^0 \frac{-\tau}{(\tau - t)^{\frac{3}{2}}} \exp\left\{-\frac{\tau^2}{4a^2(\tau - t)}\right\} \varphi_2^*(\tau)\, d\tau \\
&\le \frac{1}{2a^2} \int_t^0 \frac{-\tau}{(\tau - t)^{\frac{3}{2}}} \exp\left\{-\frac{\tau^2}{4a^2(\tau - t)}\right\} d\tau \\
&= \frac{\sqrt{-t}}{a^2} \int_0^\infty \frac{y^3}{(1+y^2)^{3/2}} \exp\left\{\frac{ty^4}{4a^2(1+y^2)}\right\} dy \\
&= \frac{\sqrt{-t}}{a^2} \int_0^\infty \left[\frac{y}{(1+y^2)^{1/2}} - \frac{y}{(1+y^2)^{3/2}}\right] \exp\left\{\frac{ty^4}{4a^2(1+y^2)}\right\} dy \\
&= \frac{\sqrt{-t}}{a^2} \int_1^\infty \exp\left\{\frac{t(z^2-1)^2}{4a^2 z^2}\right\} dz - \frac{\sqrt{-t}}{a^2} \int_0^1 \exp\left\{\frac{t(1-\zeta^2)^2}{4a^2\zeta^2}\right\} d\zeta \\
&\le \frac{\sqrt{-t}}{a^2} \exp\left\{-t/\left(2a^2\right)\right\} \int_0^\infty \exp\left\{\frac{t}{4a^2}\left(z^2 + 1/z^2\right)\right\} dz \\
&= \frac{\sqrt{-t}}{a^2} \exp\left\{-t/\left(2a^2\right)\right\} \cdot \frac{1}{2}\sqrt{\pi}\, \frac{2a}{\sqrt{-t}} \exp\left\{2t/\left(4a^2\right)\right\} = \frac{\sqrt{\pi}}{a}.
\end{aligned}$$

Here we use following substitutions:

$$y = \sqrt{\frac{-\tau}{\tau - t}}, \qquad \tau = \frac{ty^2}{1+y^2}, \qquad \tau - t = -\frac{t}{1+y^2}, \qquad d\tau = \frac{2ty\,dy}{(1+y^2)^2},$$

and

$$z = \left(1+y^2\right)^{1/2}, \qquad \zeta = \left(1+y^2\right)^{-1/2}.$$

Thus we obtain

$$-\nu_1^*(t) \le \frac{C_7}{\sqrt{-t}}, \qquad -\nu_2^*(t) \le C_8. \tag{53}$$

Using (53), we estimate the following summands of solutions $u_{11}^*(x,t)$ and $u_{12}^*(x,t)$:

$$\begin{aligned}
-u_{11}^*(x,t) &= -\frac{1}{4a^3\sqrt{\pi}} \int_t^0 \frac{x}{(\tau - t)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(\tau - t)}\right\} \nu_1^*(\tau)\, d\tau \\
&\le \frac{C_7}{4a^3\sqrt{\pi}} \int_t^0 \frac{x}{\sqrt{-\tau}(\tau - t)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(\tau - t)}\right\} d\tau
\end{aligned}$$

$$\begin{aligned}
&= \frac{C_7\sqrt{x}}{2a^3\sqrt{\pi}} \int_{\sqrt{x/(-t)}}^{\infty} \frac{y}{\sqrt{-ty^2 - x}} \exp\left\{-\frac{xy^2}{4a^2}\right\} dy \\
&= \frac{C_7}{a^2\sqrt{-\pi t}} \exp\left\{\frac{x^2}{4a^2 t}\right\} \int_{\sqrt{x/(-t)}}^{\infty} \exp\left\{\frac{x}{4a^2 t}(-ty^2 - x)\right\} d\left(\frac{\sqrt{x(-ty^2 - x)}}{2a\sqrt{-t}}\right) \\
&= \frac{C_7}{2a^2\sqrt{-t}} \exp\left\{\frac{x^2}{4a^2 t}\right\} \cdot \frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp\{-z^2\}\, dz = \frac{C_7}{2a^2} \cdot \frac{1}{\sqrt{-t}} \exp\left\{\frac{x^2}{4a^2 t}\right\},
\end{aligned}$$

where we use the substitution $y = \sqrt{\frac{x}{\tau - t}}$.

Further we have

$$\begin{aligned}
-u_{12}^*(x,t) &= -\frac{1}{4a^3\sqrt{\pi}} \int_t^0 \frac{x}{(\tau - t)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(\tau - t)}\right\} \nu_2^*(\tau)\, d\tau \\
&\leq \frac{C_8}{8a^3} \int_t^0 \frac{x}{(\tau - t)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(\tau - t)}\right\} d\tau \\
&= \frac{C_8\sqrt{\pi}}{a^2} \cdot \frac{2}{\sqrt{\pi}} \int_{\frac{1}{\sqrt{-t}}}^{\infty} \exp\left\{-\frac{x^2}{4a^2} y^2\right\} d\left(\frac{x}{2a} y\right) = \frac{C_8\sqrt{\pi}}{a^2} \cdot \operatorname{erfc}\left(\frac{x}{2a\sqrt{-t}}\right),
\end{aligned}$$

where $y = \frac{1}{\sqrt{\tau - t}}$.

Thus we have

$$\left|u_1^*(x,t)\right| \leq \left[C_9 + C_{10} \cdot \frac{1}{\sqrt{-t}} \exp\left\{\frac{x^2}{4a^2 t}\right\}\right]. \tag{54}$$

Estimates for the functions (in the order of growth they are accurate) $u_1^*(x,t)$, $u_{21}^*(x,t)$, and $u_{22}^*(x,t)$ (51), (52), and (54) determine the following estimate:

$$\left|u^*(x,t)\right| \leq C \cdot \gamma(x,t), \tag{55}$$

where

$$\gamma(x,t) = \max\left[-\frac{\sqrt{-t}}{t+x} \exp\left\{\frac{x^2}{4a^2 t}\right\}; 1 + \exp\left\{-\frac{t+x}{a^2}\right\}\right], \tag{56}$$

$\gamma(x,t) \geq 2$, $\{x,t\} \in G$, *i.e.*,

$$\gamma(x,t) = \begin{cases} -\frac{\sqrt{-t}}{t+x} \exp\{\frac{x^2}{4a^2 t}\}, & \{x,t\} \in G_1; \\ 1 + \exp\{-\frac{t+x}{a^2}\}, & \{x,t\} \in G_2 \cup S; \end{cases}$$

where

$$\begin{aligned}
S &= \left\{\{x,t\} \in G \,\middle|\, -\frac{\sqrt{-t}}{t+x} \exp\left\{\frac{x^2}{4a^2 t}\right\} = 1 + \exp\left\{-\frac{t+x}{a^2}\right\}\right\}; \\
G_1 &= \left\{\{x,t\} \in G \,\middle|\, -\frac{\sqrt{-t}}{t+x} \exp\left\{\frac{x^2}{4a^2 t}\right\} > 1 + \exp\left\{-\frac{t+x}{a^2}\right\}\right\}; \\
G_2 &= G \setminus \{G_1 \cup S\}.
\end{aligned}$$

The following is established.

Proposition 2 *For problem* L^* (21)-(22) *in class* (23)

$$\dim\{\mathrm{Ker}\{L^*\}\} = 1.$$

15 The main result. Classes of uniqueness

Theorem (The main result) *The boundary value problem* L (1)-(2) *is Noetherian, i.e.*

$$\mathrm{ind}\{L\} = \dim\{\mathrm{Ker}\{L\}\} - \dim\{\mathrm{Coker}\{L\}\} = -1.$$

From the above it follows that the classes of uniqueness for the boundary value problem (21)-(22) are determined, for example, by the following proposition.

Proposition 3 *Classes of uniqueness are*

$$|u^*(x,t)| \le C \cdot \gamma_\varepsilon(x,t), \quad \gamma_\varepsilon(x,t) \ge 2, 0 < \varepsilon < 1,$$

where

$$\gamma_\varepsilon = \max\left[\frac{\sqrt{-t}}{-t-x}\exp\left\{-\frac{x^2}{4a^2(-t)}\right\}; 1 + \exp\left\{\frac{(-t-x)^{1-\varepsilon}}{a^2}\right\}\right], \quad \{x,t\} \in G.$$

Analyzing the previous expression for the function $\gamma_\varepsilon(x,t)$ we obtain:

1. $\gamma_\varepsilon(x,t) \le \gamma(x,t)$, $\{x,t\} \in G$;
2. $\exists G_\varepsilon \subset G$, $\mathrm{meas}\{G_\varepsilon\} > 0 : \gamma_\varepsilon(x,t) < \gamma(x,t)$.

16 Conclusion

To summarize, the following is established.

- In an infinite angular domain for the homogeneous Dirichlet problem for the heat conduction equation the existence of a unique (up to a constant factor) non-trivial solution, which, however, does not belong to the class of summable functions with the weight found in the work is proved.
- For the boundary value problem adjoint to the Dirichlet problem, the existence of a unique (up to a constant factor) non-trivial solutions, which belongs to the class of essentially bounded functions with the weight is found in the work is established.
- In the weight class of summable functions it is shown that the index of the Dirichlet problem is equal to minus one.
- Weight classes of uniqueness for the boundary value problem considered in the work are found.

Problems in non-cylindrical domains, similar to those considered in this paper are highly relevant not only for modeling the processes of electrical contact apparatuses but also in the related field of the designing plasma torches. Similar problems arise in creating the new technologies, the production of crystals, laser technology and in the other branches. Mathematical modeling these processes allows one to carry out the optimal choice of parameters and operating modes of technological equipment and maximize economic and environmental benefits. Finally, we note that the results can be developed for non-homogeneous boundary value problems of heat conduction, when the data are selected from the corresponding weight classes.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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