Yu *Journal of Inequalities and Applications* (2015) 2015:317
DOI 10.1186/s13660-015-0838-6

Journal of Inequalities and Applications
a SpringerOpen Journal

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# Arcwise connected cone-quasiconvex set-valued mappings and Pareto reducibility in vector optimization

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**Abstract**

The aim of this note is twofold: first, to show that arcwise connected cone-quasiconvex set-valued mappings can be characterized in terms of classical arcwise connected quasiconvexity of certain real-valued functions, defined by Gerstewitz's scalarization function; second, by making use of the recent result concerning the Pareto reducibility in multicriteria arcwise connected cone-quasiconvex optimization problems to establish similar set-valued optimization problems under appropriate assumptions.

**MSC:** Primary 90C29; 90C46; secondary 26B25

**Keywords:** arcwise connected cone-quasiconvexity; set-valued optimization; Gerstewitz's scalarization function; Pareto reducibility

## 1 Introduction

It is well known that convexity and its various generalizations play a dominant role in optimization. In order to relax the convexity assumption, many kinds of generalized convexity have been introduced by many authors. Among the different notions of generalized convexity, quasiconvexity and arcwise connected convexity have found many important applications; see for instance [1–13]. In convex analysis, the new generalized convexity can be derived by combining two or more existing types of generalized convexity. A good example is the arcwise connected quasiconvexity, which was presented by mixing arcwise connected convexity together with quasiconvexity. In fact, the real-valued arcwise connected quasiconvex functions had already appeared in early works, such as [1]. In [11], La Torre and Popovici extended this notion to vector-valued functions taking values in real partially ordered vector spaces, and applied it to study the contractibility of efficient sets and Pareto reducibility in multicriteria optimization. The notion of Pareto reducibility, introduced by Popovici in [14], is to represent the weakly efficient solution set as the union of the sets of efficient solutions of all subproblems obtained from the original one by selecting certain criteria. Popovici in [9] extended the Pareto reducibility in multicriteria optimization problems to explicitly quasiconvex set-valued optimization; for more details related to the Pareto reducibility, refer to [8, 9, 11, 14, 15]. La Torre in [10] introduced the arcwise connected cone-quasiconvex set-valued mapping and investigated the optimality conditions for a set-valued optimization involving this type of data. A natural question arises: is the arcwise connected cone-quasiconvex set-valued optimization problem also Pareto reducible? One aim of this paper is to show, with the help of results obtained for multicriteria optimization in [11], that the answer is positive.

Another interesting topic in vector optimization is to characterize the generalized cone-convexity of the vector-valued (or set-valued) objective functions in terms of usual generalized convexity of certain real-valued functions, by means of some appropriate scalarization functionals. For instance, it was presented in [2, 16] that cone-convex and cone-quasiconvex functions can be characterized by means of the extreme directions of a polar cone and Gerstewitz's scalarization functions; similar characterizations of weakly cone-convex and weakly cone-quasiconvex functions were given in [12]; scalar characterizations of cone-convex functions in variable domination structures were proposed in [17]. For characterizations of cone-convexity and cone-quasiconvexity for set-valued maps, we refer to [3, 9] and the references therein. The second aim of this note is to show that the arcwise connected cone-quasiconvex set-valued mappings can also be characterized by means of Gerstewitz's scalarization functions.

We begin in Section 2 by recalling some definitions and preliminary results concerning arcwise connected cone-quasiconvex set-valued mappings. In addition, several properties for arcwise connected cone-quasiconvex set-valued mappings, which will be used in the sequel, are discussed. Section 3 is devoted to the characterizations of arcwise connected cone-quasiconvex set-valued mappings by means of Gerstewitz's scalarization function. Finally, in Section 4, by restricting our attention on set-valued optimization with the value of objective function in a finite dimensional Euclidean space, we get the sufficient condition for the Pareto reducibility with the help of the results derived for arcwise connected cone-quasiconvex multicriteria optimization in [11].

## 2 Preliminaries

Let $X$ be a real linear space and $Y$ be a real Banach spaces, $D \subset Y$ be a closed convex cone. Considering the partially order induced by $D$, defined as follows:

$$y_1 \leq_D y_2 \quad \text{if } y_2 - y_1 \in D.$$

From now on, we always assume that $S$ is a nonempty subset of $X$. Let us recall the notions of arcwise connected convexity for a set and arcwise connected cone-quasiconvexity for a set-valued mapping.

**Definition 2.1** [1] A subset $S \subset X$ is said to be an arcwise connected set, if for every $x_1 \in S$, $x_2 \in S$, there exists a continuous vector-valued function $H_{x_1,x_2} : [0,1] \to S$, called an arc, such that

$$H_{x_1,x_2}(0) = x_1, \qquad H_{x_1,x_2}(1) = x_2.$$

In this paper, the empty set $\emptyset$ is assumed to be an arcwise connected set. On the other hand, obviously, if a set is convex, then it is arcwise connected set. In general, the opposite is not true.

**Example 2.2** Let

$$S = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 1, x_1 > 0, x_2 > 0\}.$$

Clearly, $S$ is not convex. However, $S$ is an arcwise connected set with respect to the arc $H_{x,z}$, defined by

$$H_{x,z}(t) = \left(\sqrt{(1-t)x_1^2 + tz_1^2}, \sqrt{(1-t)x_2^2 + tz_2^2}\right), \quad \forall x = (x_1, x_2), z = (z_1, z_2) \in S, t \in [0,1].$$

Let $\varphi : S \subset X \to \mathbb{R}$ be a real-valued function. Recall that $\varphi$ is said to be arcwise connected quasiconvex, if for all $x_1, x_2 \in X$ and $t \in [0,1]$ there exists an arc $H_{x_1,x_2} : [0,1] \to S$ such that $H_{x_1,x_2}(0) = x_1$, $H_{x_1,x_2}(1) = x_2$, and

$$\varphi\left(H_{x_1,x_2}(t)\right) \leq \max\left\{\varphi(x_1), \varphi(x_2)\right\},$$

which means that for every $\lambda \in \mathbb{R}$ the following level set is arcwise connected:

$$\mathrm{Lev}_\varphi(\lambda) := \left\{x \in S : \varphi(x) \leq \lambda\right\}.$$

In the literature [5, 10], the notion of arcwise connected quasiconvexity has been extended to the vector-valued functions and set-valued mappings, respectively. For any set-valued map $F : S \to 2^Y$ and every set $A \subset Y$, we denote by $\mathrm{dom}(F) = \{x \in S : F(x) \neq \emptyset\}$ and $F^{-1}(A) := \{x \in S : F(x) \cap A \neq \emptyset\}$ efficient domain of $F$ and the inverse image of $A$ by $F$, respectively. Throughout this paper, we always assume that $\mathrm{dom} F = S$ for the set-valued mapping $F : S \subset X \to 2^Y$. A function $f : S \subset X \to Y$ is called a selection of $F$ if $f(x) \in F(x)$ for all $x \in S$.

**Definition 2.3** [10] Let $F : S \subset X \to 2^Y$ be a set-valued mapping. It is said that $F$ is arcwise connected $D$-quasiconvex if the generalized level set

$$\mathrm{Lev}_F(y) = \left\{x \in S : \text{there is } z \in F(x) \text{ such that } y - z \in D\right\}$$

is an arcwise connected set for every point $y \in Y$. Actually, $F$ is arcwise connected $D$-quasiconvex if for all $x_1, x_2 \in X$ and $t \in [0,1]$ there exists an arc $H_{x_1,x_2} : [0,1] \to S$ such that $H_{x_1,x_2}(0) = x_1$, $H_{x_1,x_2}(1) = x_2$ and

$$\left(F(x_1) + D\right) \cap \left(F(x_2) + D\right) \subset F\left(H_{x_1,x_2}(t)\right) + D.$$

If the set-valued mapping $F$ degenerates to a vector-valued function, then the definition of arcwise connected $D$-quasiconvexity coincides with the definition of 'arcwise $D$-quasiconvexity' introduced by La Torre and Popovici in [11]. In order to unify the terminology, we still use 'arcwise connected cone-quasiconvexity' in the case of single-valued functions. Let us see an example of arcwise connected cone-quasiconvex set-valued map.

**Example 2.4** Let $X = \mathbb{R}$, $Y$ be the space of all real sequences $y$ in $\mathbb{R}$

$$y = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$$

in which $\lim_{n\to\infty} \xi_n = 0$. Let $D$ be the set of all nonnegative sequences in $Y$. Define

$$\|y\| = \sup_n |\xi_n|, \quad \forall y \in Y.$$

Then $Y$ is a Banach space. For $S = \mathbb{R}_+$, the set-valued mapping $F : S \to 2^Y$ defined by

$$F(x) = \left\{ \left(x^2, \frac{x^2}{2}, \dots, \frac{x^2}{n}, \dots\right), \left(-x^2, -\frac{x^2}{2}, \dots, -\frac{x^2}{n}, \dots\right) \right\}, \quad \forall x \in S,$$

is arcwise connected $D$-quasiconvex with respect to the arc

$$H_{x,z}(t) = \sqrt{1-t} \cdot x + \sqrt{t} \cdot z, \quad \forall x, z \in S.$$

Next, some basic properties concerning arcwise connected cone-quasiconvex set-valued maps can easily be obtained and some of them will be used in the sequel.

**Proposition 2.5** *Let $F : S \subset X \to 2^Y$ be a set-valued map and $D_1$, $D_2$ be two convex cones of $Y$, $D_1 \subset D_2$. If $F$ is arcwise connected $D_1$-quasiconvex, then $F$ is also arcwise connected $D_2$-quasiconvex.*

*Proof* In fact, for any $y \in Y$, suppose that the generalized level set of $F$ with respect to the ordering cone $D_1$

$$\operatorname{Lev}_F^{D_1}(y) = \{x \in S : \text{there is } z \in F(x) \text{ such that } y - z \in D_1\}$$

is arcwise connected. Then, for any $x_1, x_2 \in \{x \in S : y \in F(x) + D_1\}$, there exists an arc $H_{x_1,x_2}[0,1] \to S$ such that $H_{x_1,x_2}(0) = x_1$, $H_{x_1,x_2}(1) = x_2$, and $H_{x_1,x_2}(t) \in \operatorname{Lev}_F^{D_1}(y)$ for all $t \in ]0,1[$. Therefore, we get $y \in F(H_{x_1,x_2}(t)) + D_1 \subset F(H_{x_1,x_2}(t)) + D_2$ for all $t \in [0,1]$. This shows that

$$\operatorname{Lev}_F^{D_2}(y) = \{x \in S : \text{there is } z \in F(x) \text{ such that } y - z \in D_2\}$$

is arcwise connected, as desired. □

**Proposition 2.6** *If $F : S \subset X \to 2^Y$ is arcwise connected $D$-quasiconvex and the ordering cone $D$ generates the space $Y$, i.e. $D - D = Y$, then $S$ is arcwise connected.*

*Proof* Taking $x_1, x_2 \in S$ arbitrary, there exist $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$. Since $D$ generates the space $Y$, there exists $y \in Y$ such that $y_1 \leq_D y$ and $y_2 \leq_D y$. This means

$$x_1 \in \operatorname{Lev}_F(y) \quad \text{and} \quad x_2 \in \operatorname{Lev}_F(y).$$

Then we see from the arcwise connected $D$-quasiconvexity of $F$ that there exists an arc $H_{x_1,x_2} : [0,1] \to S$ such that $H_{x_1,x_2}(0) = x_1$, $H_{x_1,x_2}(2) = x_2$, and $H_{x_1,x_2}(t) \in \operatorname{Lev}_F(y) \subset S$ for all $t \in ]0,1[$. Hence, $S$ is arcwise connected. □

**Proposition 2.7** *Let $F : S \subset X \to 2^Y$ be a set-valued mapping. Suppose that $f$ is a selection of $F$ and $F(x) \subset f(x) + D$ for all $x \in S$. If $f$ is arcwise connected $D$-quasiconvex, then $F$ is arcwise connected $D$-quasiconvex.*

*Proof* In fact, for $x_1, x_2 \in X$ there exists an arc $H_{x_1,x_2} : [0,1] \to S$ such that $H_{x_1,x_2}(0) = x_1$, $H_{x_1,x_2}(1) = x_2$, and

$$\big(f(x_1) + D\big) \cap \big(f(x_2) + D\big) \subset f\big(H_{x_1,x_2}(t)\big) + D$$

for any $t \in ]0,1[$. Hence, it follows that

$$\begin{aligned}\big(F(x_1) + D\big) \cap \big(F(x_2) + D\big) &\subset \big(f(x_1) + D + D\big) \cap \big(f(x_2) + D + D\big)\\ &\subset \big(f(x_1) + D\big) \cap \big(f(x_2) + D\big)\\ &\subset f\big(H_{x_1,x_2}(t)\big) + D\\ &\subset F\big(H_{x_1,x_2}(t)\big) + D.\end{aligned}$$

□

**Proposition 2.8** *Let $F : S \subset X \to 2^Y$ be a set-valued mapping. Suppose that $f$ is a selection of $F$ and $F(x) \subset f(x) + D$ for all $x \in S$. If $F$ is arcwise connected D-quasiconvex, then $f$ is arcwise connected D-quasiconvex.*

*Proof* In fact, for $x_1, x_2 \in X$ and $t \in [0,1]$ there exists an arc $H_{x_1,x_2} : [0,1] \to S$ such that $H_{x_1,x_2}(0) = x_1$, $H_{x_1,x_2}(2) = x_2$, and

$$\begin{aligned}\big(f(x_1) + D\big) \cap \big(f(x_2) + D\big) &\subset \big(F(x_1) + D\big) \cap \big(F(x_2) + D\big)\\ &\subset F\big(H_{x_1,x_2}(t)\big) + D\\ &\subset f\big(H_{x_1,x_2}(t)\big) + D + D\\ &\subset f\big(H_{x_1,x_2}(t)\big) + D.\end{aligned}$$

□

**Remark 2.9** In fact, the assumption $F(x) \subset f(x) + D$ for all $x \in S$ had been used in [10] by La Torre to derive the similar characterization for arcwise connected cone-convex set-valued maps; see Theorem 4 and Theorem 5 in [10]. Let us give a simple example, in which the assumption $F(x) \subset f(x) + D$ is satisfied. Let $X = Y = \mathbb{R}$, $D = \mathbb{R}_+$, and $F : X \to 2^Y$ be defined by

$$F(x) := \{y \in Y : y \geq x^2\}, \quad \text{for all } x \in X.$$

It is obviously that $f(x) = x^2$ is a selection of $F$ and $F(x) \subset f(x) + D$.

Corollary 2.10 shows that an acwise connected cone-quasiconvex set-valued mapping $FS \subset X \to 2^Y$ is characterized by a selection $f$ satisfying $F(x) \subset f(x) + D$ for all $x \in S$, which can be obtained directly from Proposition 2.7 and Proposition 2.8.

**Corollary 2.10** *Let $FS \subset X \to 2^Y$ be a set-valued mapping. Suppose that $f$ is a selection of $F$ satisfying $F(x) \subset f(x) + D$ for all $x \in S$. Then $F$ is arcwise connected D-quasiconvex if and only if $f$ is arcwise connected D-quasiconvex.*

## 3 Scalarization by means of Gerstewitz's function

In this section, we assume that the closed convex cone $D \subset Y$ is solid, *i.e.* $\operatorname{int} D \neq \emptyset$. For a fixed $e \in \operatorname{int} D$, $v \in Y$ and any $y \in Y$ the set $\{t \in \mathbb{R} : y \in te + v - D\}$ is nonempty, closed and bounded from below (see [5]). The well-known Gerstewitz's function $h_{e,v} : Y \to \mathbb{R}$ is defined by

$$h_{e,v}(y) := \min\{t \in \mathbb{R} : y \in te + v - D\}, \quad \text{for all } y \in Y.$$

We need its following salient property.

**Lemma 3.1** [18] For fixed $e \in \operatorname{int} D$, any $v \in Y$ and $r \in \mathbb{R}$, we have $h_{e,v}(y) \leq r \Leftrightarrow y \in re + v - D$.

**Proposition 3.2** *Let $F : S \subset X \to 2^Y$ be arcwise connected D-quasiconvex set-valued mapping. Then for any $v \in Y$, $h_{e,v} \circ F$ is arcwise connected $\mathbb{R}_+$-quasiconvex, where $h_{e,v} \circ F(x) := h_{e,v}(F(x)) = \bigcup_{y \in F(x)} h_{e,v}(y)$, for all $x \in S$.*

*Proof* For any $v \in Y$, we have to check, for any $\lambda_0 \in \mathbb{R}$, whether the set

$$\operatorname{Lev}_{(h_{e,v} \circ F)}(\lambda_0) = \{x \in S : \text{there exists } \lambda \in h_{e,v}(F(x)) \text{ such that } \lambda \leq \lambda_0\}$$

is arcwise connected set. Without loss of generality, we suppose that $\operatorname{Lev}_{(h_{e,v} \circ F)}(\lambda_0) \neq \emptyset$, let $x_1$, $x_2$ be any two points of the level set $\operatorname{Lev}_{(h_{e,v} \circ F)}(\lambda_0)$ and $t \in [0,1]$. Then there exist $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ such that $\lambda_1 = h_{e,v}(y_1) \leq \lambda_0$ and $\lambda_2 = h_{e,v}(y_2) \leq \lambda_0$. Noticing that $D$ is a closed convex cone, then we get

$$y_1 \in v + \lambda_1 \cdot e - D \quad \text{and} \quad y_2 \in v + \lambda_2 \cdot e - D.$$

Since $\lambda_1 = h_{e,v}(y_1) \leq \lambda_0$ and $\lambda_2 = h_{e,v}(y_2) \leq \lambda_0$, it follows from Lemma 3.1 that

$$y_1 \in v + \lambda_0 \cdot e - D \quad \text{and} \quad y_2 \in v + \lambda_0 \cdot e - D,$$

which means

$$y_1 \leq_D v + \lambda_0 \cdot e \quad \text{and} \quad y_2 \leq_D v + \lambda_0 \cdot e.$$

By the arcwise connected $D$-quasiconvexity of $F$, there exist an arc $H_{x_1,x_2} : [0,1] \to S$ and $y_t \in F(H_{x_1,x_2}(t))$ such that

$$y_t \leq_D v + \lambda_0 \cdot e, \quad t \in [0,1],$$

that is,

$$y_t \in v + \lambda_0 \cdot e - D, \quad t \in [0,1].$$

By Lemma 3.1 again, we get

$$h_{e,v}(y_t) \leq \lambda_0, \quad t \in [0,1].$$

Therefore, $H_{x_1,x_2}(t) \in \mathrm{Lev}_{(h_{e,v}\circ F)}(\lambda_0)$, which shows that $\mathrm{Lev}_{(h_{e,v}\circ F)}(\lambda_0)$ is an arcwise connected set. □

**Proposition 3.3** *Let $F : S \subset X \to 2^Y$ be a set-valued mapping. If for any $v \in Y$, $h_{e,v} \circ F$ is arcwise connected $\mathbb{R}_+$-quasiconvex, then $F$ is arcwise connected $D$-quasiconvex.*

*Proof* We proceed by contradiction. Suppose that $F$ is not arcwise connected $D$-quasiconvex, *i.e.* there exist $x_1, x_2 \in X$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $y \in Y$ with $y_1 \in y - D$ and $y_2 \in y - D$, and for any arc $H_{x_1,x_2} : [0,1] \to S$ and $t \in [0,1]$, there exists $y_t \in F(H_{x_1,x_2}(t))$ such that

$$y_t \notin y - D.$$

Taking $v = y$ and noticing that $y_1 \in y - D$ and $y_2 \in y - D$, we get from Lemma 3.1

$$h_{e,y}(y_1) \le 0 \quad \text{and} \quad h_{e,y}(y_2) \le 0.$$

On the other hand, one finds from $y_t \in F(H_{x_1,x_2}(t))$ with $y_t \notin y - D$ and Lemma 3.1 that

$$h_{e,y}(y_t) > 0,$$

which shows that $h_{e,y} \circ F$ is not arcwise connected $\mathbb{R}_+$-quasiconvex. This is a contradiction. □

According to Proposition 3.2 and Proposition 3.3, we conclude this section by presenting Corollary 3.4, which is a characterization of arcwise cone-quasiconvex set-valued maps in terms of scalar arcwise connected quasiconvexity.

**Corollary 3.4** *Let $F : S \subset X \to 2^Y$ be a set-valued map and* $\mathrm{int}\, D \neq \emptyset$. *For every $e \in \mathrm{int}\, D$ the following assertions are equivalent*:

(I) *$f$ is arcwise connected $D$-quasiconvex.*

(II) *For every point $v \in Y$ the composite mapping $h_{e,v} \circ F$ is arcwise connected $\mathbb{R}_+$-quasiconvex.*

**Remark 3.5** The Gerstewitz scalarizing function plays a very important role in vector optimization with set-valued mappings. Recently, it has been extended to functions mapping from the family of nonempty subsets of $Y$ to $\mathbb{R}$, and whether the extended Gerstewitz functions can characterize the generalized convex set-valued mappings is also an interesting question; for more details related to the extended Gerstewitz functions, we refer to [19–22].

## 4 Set-valued optimization problems

In this section, we will restrict our attention to the particular case where $Y = \mathbb{R}^n$ is the $n$-dimensional Euclidean space with $n \ge 2$, partially ordered by the standard ordering cone $D = \mathbb{R}^n_+$. For any subset $A$ of $\mathbb{R}^n$ we denote by

$$\mathrm{Min}\, A := \{ y \in Y : A \cap (y - \mathbb{R}^n_+) = \{y\} \},$$

$$\mathrm{WMin}\, A := \{ y \in Y : A \cap (y - \mathrm{int}\, \mathbb{R}^n_+) = \emptyset \}$$

the sets of efficient points and weakly efficient points of $A$, respectively. Let $F : S \subset X \to 2^{\mathbb{R}^n}$, considering the set-valued optimization problem

$$\text{(SVOP)} \quad \begin{cases} \min \quad F(x) \qquad \text{w.r.t. } \mathbb{R}^n_+, \\ \text{s.t.} \quad x \in S \subset X. \end{cases}$$

The efficient solutions and the weakly efficient solutions of problem (SVOP) are defined as the following sets, respectively:

$$\operatorname{Eff}(S|F) := F^{-1}\bigl(\operatorname{Min} F(S)\bigr) \quad \text{and} \quad \operatorname{WEff}(S|F) := F^{-1}\bigl(\operatorname{WMin} F(S)\bigr).$$

Let $I_n := \{1, 2, \ldots, n\}$ be the set of indices, for every selection of indices, $\emptyset \neq I \subset I_n$, we consider the polyhedral cone:

$$D_I := \bigl\{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : y_i \geq 0, \forall i \in I\bigr\}.$$

For any subset $A$ of $\mathbb{R}^n$, the set of efficient points of $A$ with respect to $D_I$ is defined by

$$\begin{aligned} \operatorname{Min}_I A : &= \bigl\{y \in A : A \cap (y - D_I) \subset y + D_I\bigr\} \\ &= \bigl\{y \in A : \forall y' \in A : \bigl(y'_i \leq y_i, \forall i \in I\bigr) \Rightarrow \bigl(y'_i = y_i, \forall i \in I\bigr)\bigr\}. \end{aligned}$$

Then the set $\operatorname{Eff}_I(S|F) := F^{-1}(\operatorname{Min}_I F(S))$ represents the set of efficient solutions of the following set-valued optimization problem associated to (SVOP):

$$\text{(SVOP)}_I \quad \begin{cases} \min \quad F(x) \qquad \text{w.r.t. } D_I, \\ \text{s.t.} \quad x \in S \subset X. \end{cases}$$

**Definition 4.1** [9] It is said that problem (SVOP) is Pareto reducible if its weakly efficient solutions can be represented as the union of the efficient solutions of all associated problems of type $\text{(SVOP)}_I$, *i.e.*

$$\operatorname{WEff}(S|F) = \bigcup_{\emptyset \neq I \subset I_n} \operatorname{Eff}_I(S|F).$$

In the literature [11], for every $i \in I_n$, the convex cone $K_i$ in $\mathbb{R}^n$, defined by

$$K_i := \mathbb{R}^n_+ \cup \operatorname{int} D_{I_n \setminus \{i\}},$$

was introduced. Let us recall some basic definitions in vector optimization. A set $\Omega \subset \mathbb{R}^n$ is called:

1° upward, if $\Omega + \mathbb{R}^n_+ = \Omega$;
2° $K$-radiant, where $K$ is a cone of $\mathbb{R}^n$, if

$$\operatorname{ray}(y_1, y_2) := y_1 + \mathbb{R}_+(y_2 - y_1) \subset \Omega \quad \text{for all } y_1, y_2 \in \Omega, y_1 \leq_K y_2.$$

**Remark 4.2** Let $K$ be a cone of $\mathbb{R}^n$, $\Omega_1$, $\Omega_2$ be two subsets of $\mathbb{R}^n$, and $\Omega_1 \subset \Omega_2$. From the definition of $K$-radiant, it is easy to see that if $\Omega_1$ is $K$-radiant then $\Omega_2$ is $K$-radiant.

**Lemma 4.3** [11] Let $f : S \to \mathbb{R}^n$ be a function. If $f$ is continuous and arcwise connected $K_i$-quasiconvex for every $i \in I_n$, then $\mathrm{WMin}(f(S) + \mathbb{R}^n_+)$ is $\mathbb{R}^n_+$-radiant.

**Lemma 4.4** [9] In problem (SVOP), if $\mathrm{WMin}(F(S) + \mathbb{R}^n_+)$ is $\mathbb{R}^n_+$-radiant, then the problem (SVOP) is Pareto reducible.

**Proposition 4.5** *Let $F : S \subset X \to 2^{\mathbb{R}^n}$ be a set-valued mapping. Suppose that $f$ is a continuous selection of $F$ and $F(x) \subset f(x) + \mathbb{R}^n_+$. If $F$ is arcwise connected $K_i$-quasiconvex for every $i \in I_n$, then* $\mathrm{WMin}(F(S) + \mathbb{R}^n_+)$ *is $\mathbb{R}^n_+$-radiant.*

*Proof* It follows from Proposition 2.8 that the continuous function $f$ is arcwise connected $K_i$-quasiconvex for every $i \in I_n$. Then we see from Lemma 4.3 that $\mathrm{WMin}(f(S) + \mathbb{R}^n_+)$ is $\mathbb{R}^n_+$-radiant. Finally, we see from Remark 4.2 that $\mathrm{WMin}(F(S) + \mathbb{R}^n_+)$ is $\mathbb{R}^n_+$-radiant. □

**Proposition 4.6** *Let $F : S \subset X \to 2^{\mathbb{R}^n}$ be a set-valued mapping. Suppose that $f$ is a continuous selection of $F$ and $F(x) \subset f(x) + \mathbb{R}^n_+$. If $F$ is arcwise connected $K_i$-quasiconvex for every $i \in I_n$, then the problem (SVOP) is Pareto reducible.*

*Proof* It is a straightforward consequence of Lemma 4.4 and Proposition 4.5. □

## 5 Conclusions

In this note, some properties of the arcwise connected cone-quasiconvex set-valued mapping have been carried out. We point out that an arcwise connected cone-quasiconvex set-valued mapping is characterized by a selection satisfying suitable conditions. On the other hand, we show that the arcwise connected cone-quasiconvex set-valued mappings can also be characterized by means of Gerstewitz scalarization functions. Finally, in the setting of finite dimensional Euclidean space, the sufficient condition for the Pareto reducibility for arcwise connected cone-quasiconvex multicriteria optimization is presented.

**Competing interests**

The author declares that he has no competing interests.

**Acknowledgements**

This research was supported by Natural Science Foundation of China under Grant No. 11361001; Natural Science Foundation of Ningxia under Grant No. NZ14101. The author would like to extend sincere gratitude to Prof. Dai Yuhong (Academy of Mathematics and System Science, Chinese Academy of Sciences, China) for his free-handed assistance. The author is grateful to the anonymous referees who have contributed to improve the quality of the paper.

Received: 10 June 2015 Accepted: 23 September 2015 Published online: 06 October 2015

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