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C^*-Valued G-contractions and fixed points

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Abstract

Recently, Ma *et al.* introduced the notion of C^*-valued metric spaces and extended the Banach contraction principle for self-mappings on C^*-valued metric spaces. Motivated by the work of Jachymski, in this paper we extend and improve the result of Ma *et al.* by proving a fixed point theorem for self-mappings on C^*-valued metric spaces satisfying the contractive condition for those pairs of elements from the metric space which form edges of a graph in the metric space. Our result generalizes and extends the main result of Jachymski and Ma *et al.* We also establish some examples to elaborate our new notions and to substantiate our result.

1 Preliminaries and introduction

In this section we recollect some basic definitions and notions and fix our terminology to be used throughout the paper.

For the last few decades the Banach contraction principle [1] has become a powerful tool in solving non-linear phenomena. The Banach contraction principle says that if T is a mapping from a complete metric space (X, d) to itself and there exists $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad x, y \in X, \tag{1}$$

then T has a unique fixed point. Note that the Banach contraction principle requires that T satisfies the contractive condition on each point of $X \times X$. The question arises that if the mapping T does not satisfy the contraction condition (1) on the whole of $X \times X$ but it satisfies the contraction condition on some subset of $X \times X$, then does such a mapping have a fixed point? Ran and Reurings [2] showed that if in addition X is a partially ordered set and the contraction condition holds for those pair of elements from X that are comparable, then the mapping T still has a fixed point provided that $x_n \preceq x$, for all $n \in \mathbb{N}$, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$.

Afterwards, many results appeared in the literature for fixed points of mappings on partially ordered metric spaces by various authors, see for example, Bashkar and Lakshmikanthm [3], Nieto and Rodriguez-Lopez [4], Petrusel and Rus [5], and Nieto *et al.* [6, 7]. By using graph theory, Jachymski [8] unified and extended the results by the above-mentioned authors.

Now, we recall some basic definitions from graph theory that may be found in any standard text on graph theory, for example [9]. Following Jachymski [8], Δ denotes the diagonal of the Cartesian product $X \times X$ in a metric space (X, d), G is a directed graph such that the set $V(G)$ of its vertices coincide with X and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of the graph G, the graph obtained from G by reversing the direction of the edges. Thus we have $E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}$. $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of the edges. Actually it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric; under this convention, $E(\tilde{G}) = E(G) \cup E(G^{-1})$. Jachymski [8] proved the existence of a fixed point of the mapping T if the contraction condition holds for those pairs of elements from X that form edges of a graph in X provided the following condition holds:

($\mathcal{P}$) for any $\{x_n\}$ in X such that $x_n \to x$ with $(x_{n+1}, x_n) \in E(G)$ for all $n \geq 1$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x, x_{n_k}) \in E(G)$.

Some interesting fixed point theorems for Banach type contractions endowed with the graph G or ordering are obtained by the authors in [10–19]. Samreen and Kamran [20] also obtained useful results for such mappings by using the following condition, which is weaker than ($\mathcal{P}$)

($\mathcal{P}'$) for any $\{f^n x\}$ in X such that $f^n x \to y \in X$ with $(f^{n+1}x, f^n x) \in E(G)$ there exist a subsequence $\{f^{n_k}x\}$ of $\{f^n x\}$ and $n_0 \in \mathbb{N}$ such that $(y, f^{n_k}x) \in E(G)$ for all $k \geq n_0$.

Example 1.1 [20] Let $X = [0,1]$ endowed with usual metric $d(x,y) = |x - y|$. Consider a graph G consisting of $V(G) := X$ and $E(G) := \{(\frac{n}{n+1}, \frac{n+1}{n+2}) : n \in \mathbb{N}\} \cup \{(\frac{x}{2^n}, \frac{x}{2^{n+1}}) : n \in \mathbb{N}, x \in [0,1]\} \cup \{(\frac{x}{2^{2n}}, 0) : n \in \mathbb{N}, x \in [0,1]\}$. Note that G does not satisfy property ($\mathcal{P}$) as $\frac{n}{n+1} \to 1$. By defining $f : X \to X$ as $fx = \frac{x}{2}$, G satisfies property ($\mathcal{P}'$). We have $f^n x = \frac{x}{2^n} \to 0$ as $n \to \infty$.

Recently, Ma *et al.* [21] introduced the notion of C^*-valued metric spaces and extended the Banach contraction principle for self-mappings on C^*-valued metric spaces. Before giving the definition and result by Ma *et al.* [21] let us recall some notions from C^*-algebra that may be found in [22, 23]. A $*$-algebra $\mathbb{A}$ is a complex algebra with conjugate linear involution $*$ such that for any $x, y \in \mathbb{A}$, $x^{**} = x$ and $(xy)^* = y^* x^*$. In addition, if $\mathbb{A}$ is a Banach space and for $x \in \mathbb{A}$, $\|x^* x\| = \|x\|^2$, then $\mathbb{A}$ is called a C^*-algebra. The set $\sigma(x) = \{\lambda \in \mathbb{C} : \lambda I - x \text{ is not invertible}\}$ is called the spectrum of an element $x \in \mathbb{A}$. An element $x \in \mathbb{A}$ is called a positive element of $\mathbb{A}$ if x is self-adjoint *i.e.*, $x = x^*$ and $\sigma(x) \subset [0, \infty)$. The set $\mathbb{A}_+$ denotes the set of positive elements in $\mathbb{A}$. We will write $x \succeq y$ iff $x - y \in \mathbb{A}_+$. Each positive element x of a C^*-algebra has a unique positive square root. If x and y are self-conjugate elements of a C^*-algebra and $\theta \preceq x \preceq y$ then $\|x\| \preceq \|y\|$, where θ is the zero element of the C^*-algebra $\mathbb{A}$.

Definition 1.2 [21] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{A}$ is called a C^*-valued metric on X if it satisfies the following conditions:

(i) $d(x,y) \succeq \theta$, for all $x, y \in X$;
(ii) $d(x,y) = \theta \Leftrightarrow x = y$;
(iii) $d(x,y) = d(y,x)$, for all $x, y \in X$;
(iv) $d(x,y) \preceq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

The tuple $(X, \mathbb{A}, d)$ is called a C^*-valued metric space. Let $x \in (X, \mathbb{A}, d)$. A sequence $\{x_n\}$ in $(X, \mathbb{A}, d)$ is said to be convergent with respect to $\mathbb{A}$, if for any $\epsilon > 0$ there exists a positive integer N such that $\|d(x_n,x)\| \leq \epsilon$ for all $n > N$. A sequence $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$ if for any $\epsilon > 0$ there exists a positive integer N such that $\|d(x_n,x_m)\| \leq \epsilon$ for all $n,m > N$. If every Cauchy sequence with respect to $\mathbb{A}$ is convergent, then $(X,\mathbb{A},d)$ is said to be a complete C^*-valued metric space.

Definition 1.3 [21] Let $(X,\mathbb{A},d)$ be a C^*-valued metric space. A mapping $T: X \to X$ is said to be a C^*-valued contraction mapping on X if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx,Ty) \preceq A^* d(x,y)A, \quad \text{for all } x,y \in X. \tag{2}$$

Theorem 1.4 [21] *If $(X,\mathbb{A},d)$ is a C^*-algebra valued metric space and T satisfies* (2), *then T has a unique fixed point in X.*

It is natural question to ask whether the mapping T, considered above, has a fixed point if the contraction condition holds for those pair of elements that form edges of the graph as defined by Jachymski [8]. In this paper, we give a positive answer to this question by introducing the notion of C^*-valued G-contractions and then proving a fixed point theorem for such contractions. We construct some examples to elaborate the generalities of our notion and result.

2 Main results

We begin this section by introducing the notion of a C^*-valued G-contraction which is weaker than the notion of a C^*-valued contraction, Definition 1.3.

Definition 2.1 Suppose $(X,\mathbb{A},d)$ be a C^*-valued metric space endowed with the graph $G = (V(G),E(G))$. A mapping $T: X \to X$ is called a C^*-valued G-contraction on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx,Ty) \preceq A^* d(x,y)A, \quad \forall (x,y) \in E(G). \tag{3}$$

Remark 2.2 By taking $G_1 = (X, X \times X)$, we see that a C^*-valued contraction is a C^*-valued G_1-contraction.

The following example shows that the converse of the above statement is not true in general.

Example 2.3 Consider the algebra, $\mathbb{A} = M_{2\times 2}(\mathbb{R})$, of all 2×2 matrices with the usual operations of addition, scalar multiplication, and matrix multiplication. Note that $\|A\| = (\sum_{i,j=1}^{2} |a_{ij}|^2)^{\frac{1}{2}}$ defines a norm on $\mathbb{A}$ and $*: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, given by $A^* = A$, defines a convolution on $M_{2\times 2}(\mathbb{R})$. Thus $\mathbb{A}$ becomes a C^*-algebra. For

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

we denote

$$A \preceq B \quad \text{if and only if} \quad (a_{ij} - b_{ij}) \leq 0, \quad \text{for all } i,j = 1,2. \tag{4}$$

It is straightforward to see that $\preceq$ given by (4) is a partial order on $M_{2\times 2}(\mathbb{R})$. Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{A}$ by

$$d(x,y) = \begin{pmatrix} |x-y| & 0 \\ 0 & |x-y| \end{pmatrix}. \tag{5}$$

It is easy to check that d satisfies all conditions of Definition 1.2. Therefore, $(\mathbb{R}, \mathbb{A}, d)$ is a C^*-valued metric space. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = \frac{x^2}{3}$ and consider the graph $G = (V(G), E(G))$, where $V(G) = \mathbb{R}$ and

$$E(G) = \left\{ \left(\frac{1}{3^n}, \frac{1}{3^{2n+1}} \right) : n = 1, 2, \dots \right\} \cup \left\{ (x,x) : x \in \mathbb{R} \right\}. \tag{6}$$

Note that, for each $n \in \mathbb{N}$,

$$\left(T\frac{1}{3^n}, T\frac{1}{3^{2n+1}} \right) = \left(\frac{1}{3^{2n+1}}, \frac{1}{3^{4n+3}} \right) \in E(G).$$

Also, for each $x \in \mathbb{R}$, $(Tx, Tx) = (\frac{x^2}{3}, \frac{x^2}{3})$, which is again an edge in the graph G. Moreover, by taking $A = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$, we have $\|A\| < 1$, and one can check that the contractive condition (3) holds for all edges that belong to the graph G. Furthermore, the contractive condition (2) is not satisfied, for example, at $x = 1$, $y = 7$. Hence T is C^*-valued G-contraction but not a C^*-valued contraction.

The following lemma is straightforward.

Lemma 2.4 *Let $\mathbb{A}$ be a C^*-algebra and $x \in \mathbb{A}$ such that $\|x\| < 1$, then*

$$\lim_{m\to\infty} \sum_{k=m}^{n} \|x\|^k = 0. \tag{7}$$

The proof of (7) follows from the fact that $\sum_{k=m}^{n} \|x\|^k$ is a geometric series with the common ratio $\|x\| < 1$ and $m \to \infty$ implies $\|x\|^m \to 0$.

Theorem 2.5 *Let $(X, \mathbb{A}, d)$ be a C^*-valued metric space endowed with the graph $G = (V(G), E(G))$. Suppose $T : X \to X$ is a C^*-valued G-contraction on X satisfying property $(\mathcal{P}')$ and the following conditions*:

(I) *if $(x,y) \in E(G)$ then $(Tx, Ty) \in E(G)$;*

(II) *there exists an $x_\circ \in X$ such that $(x_\circ, Tx_\circ) \in E(G)$.*

Then T has a fixed point. Moreover, if y, z are two fixed points of T and $(y,z) \in E(G)$ then $y = z$.

Proof It is clear that if $A = \theta$ then T maps X into a single point, since $\theta \preceq d(Tx, Ty) \preceq \theta$ for all $(x,y) \in E(G)$. Thus without loss of generality, we assume that $A \neq \theta$. From hypothesis (II), we have an $x_\circ \in X$ such that $(x_\circ, Tx_\circ) \in E(G)$, then by using assumption (I), we get $(Tx_\circ, T^2x_\circ) \in E(G)$. Continuing in the same way we get a sequence $\{x_n\}$ such that $x_{n+1} =$

$Tx_n = T^{n+1}x_\circ$ and $(x_{n-1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$. Let us denote $d(x_\circ, x_1)$ by $P \in \mathbb{A}$. From (3), we have

$$\begin{aligned} d(x_{n+1}, x_n) &\preceq A^* d(x_n, x_{n-1})A \\ &\preceq (A^*)^2 d(x_{n-1}, x_{n-2})A^2 \\ &\vdots \\ &\preceq (A^*)^n d(x_\circ, x_1)A^n \\ &= (A^*)^n PA^n. \end{aligned}$$

For $n + 1 > m$, we get

$$\begin{aligned} d(x_{n+1}, x_m) &\preceq d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \\ &= \sum_{k=m}^{n} (A^*)^k PA^k \\ &= \sum_{k=m}^{n} (P^{\frac{1}{2}}A^k)^* (P^{\frac{1}{2}}A^k) \\ &= \sum_{k=m}^{n} |P^{\frac{1}{2}}A^k|^2 \\ &\preceq \sum_{k=m}^{n} \| |P^{\frac{1}{2}}A^k|^2 \| \\ &= \|P^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|A^{2k}\|. \end{aligned}$$

Since $\|A\| < 1$, it follows from Lemma 2.4 that $d(T^{n+1}x_\circ, T^m x_\circ) \to \theta$ as $m \to \infty$. This shows that $(T^n x_\circ)$ is a Cauchy sequence with respect to $\mathbb{A}$. Further, completeness of $(X, \mathbb{A}, d)$ implies that there exists $y \in X = V(G)$ such that

$$\lim_{n \to \infty} T^n x_\circ = y.$$

As $T^n x_\circ \to y$ and $(T^{n+1}x, T^n x) \in E(G)$, for all $n \in \mathbb{N}$, therefore by property $(\mathcal{P}')$ there exist a subsequence $(T^{n_k}x_\circ)$ and $n_0 \in \mathbb{N}$ such that $(T^{n_k}x_\circ, y) \in E(G)$ for all $k \geq n_0$. It follows that

$$\begin{aligned} \theta \preceq d(Ty, y) &\preceq d(Ty, T^{n_{k+1}}x_\circ) + d(T^{n_{k+1}}x_\circ, y) \\ &\preceq A^* d(y, x_{n_{k+1}})A + d(x_{n_{k+2}}, y) \to \theta \quad \text{as } k \to \infty. \end{aligned}$$

Thus $Ty = y$. Suppose $z \in V(G)$ is another fixed point of T such that $(z, y) \in E(G)$; then

$$\begin{aligned} 0 \leq \|d(z, y)\| = \|d(Tz, Ty)\| &\leq \|A^* d(z, y)A\| \\ &\leq \|A\|^2 \|d(z, y)\| \\ &< \|d(z, y)\|, \end{aligned}$$

since $\|A\| \leq 1$. This is possible only if $\|d(z,y)\| = 0$. This implies $d(z,y) = \theta$. Hence $z = y$. □

Remark 2.6 By taking $G = (X, X \times X)$, we see that Theorem 1.4 is a special case of Theorem 2.5. Moreover, [8], Theorem 3.2, is a special case of Theorem 2.5 when $\mathbb{A} = \mathbb{R}$.

Example 2.7 Consider the algebra, $\mathbb{A} = M_{2\times2}(\mathbb{R})$, of all 2×2 matrices with usual operations of addition, scalar multiplication, and matrix multiplication. Note that $\|A\| = (\sum_{i,j=1}^{2} |a_{ij}|^2)^{\frac{1}{2}}$ defines a norm on $\mathbb{A}$ and $* : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$, given by $A^* = A$, defines a convolution on $M_{2\times2}(\mathbb{R})$. Thus $\mathbb{A}$ becomes a C^*-algebra. For

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_{2\times2}(\mathbb{R}),$$

we denote

$$A \preceq B \quad \text{if and only if} \quad (a_{ij} - b_{ij}) \leq 0, \quad \text{for all } i,j = 1,2. \tag{8}$$

It is straightforward to see that $\preceq$ given by (8) is a partial order on $M_{2\times2}(\mathbb{R})$. Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{A}$ by

$$d(x,y) = \begin{pmatrix} |x-y| & 0 \\ 0 & |x-y| \end{pmatrix}. \tag{9}$$

It is easy to check that d satisfies all conditions of Definition 1.2. Therefore, $(\mathbb{R}, \mathbb{A}, d)$ is a C^*-valued metric space. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = \frac{x^2}{c}$, where $c > 3$ is a fixed real number, and consider the graph $G = (V(G), E(G))$, where $V(G) = \mathbb{R}$ and

$$E(G) = \left\{ \left(\frac{1}{c^n}, \frac{1}{c^{2n+1}} \right) : n = 1,2,\ldots \right\} \cup \left\{ \left(\frac{1}{c^n}, 0 \right) : n = 1,2,\ldots \right\} \cup \left\{ (x,x) : x \in \mathbb{R} \right\}. \tag{10}$$

Note that, for each $n \in \mathbb{N}$,

$$\left(T\frac{1}{c^n}, T\frac{1}{c^{2n+1}} \right) = \left(\frac{1}{c^{2n+1}}, \frac{1}{c^{4n+3}} \right) \in E(G), \quad \text{and} \quad \left(T\frac{1}{c^n}, T0 \right) = \left(\frac{1}{c^{2n+1}}, 0 \right) \in E(G).$$

Also, for each $x \in \mathbb{R}$, $(Tx, Tx) = (\frac{x^2}{c}, \frac{x^2}{c})$, which is again an edge in the graph G. Moreover, by taking $A = \begin{pmatrix} \frac{1}{\sqrt{c}} & 0 \\ 0 & \frac{1}{\sqrt{c}} \end{pmatrix}$, we have $\|A\| < 1$, and taking $x_0 = \frac{1}{c}$ one can check that all conditions of Theorem 2.5 are satisfied and 0 is fixed point of T. Note that Theorem 1.4 is not applicable here, since the contractive condition (2) does not hold, for example, at $x = 0$, $y = 4$.

3 Conclusion

Let E be a real Banach space. A cone P in E defines a partial ordering in E as follows: let $x, y \in E$; we denote $x \preceq y$ if $y - x \in P$. Using this partial ordering Huang and Zhang [24] introduced the notion of a cone metric space. A cone metric on a nonempty set X is a mapping $d_c : X \times X \to E$ satisfying: (i) $d_c(x,y) > 0$ for all $x, y \in X$ and $d_c(x,y) = 0$ if and only if $x = y$; (ii) $d_c(x,y) = d_c(y,x)$ for all $x, y \in X$; (iii) $d_c(x,z) \preceq d_c(x,y) + d_c(y,z)$ for all $x, y, z \in X$.

In fact this notion is not new and was initially defined by Kantorovich [25] as a K-metric space [25, 26]. Huang and Zhang [24] called a mapping $f : X \to X$ a cone contraction if it satisfies the following condition:

$$d_c(fx, fy) \leq kd_c(x, y) \quad \forall x, y \in X \text{ for some } k \in (0, 1). \tag{11}$$

Then they generalized the Banach contraction principle in the context of cone metric spaces [24], Theorem 1. Note that $d_c(x, y)$ is an element of the Banach space E and the right hand side of (11) is defined, since E is a real Banach space. The set of positive elements in a C^*-algebra forms a positive cone in the C^*-algebra but the underlying vector space is not a real vector space, in general. Therefore, the notion of a C^*-valued metric space seems to be more general than the notion of a cone metric space. For example if we consider the set $\mathbb{A}$ of all 2×2 matrices having entries from complex numbers, then $\mathbb{A}$ is a vector space over the field of complex numbers. Also, $\mathbb{A}$ is a C^*-algebra with Euclidean norm. A mapping $T : X \to X$ is said to be a C^*-valued contraction mapping on X, by Ma *et al.* (Definition 1.2), if there exists an A in a C^*-algebra $\mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x, y)A, \quad \text{for all } x, y \in X. \tag{12}$$

Observe that the right hand side of (12) is defined because $\mathbb{A}$ is an algebra, not necessarily real. Also, observe that it is not necessary that one can define an involution '$*$' on a normed space. Thus it seems to be difficult that the inequality (12) can be reduced to the inequality (11). Further note that the proof of the main result by Ma *et al.* [21] depends on the machinery of C^* algebras. Thus we conclude that the main results of Ma *et al.* and ours may not follow from the corresponding results of cone metric spaces.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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