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Mann-type hybrid steepest-descent method for three nonlinear problems

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Abstract
We introduce a Mann-type hybrid steepest-descent iterative algorithm for finding a common element of the set of solutions of a general mixed equilibrium problem, the set of solutions of general system of variational inequalities, the set of solutions of finitely many variational inequalities, and the set of common fixed points of finitely many nonexpansive mappings and a strict pseudocontraction in a real Hilbert space. We derive the strong convergence of the iterative algorithm to a common element of these sets, which also solves some hierarchical variational inequality.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $A : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \to C$ is called ξ-strictly pseudocontractive if there exists a constant $\xi \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \xi \left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in C.$$

Let $A : C \to H$ be a nonlinear mapping on C. We consider the following variational inequality problem (VIP) [1]: find a point $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, A)$.

The general mixed equilibrium problem (GMEP) (see, *e.g.*, [2]) is to find $x \in C$ such that

$$\Theta(x, y) + h(x, y) \geq 0, \quad \forall y \in C, \tag{1.2}$$

where $\Theta, h : C \times C \to \mathbf{R}$ are two bi-functions. We denote the set of solutions of GMEP (1.2) by GMEP(Θ, h). We assume as in [3] that $\Theta : C \times C \to \mathbf{R}$ is a bi-function satisfying conditions $(\theta 1)$-$(\theta 3)$ and $h : C \times C \to \mathbf{R}$ is a bi-function with restrictions (h1)-(h3), where

$(\theta 1)$ $\Theta(x,x) = 0$ for all $x \in C$;

$(\theta 2)$ Θ is monotone (*i.e.*, $\Theta(x,y) + \Theta(y,x) \le 0$, $\forall x,y \in C$) and upper hemicontinuous in the first variable, *i.e.*, for each $x,y,z \in C$,

$$\limsup_{t \to 0^+} \Theta\bigl(tz + (1-t)x, y\bigr) \le \Theta(x,y);$$

$(\theta 3)$ Θ is lower semicontinuous and convex in the second variable;

(h1) $h(x,x) = 0$ for all $x \in C$;

(h2) h is monotone and weakly upper semicontinuous in the first variable;

(h3) h is convex in the second variable.

For $r > 0$ and $x \in H$, let $T_r : H \to 2^C$ be a mapping defined by

$$T_r x = \left\{ z \in C : \Theta(z,y) + h(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\}$$

called the resolvent of Θ and h.

Let $F_1, F_2 : C \to H$ be two mappings. Consider the following general system of variational inequalities (GSVI) of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 F_1 y^* + x^* - y^*, x - x^* \rangle \ge 0, & \forall x \in C, \\ \langle \nu_2 F_2 x^* + y^* - x^*, x - y^* \rangle \ge 0, & \forall x \in C, \end{cases} \tag{1.3}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. The solution set of GSVI (1.3) is denoted by GSVI(C, F_1, F_2).

If C is the fixed point set Fix(T) of a nonexpansive mapping T and S is another nonexpansive mapping (not necessarily with fixed points), then VIP (1.1) becomes the variational inequality problem of finding $x^* \in \mathrm{Fix}(T)$ such that

$$\bigl\langle (I - S)x^*, x - x^* \bigr\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.4}$$

This problem, introduced by Mainge and Moudafi [4, 5], is called the hierarchical fixed point problem. It is clear that if S has fixed points, then they are solutions of VIP (1.4).

During the 1980s and 1990s, the system of variational inequalities used as tools to solve Nash equilibrium problems. See, for example, [6–8] and the references therein. On the similar lines, the results of this paper can be applicable to solve Nash equilibrium problem for two person game. In the recent past, several iterative methods have been proposed and analyzed to three nonlinear problems, namely, system of variational inequalities, generalized mixed equilibrium problems and variational inequalities; see, for example, [9–11] and the references therein.

In this paper, we will introduce a Mann-type hybrid steepest-descent iterative algorithm for finding a common element of the solution set GMEP(Θ, h) of GMEP (1.2), the solution set GSVI(C, F_1, F_2) (*i.e.*, Ξ) of GSVI (1.3), the solution set $\bigcap_{k=1}^{M} \mathrm{VI}(C, A_k)$ of finitely many variational inequalities for inverse-strongly monotone mappings $A_k : C \to H$, $k = 1, \dots, M$, and the common fixed point set $\bigcap_{i=1}^N \operatorname{Fix}(S_i) \cap \operatorname{Fix}(T)$ of finitely many nonexpansive mappings $S_i : C \to C$, $i = 1, \dots, N$ and a strictly pseudocontractive mapping $T : C \to C$, in the setting of the infinite-dimensional Hilbert space. The iterative algorithm is based on Korpelevich's extragradient method, the viscosity approximation method [12], Mann's iteration method, and the hybrid steepest-descent method. Our aim is to prove that the iterative algorithm converges strongly to a common element of these sets, which also solves some hierarchical variational inequality. We observe that related results have been derived in [4, 5, 13, 14].

2 Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty, closed, and convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$ and $\omega_s(x_n)$ to denote the strong ω-limit set of the sequence $\{x_n\}$, *i.e.*,

$$\omega_w(x_n) := \{ x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\} \}$$

and

$$\omega_s(x_n) := \{ x \in H : x_{n_i} \to x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\} \}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\| x - P_C x \| = \inf_{y \in C} \| x - y \| =: d(x, C).$$

The following properties of projections are useful for our purpose.

Proposition 2.1 *Given any $x \in H$ and $z \in C$. One has*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0$, $\forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \| x - z \|^2 \leq \| x - y \|^2 - \| y - z \|^2$, $\forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y \rangle \geq \| P_C x - P_C y \|^2$, $\forall y \in H$, *which hence implies that P_C is nonexpansive and monotone*.

Definition 2.1 A mapping $T : H \to H$ is said to be

(a) nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|, \quad \forall x, y \in H;$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse-strongly monotone (1-ism),

$$\langle x - y, Tx - Ty \rangle \geq \| Tx - Ty \|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.2 A mapping $A : C \to H$ is said to be

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that if $A : C \to H$ is α-inverse-strongly monotone, then A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. Moreover, we also have, for all $u, v \in C$ and $\lambda > 0$,

$$\big\|(I - \lambda A)u - (I - \lambda A)v\big\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \tag{2.1}$$

Proposition 2.2 (see [15]) *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of the GSVI* (1.3) *if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x, \quad \forall x \in C,$$

where $\bar{y} = P_C(I - \nu_2 F_2)\bar{x}$.

In particular, if the mapping $F_j : C \to H$ is ζ_j-inverse-strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1, 2$. We denote by Ξ the fixed point set of the mapping G.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1 *Let X be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following hold*:

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all $x, y \in H$*;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all $x, y \in H$ and $\lambda, \mu \in [0, 1]$ with* $\lambda + \mu = 1$;

(c) *if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C,$$

where $F : C \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies the conditions:

$$\|Fx - Fy\| \leq \kappa\|x - y\| \quad \textit{and} \quad \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$$

for all $x, y \in C$.

In the sequel, we let $\mathrm{GMEP}(\Theta, h)$ denote the solution set of GMEP (1.2).

Lemma 2.3 (see [3]) *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bi-function satisfying conditions* (θ1)-(θ3) *and $h : C \times C \to \mathbf{R}$ is a bi-function with restrictions* (h1)-(h3). *Moreover, let us suppose that*

(H) *for fixed $r > 0$ and $x \in C$, there exist a bounded $K \subset C$ and $\hat{x} \in K$ such that for all $z \in C \setminus K$, $-\Theta(\hat{x}, z) + h(z, \hat{x}) + \frac{1}{r}\langle \hat{x} - z, z - x\rangle < 0$.*

For $r > 0$ and $x \in H$, the mapping $T_r : H \to 2^C$ (i.e., the resolvent of Θ and h) has the following properties:

(i) $T_r x \neq \emptyset$;

(ii) *$T_r x$ is a singleton*;

(iii) *T_r is firmly nonexpansive*;

(iv) $\mathrm{GMEP}(\Theta, h) = \mathrm{Fix}(T_r)$ *and it is closed and convex.*

Recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$ and $g \in Ty$ imply

$$\langle f - g, x - y\rangle \geq 0.$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

3 Main results

We now propose the following Mann-type hybrid steepest-descent iterative scheme:

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1}S_1 u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}S_i u_n + (1 - \beta_{n,i})y_{n,i-1}, \quad i = 2, \ldots, N, \\ y_n = P_C[\alpha_n \gamma f(y_{n,N}) + (I - \alpha_n \mu F)G y_{n,N}], \\ x_{n+1} = \beta_n x_n + \gamma_n \Lambda_n^M y_n + \delta_n T \Lambda_n^M y_n \end{cases} \tag{3.1}$$

for all $n \geq 0$, where

$F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ and $f : C \to C$ is an l-Lipschitzian mapping with constant $l \geq 0$;
$A_k : C \to H$ is η_k-inverse-strongly monotone, $\{\lambda_{k,n}\} \subset [a_k, b_k] \subset (0, 2\eta_k)$, $\forall k \in \{1, \dots, M\}$, and $\Lambda_n^M := P_C(I - \lambda_{M,n}A_M) \cdots P_C(I - \lambda_{1,n}A_1)$;
$F_j : C \to H$ is ζ_j-inverse-strongly monotone and $G := P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$ with $\nu_j \in (0, 2\zeta_j)$ for $j = 1, 2$;
$T : C \to C$ is a ξ-strict pseudocontraction and $S_i : C \to C$ is a nonexpansive mapping for each $i = 1, \dots, N$;
$\Theta, h : C \times C \to \mathbf{R}$ are two bi-functions satisfying the hypotheses of Lemma 2.3;
$0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$ with $\tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
$\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
$\{\gamma_n\}$, $\{\delta_n\}$ are sequences in $[0, 1]$ with $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \geq 0$;
$\{\beta_{n,i}\}_{i=1}^N$ are sequences in $(0, 1)$ and $(\gamma_n + \delta_n)\xi \leq \gamma_n$, $\forall n \geq 0$;
$\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\liminf_{n\to\infty} \delta_n > 0$.

We start our main result from the following series of propositions.

Proposition 3.1 *Let us suppose that* $\Omega = \operatorname{Fix}(T) \cap \bigcap_{i=1}^N \operatorname{Fix}(S_i) \cap \bigcap_{k=1}^M \operatorname{VI}(C, A_k) \cap \operatorname{GMEP}(\Theta, h) \cap \Xi \neq \emptyset$. *Then the sequences* $\{x_n\}$, $\{y_n\}$, $\{y_{n,i}\}$ *for all* i, $\{u_n\}$ *are bounded.*

Proof Since $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [c, d] \subset (0, 1)$. For simplicity, we write

$$v_n = \alpha_n \gamma f(y_{n,N}) + (I - \alpha_n \mu F)G y_{n,N}$$

for all $n \geq 0$. Then $y_n = P_C v_n$. Also, we set $\tilde{y}_n = \Lambda_n^M y_n$,

$$\Lambda_n^k = P_C(I - \lambda_{k,n}A_k)P_C(I - \lambda_{k-1,n}A_{k-1}) \cdots P_C(I - \lambda_{1,n}A_1)$$

for all $k \in \{1, 2, \dots, M\}$ and $n \geq 0$, and $\Lambda_n^0 = I$, where I is the identity mapping on H.

First of all, take a fixed $p \in \Omega$ arbitrarily. We observe that

$$\|y_{n,1} - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

For all i from $i = 2$ to $i = N$, by induction, one proves that

$$\begin{aligned} \|y_{n,i} - p\| &\leq \beta_{n,i}\|u_n - p\| + (1 - \beta_{n,i})\|y_{n,i-1} - p\| \\ &\leq \|u_n - p\| \leq \|x_n - p\|. \end{aligned}$$

Thus we obtain that for every $i = 1, \dots, N$,

$$\|y_{n,i} - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.2}$$

Since for each $k \in \{1, \dots, M\}$, $I - \lambda_{k,n}A_k$ is nonexpansive and $p = P_C(I - \lambda_{k,n}A_k)p$, we have

$$\begin{aligned} \|\tilde{y}_n - p\| &= \left\|P_C(I - \lambda_{M,n}A_M)\Lambda_n^{M-1} y_n - P_C(I - \lambda_{M,n}A_M)\Lambda_n^{M-1} p\right\| \\ &\leq \left\|(I - \lambda_{M,n}A_M)\Lambda_n^{M-1} y_n - (I - \lambda_{M,n}A_M)\Lambda_n^{M-1} p\right\| \end{aligned}$$

$$
\begin{aligned}
&\leq \left\| \Lambda_n^{M-1} y_n - \Lambda_n^{M-1} p \right\| \\
&\quad \cdots \\
&\leq \left\| \Lambda_n^0 y_n - \Lambda_n^0 p \right\| \\
&= \|y_n - p\|.
\end{aligned} \tag{3.3}
$$

For simplicity, we write $\tilde{p} = P_C(p - \nu_2 F_2 p)$, $\tilde{y}_{n,N} = P_C(y_{n,N} - \nu_2 F_2 y_{n,N})$ and $z_n = P_C(\tilde{y}_{n,N} - \nu_1 F_1 \tilde{y}_{n,N})$ for each $n \geq 0$. Then $z_n = G y_{n,N}$ and

$$
p = P_C(I - \nu_1 F_1)\tilde{p} = P_C(I - \nu_1 F_1) P_C(I - \nu_2 F_2) p = Gp.
$$

Since $F_j : C \to H$ is ζ_j-inverse-strongly monotone and $0 < \nu_j < 2\zeta_j$ for each $j = 1, 2$, we know that, for all $n \geq 0$,

$$
\begin{aligned}
\|z_n - p\|^2 &= \|G y_{n,N} - p\|^2 \\
&= \left\| P_C(I - \nu_1 F_1) P_C(I - \nu_2 F_2) y_{n,N} - P_C(I - \nu_1 F_1) P_C(I - \nu_2 F_2) p \right\|^2 \\
&\leq \left\| (I - \nu_1 F_1) P_C(I - \nu_2 F_2) y_{n,N} - (I - \nu_1 F_1) P_C(I - \nu_2 F_2) p \right\|^2 \\
&= \left\| \left[P_C(I - \nu_2 F_2) y_{n,N} - P_C(I - \nu_2 F_2) p \right] \right. \\
&\quad \left. - \nu_1 \left[F_1 P_C(I - \nu_2 F_2) y_{n,N} - F_1 P_C(I - \nu_2 F_2) p \right] \right\|^2 \\
&\leq \left\| P_C(I - \nu_2 F_2) y_{n,N} - P_C(I - \nu_2 F_2) p \right\|^2 \\
&\quad + \nu_1(\nu_1 - 2\zeta_1) \left\| F_1 P_C(I - \nu_2 F_2) y_{n,N} - F_1 P_C(I - \nu_2 F_2) p \right\|^2 \\
&\leq \left\| P_C(I - \nu_2 F_2) y_{n,N} - P_C(I - \nu_2 F_2) p \right\|^2 \\
&\leq \left\| (I - \nu_2 F_2) y_{n,N} - (I - \nu_2 F_2) p \right\|^2 \\
&= \left\| (y_{n,N} - p) - \nu_2 (F_2 y_{n,N} - F_2 p) \right\|^2 \\
&\leq \|y_{n,N} - p\|^2 + \nu_2(\nu_2 - 2\zeta_2) \|F_2 y_{n,N} - F_2 p\|^2 \\
&\leq \|y_{n,N} - p\|^2 \leq \|u_n - p\|^2 \leq \|x_n - p\|^2.
\end{aligned} \tag{3.4}
$$

Also, since $Gp = p$ and G is nonexpansive, utilizing Lemma 3.1 of [16] we have from (3.1) and (3.4)

$$
\begin{aligned}
\|y_n - p\| &= \|P_C v_n - p\| \\
&\leq \left\| \alpha_n \gamma \big(f(y_{n,N}) - f(p) \big) + (I - \alpha_n \mu F) G y_{n,N} - (I - \alpha_n \mu F) p + \alpha_n (\gamma f - \mu F) p \right\| \\
&\leq \alpha_n \gamma \left\| f(y_{n,N}) - f(p) \right\| + \left\| (I - \alpha_n \mu F) G y_{n,N} - (I - \alpha_n \mu F) p \right\| + \alpha_n \left\| (\gamma f - \mu F) p \right\| \\
&\leq \alpha_n \gamma l \|y_{n,N} - p\| + (1 - \alpha_n \tau) \|y_{n,N} - p\| + \alpha_n \left\| (\gamma f - \mu F) p \right\| \\
&= \big(1 - \alpha_n (\tau - \gamma l) \big) \|y_{n,N} - p\| + \alpha_n \left\| (\gamma f - \mu F) p \right\| \\
&= \big(1 - \alpha_n (\tau - \gamma l) \big) \|y_{n,N} - p\| + \alpha_n (\tau - \gamma l) \frac{\|(\gamma f - \mu F) p\|}{\tau - \gamma l} \\
&\leq \max \left\{ \|y_{n,N} - p\|, \frac{\|(\gamma f - \mu F) p\|}{\tau - \gamma l} \right\} \\
&\leq \max \left\{ \|x_n - p\|, \frac{\|(\gamma f - \mu F) p\|}{\tau - \gamma l} \right\}.
\end{aligned} \tag{3.5}
$$

Taking into account $(\gamma_n + \delta_n)\xi \le \gamma_n$ and utilizing [17], we obtain from (3.1), (3.3), and (3.5) that

$$\begin{aligned}\|x_{n+1} - p\| &= \big\|\beta_n(x_n - p) + \gamma_n(\tilde{y}_n - p) + \delta_n(T\tilde{y}_n - p)\big\| \\ &\le \beta_n\|x_n - p\| + \big\|\gamma_n(\tilde{y}_n - p) + \delta_n(T\tilde{y}_n - p)\big\| \\ &\le \beta_n\|x_n - p\| + (\gamma_n + \delta_n)\|\tilde{y}_n - p\| \\ &\le \beta_n\|x_n - p\| + (\gamma_n + \delta_n)\|y_n - p\| \\ &\le \beta_n\|x_n - p\| + (\gamma_n + \delta_n)\max\left\{\|x_n - p\|, \frac{\|(\gamma f - \mu F)p\|}{\tau - \gamma l}\right\} \\ &\le \max\left\{\|x_n - p\|, \frac{\|(\gamma f - \mu F)p\|}{\tau - \gamma l}\right\}. \end{aligned} \tag{3.6}$$

By induction, we get

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|(\gamma f - \mu F)p\|}{\tau - \gamma l}\right\}, \quad \forall n \ge 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{F_2 y_{n,N}\}$, $\{F_1\tilde{y}_{n,N}\}$, $\{\tilde{y}_{n,N}\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, $\{y_{n,i}\}$ for each $i = 1, \ldots, N$. Since $\|T\tilde{y}_n - p\| \le \frac{1+\xi}{1-\xi}\|\tilde{y}_n - p\|$, $\{T\tilde{y}_n\}$ is also bounded. □

Proposition 3.2 *Let us suppose that $\Omega \neq \emptyset$. Moreover, let us suppose that the following hold*:

- (H0) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;
- (H1) $\sum_{n=1}^{\infty}|\lambda_{k,n} - \lambda_{k,n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{|\lambda_{k,n}-\lambda_{k,n-1}|}{\alpha_n} = 0$ *for each* $k = 1, \ldots, M$;
- (H2) $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n} = 0$;
- (H3) $\sum_{n=1}^{\infty}|\beta_{n,i} - \beta_{n-1,i}| < \infty$ *or* $\lim_{n\to\infty}\frac{|\beta_{n,i}-\beta_{n-1,i}|}{\alpha_n} = 0$ *for each* $i = 1, \ldots, N$;
- (H4) $\sum_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{|r_n-r_{n-1}|}{\alpha_n} = 0$;
- (H5) $\sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n} = 0$;
- (H6) $\sum_{n=1}^{\infty}|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}| < \infty$ *or* $\lim_{n\to\infty}\frac{1}{\alpha_n}|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}| = 0$.

Then $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, *i.e.*, $\{x_n\}$ *is asymptotically regular.*

Proof First, it is known that $\{\beta_n\} \subset [c, d] \subset (0, 1)$ as in the proof of Proposition 3.1. Taking into account $\liminf_{n\to\infty} r_n > 0$, we may assume, without loss of generality, that $\{r_n\} \subset [\epsilon, \infty)$ for some $\epsilon > 0$. First, we write $x_n = \beta_{n-1}x_{n-1} + (1 - \beta_{n-1})w_{n-1}$, $\forall n \ge 1$, where $w_{n-1} = \frac{x_n - \beta_{n-1}x_{n-1}}{1-\beta_{n-1}}$. It follows that for all $n \ge 1$,

$$\begin{aligned} w_n - w_{n-1} &= \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} - \frac{x_n - \beta_{n-1}x_{n-1}}{1 - \beta_{n-1}} \\ &= \frac{\gamma_n\tilde{y}_n + \delta_n T\tilde{y}_n}{1 - \beta_n} - \frac{\gamma_{n-1}\tilde{y}_{n-1} + \delta_{n-1}T\tilde{y}_{n-1}}{1 - \beta_{n-1}} \\ &= \frac{\gamma_n(\tilde{y}_n - \tilde{y}_{n-1}) + \delta_n(T\tilde{y}_n - T\tilde{y}_{n-1})}{1 - \beta_n} \\ &\quad + \left(\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right)\tilde{y}_{n-1} + \left(\frac{\delta_n}{1 - \beta_n} - \frac{\delta_{n-1}}{1 - \beta_{n-1}}\right)T\tilde{y}_{n-1}. \end{aligned} \tag{3.7}$$

Since $(\gamma_n + \delta_n)\xi \le \gamma_n$ for all $n \ge 0$, we have

$$\left\| \gamma_n(\tilde{y}_n - \tilde{y}_{n-1}) + \delta_n(T\tilde{y}_n - T\tilde{y}_{n-1}) \right\| \le (\gamma_n + \delta_n)\|\tilde{y}_n - \tilde{y}_{n-1}\|. \tag{3.8}$$

Next, we estimate $\|y_n - y_{n-1}\|$. Observe that

$$\begin{aligned}
\|z_n - z_{n-1}\|^2 &= \left\| P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)y_{n,N} - P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)y_{n-1,N} \right\|^2 \\
&\le \left\| (I - \nu_1 F_1)P_C(I - \nu_2 F_2)y_{n,N} - (I - \nu_1 F_1)P_C(I - \nu_2 F_2)y_{n-1,N} \right\|^2 \\
&= \left\| \left[P_C(I - \nu_2 F_2)y_{n,N} - P_C(I - \nu_2 F_2)y_{n-1,N}\right] \right. \\
&\quad \left. - \nu_1\left[F_1 P_C(I - \nu_2 F_2)y_{n,N} - F_1 P_C(I - \nu_2 F_2)y_{n-1,N}\right] \right\|^2 \\
&\le \left\| P_C(I - \nu_2 F_2)y_{n,N} - P_C(I - \nu_2 F_2)y_{n-1,N} \right\|^2 \\
&\quad - \nu_1(2\zeta_1 - \nu_1)\left\| F_1 P_C(I - \nu_2 F_2)y_{n,N} - F_1 P_C(I - \nu_2 F_2)y_{n-1,N} \right\|^2 \\
&\le \left\| P_C(I - \nu_2 F_2)y_{n,N} - P_C(I - \nu_2 F_2)y_{n-1,N} \right\|^2 \\
&\le \left\| (I - \nu_2 F_2)y_{n,N} - (I - \nu_2 F_2)y_{n-1,N} \right\|^2 \\
&= \left\| (y_{n,N} - y_{n-1,N}) - \nu_2(F_2 y_{n,N} - F_2 y_{n-1,N}) \right\|^2 \\
&\le \|y_{n,N} - y_{n-1,N}\|^2 - \nu_2(2\zeta_2 - \nu_2)\|F_2 y_{n,N} - F_2 y_{n-1,N}\|^2 \\
&\le \|y_{n,N} - y_{n-1,N}\|^2.
\end{aligned} \tag{3.9}$$

Also, we observe that

$$\begin{cases} v_n = \alpha_n \gamma f(y_{n,N}) + (I - \alpha_n \mu F)z_n, \\ v_{n-1} = \alpha_{n-1}\gamma f(y_{n-1,N}) + (I - \alpha_{n-1}\mu F)z_{n-1}, \quad \forall n \ge 1. \end{cases}$$

Simple calculations show that

$$\begin{aligned}
v_n - v_{n-1} &= (I - \alpha_n \mu F)z_n - (I - \alpha_n \mu F)z_{n-1} + (\alpha_n - \alpha_{n-1})\left(\gamma f(y_{n-1,N}) - \mu F z_{n-1}\right) \\
&\quad + \alpha_n \gamma\left(f(y_{n,N}) - f(y_{n-1,N})\right).
\end{aligned}$$

Then, passing to the norm we get from (3.9)

$$\begin{aligned}
\|v_n - v_{n-1}\| &\le \left\| (I - \alpha_n \mu F)z_n - (I - \alpha_n \mu F)z_{n-1} \right\| + |\alpha_n - \alpha_{n-1}| \left\| \gamma f(y_{n-1,N}) - \mu F z_{n-1} \right\| \\
&\quad + \alpha_n \gamma \left\| f(y_{n,N}) - f(y_{n-1,N}) \right\| \\
&\le (1 - \alpha_n \tau)\|z_n - z_{n-1}\| + \widetilde{M}|\alpha_n - \alpha_{n-1}| + \alpha_n \gamma l \|y_{n,N} - y_{n-1,N}\| \\
&\le (1 - \alpha_n \tau)\|y_{n,N} - y_{n-1,N}\| + \widetilde{M}|\alpha_n - \alpha_{n-1}| + \alpha_n \gamma l \|y_{n,N} - y_{n-1,N}\| \\
&= \left(1 - \alpha_n(\tau - \gamma l)\right)\|y_{n,N} - y_{n-1,N}\| + \widetilde{M}|\alpha_n - \alpha_{n-1}|,
\end{aligned} \tag{3.10}$$

where $\sup_{n \ge 0} \|\gamma f(y_{n,N}) - \mu F z_n\| \le \widetilde{M}$ for some $\widetilde{M} > 0$. In the meantime, by the definition of $y_{n,i}$ one obtains, for all $i = N, \dots, 2$,

$$\begin{aligned}
\|y_{n,i} - y_{n-1,i}\| &\le \beta_{n,i}\|u_n - u_{n-1}\| + \|S_i u_{n-1} - y_{n-1,i-1}\| |\beta_{n,i} - \beta_{n-1,i}| \\
&\quad + (1 - \beta_{n,i})\|y_{n,i-1} - y_{n-1,i-1}\|.
\end{aligned} \tag{3.11}$$

In the case $i = 1$, we have

$$\begin{aligned}\|y_{n,1} - y_{n-1,1}\| &\leq \beta_{n,1}\|u_n - u_{n-1}\| + \|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}| \\ &\quad + (1 - \beta_{n,1})\|u_n - u_{n-1}\| \\ &= \|u_n - u_{n-1}\| + \|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|. \end{aligned} \tag{3.12}$$

Substituting (3.12) in all (3.11)-type expressions one obtains for $i = 2, \dots, N$,

$$\begin{aligned}\|y_{n,i} - y_{n-1,i}\| &\leq \|u_n - u_{n-1}\| + \sum_{k=2}^{i}\|S_k u_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\quad + \|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|.\end{aligned}$$

This together with (3.10) implies that

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|P_C v_n - P_C v_{n-1}\| \\ &\leq \bigl(1 - \alpha_n(\tau - \gamma l)\bigr)\|y_{n,N} - y_{n-1,N}\| + \widetilde{M}|\alpha_n - \alpha_{n-1}| \\ &\leq \bigl(1 - \alpha_n(\tau - \gamma l)\bigr)\Biggl[\|u_n - u_{n-1}\| + \sum_{k=2}^{N}\|S_k u_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\quad + \|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|\Biggr] + \widetilde{M}|\alpha_n - \alpha_{n-1}| \\ &\leq \bigl(1 - \alpha_n(\tau - \gamma l)\bigr)\|u_n - u_{n-1}\| + \sum_{k=2}^{N}\|S_k u_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\quad + \|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}| + \widetilde{M}|\alpha_n - \alpha_{n-1}|.\end{aligned} \tag{3.13}$$

Furthermore, utilizing (2.1), we obtain

$$\begin{aligned}\|\tilde{y}_n - \tilde{y}_{n-1}\| &= \bigl\|\Lambda_n^M y_n - \Lambda_{n-1}^M y_{n-1}\bigr\| \\ &= \bigl\|P_C(I - \lambda_{M,n}A_M)\Lambda_n^{M-1}y_n - P_C(I - \lambda_{M,n-1}A_M)\Lambda_{n-1}^{M-1}y_{n-1}\bigr\| \\ &\leq \bigl\|P_C(I - \lambda_{M,n}A_M)\Lambda_n^{M-1}y_n - P_C(I - \lambda_{M,n-1}A_M)\Lambda_n^{M-1}y_n\bigr\| \\ &\quad + \bigl\|P_C(I - \lambda_{M,n-1}A_M)\Lambda_n^{M-1}y_n - P_C(I - \lambda_{M,n-1}A_M)\Lambda_{n-1}^{M-1}y_{n-1}\bigr\| \\ &\leq \bigl\|(I - \lambda_{M,n}A_M)\Lambda_n^{M-1}y_n - (I - \lambda_{M,n-1}A_M)\Lambda_n^{M-1}y_n\bigr\| \\ &\quad + \bigl\|(I - \lambda_{M,n-1}A_M)\Lambda_n^{M-1}y_n - (I - \lambda_{M,n-1}A_M)\Lambda_{n-1}^{M-1}y_{n-1}\bigr\| \\ &\leq |\lambda_{M,n} - \lambda_{M,n-1}|\bigl\|A_M\Lambda_n^{M-1}y_n\bigr\| + \bigl\|\Lambda_n^{M-1}y_n - \Lambda_{n-1}^{M-1}y_{n-1}\bigr\| \\ &\leq |\lambda_{M,n} - \lambda_{M,n-1}|\bigl\|A_M\Lambda_n^{M-1}y_n\bigr\| + |\lambda_{M-1,n} - \lambda_{M,n-1}|\bigl\|A_{M-1}\Lambda_n^{M-2}y_n\bigr\| \\ &\quad + \bigl\|\Lambda_n^{M-2}y_n - \Lambda_{n-1}^{M-2}y_{n-1}\bigr\| \\ &\leq \cdots \\ &\leq |\lambda_{M,n} - \lambda_{M,n-1}|\bigl\|A_M\Lambda_n^{M-1}y_n\bigr\| \\ &\quad + |\lambda_{M-1,n} - \lambda_{M-1,n-1}|\bigl\|A_{M-1}\Lambda_n^{M-2}y_n\bigr\|\end{aligned}$$

$$
\begin{aligned}
&+\cdots+|\lambda_{1,n}-\lambda_{1,n-1}|\bigl\|A_1\Lambda_n^0 y_n\bigr\|+\bigl\|\Lambda_n^0 y_n-\Lambda_{n-1}^0 y_{n-1}\bigr\| \\
&\leq \widetilde{M}_0\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|+\|y_n-y_{n-1}\|, \qquad (3.14)
\end{aligned}
$$

where $\sup_{n\geq 1}\{\sum_{k=1}^{M}\|A_k\Lambda_n^{k-1}y_n\|\}\leq \widetilde{M}_0$ for some $\widetilde{M}_0>0$.

By [3], we know that

$$
\|u_n-u_{n-1}\|\leq\|x_n-x_{n-1}\|+L\left|1-\frac{r_{n-1}}{r_n}\right|, \qquad (3.15)
$$

where $L=\sup_{n\geq 0}\|u_n-x_n\|$. So, substituting (3.15) in (3.13), we obtain

$$
\begin{aligned}
&\|y_n-y_{n-1}\| \\
&\quad\leq\bigl(1-\alpha_n(\tau-\gamma l)\bigr)\left(\|x_n-x_{n-1}\|+L\left|1-\frac{r_{n-1}}{r_n}\right|\right)+\sum_{k=2}^{N}\|S_k u_{n-1}-y_{n-1,k-1}\||\beta_{n,k}-\beta_{n-1,k}| \\
&\qquad+\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}|+\widetilde{M}|\alpha_n-\alpha_{n-1}| \\
&\quad\leq\bigl(1-\alpha_n(\tau-\gamma l)\bigr)\|x_n-x_{n-1}\|+L\frac{|r_n-r_{n-1}|}{r_n}+\sum_{k=2}^{N}\|S_k u_{n-1}-y_{n-1,k-1}\||\beta_{n,k}-\beta_{n-1,k}| \\
&\qquad+\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}|+\widetilde{M}|\alpha_n-\alpha_{n-1}| \\
&\quad\leq\bigl(1-\alpha_n(\tau-\gamma l)\bigr)\|x_n-x_{n-1}\|+\widetilde{M}_1\Biggl[\frac{|r_n-r_{n-1}|}{r_n}+\sum_{k=2}^{N}|\beta_{n,k}-\beta_{n-1,k}| \\
&\qquad+|\beta_{n,1}-\beta_{n-1,1}|+|\alpha_n-\alpha_{n-1}|\Biggr] \\
&\quad\leq\bigl(1-(\tau-\gamma l)\alpha_n\bigr)\|x_n-x_{n-1}\|+\widetilde{M}_1\left[\frac{|r_n-r_{n-1}|}{\epsilon}+\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|+|\alpha_n-\alpha_{n-1}|\right],
\end{aligned}
$$

where $\sup_{n\geq 0}\{L+\widetilde{M}+\sum_{k=2}^{N}\|S_ku_n-y_{n,k-1}\|+\|S_1u_n-u_n\|\}\leq\widetilde{M}_1$ for some $\widetilde{M}_1>0$. This together with (3.7), (3.8), and (3.14), implies that

$$
\begin{aligned}
&\|w_n-w_{n-1}\| \\
&\quad\leq\frac{\|\gamma_n(\tilde{y}_n-\tilde{y}_{n-1})+\delta_n(T\tilde{y}_n-T\tilde{y}_{n-1})\|}{1-\beta_n} \\
&\qquad+\left|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\right|\|\tilde{y}_{n-1}\|+\left|\frac{\delta_n}{1-\beta_n}-\frac{\delta_{n-1}}{1-\beta_{n-1}}\right|\|T\tilde{y}_{n-1}\| \\
&\quad\leq\frac{(\gamma_n+\delta_n)\|\tilde{y}_n-\tilde{y}_{n-1}\|}{1-\beta_n}+\left|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\right|\|\tilde{y}_{n-1}\|+\left|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\right|\|T\tilde{y}_{n-1}\| \\
&\quad=\|\tilde{y}_n-\tilde{y}_{n-1}\|+\left|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\right|\bigl(\|\tilde{y}_{n-1}\|+\|T\tilde{y}_{n-1}\|\bigr) \\
&\quad\leq\widetilde{M}_0\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|+\|y_n-y_{n-1}\|+\left|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\right|\bigl(\|\tilde{y}_{n-1}\|+\|T\tilde{y}_{n-1}\|\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&\le \widetilde{M}_0 \sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}| + \big(1 - (\tau - \gamma l)\alpha_n\big)\|x_n - x_{n-1}\| + \widetilde{M}_1 \Bigg[\frac{|r_n - r_{n-1}|}{\epsilon} \\
&\quad + \sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}| + |\alpha_n - \alpha_{n-1}| \Bigg] + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \big(\|\tilde{y}_{n-1}\| + \|T\tilde{y}_{n-1}\|\big) \\
&\le \big(1 - (\tau - \gamma l)\alpha_n\big)\|x_n - x_{n-1}\| + \widetilde{M}_2 \Bigg[\frac{|r_n - r_{n-1}|}{\epsilon} + \sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}| \\
&\quad + \sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}| + |\alpha_n - \alpha_{n-1}| + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \Bigg],
\end{aligned} \tag{3.16}
$$

where $\sup_{n\ge 0}\{\widetilde{M}_0 + \widetilde{M}_1 + \|\tilde{y}_n\| + \|T\tilde{y}_n\|\} \le \widetilde{M}_2$ for some $\widetilde{M}_2 > 0$.

Further, we observe that

$$
\begin{cases} x_{n+1} = \beta_n x_n + (1-\beta_n) w_n, \\ x_n = \beta_{n-1} x_{n-1} + (1-\beta_{n-1}) w_{n-1}, \quad \forall n \ge 1. \end{cases}
$$

Simple calculations show that

$$
x_{n+1} - x_n = (1-\beta_n)(w_n - w_{n-1}) + (\beta_n - \beta_{n-1})(x_{n-1} - w_{n-1}) + \beta_n(x_n - x_{n-1}).
$$

Then, passing to the norm we get from (3.16)

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le (1-\beta_n)\|w_n - w_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1} - w_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\
&\quad \le (1-\beta_n)\Bigg\{ \big(1 - \alpha_n(\tau - \gamma l)\big)\|x_n - x_{n-1}\| + \widetilde{M}_2 \Bigg[\frac{|r_n - r_{n-1}|}{\epsilon} + \sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}| \\
&\qquad + \sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}| + |\alpha_n - \alpha_{n-1}| + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \Bigg] \Bigg\} \\
&\qquad + |\beta_n - \beta_{n-1}|\|x_{n-1} - w_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\
&\quad \le \big(1 - (\tau - \gamma l)(1-\beta_n)\alpha_n\big)\|x_n - x_{n-1}\| + \widetilde{M}_2 \Bigg[\frac{|r_n - r_{n-1}|}{\epsilon} + \sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}| \\
&\qquad + \sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}| + |\alpha_n - \alpha_{n-1}| + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \Bigg] \\
&\qquad + |\beta_n - \beta_{n-1}|\|x_{n-1} - w_{n-1}\| \\
&\quad \le \big(1 - (\tau - \gamma l)(1-d)\alpha_n\big)\|x_n - x_{n-1}\| + \widetilde{M}_3 \Bigg[\frac{|r_n - r_{n-1}|}{\epsilon} + \sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}| \\
&\qquad + \sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}| + |\alpha_n - \alpha_{n-1}| + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| + |\beta_n - \beta_{n-1}| \Bigg],
\end{aligned} \tag{3.17}
$$

where $\sup_{n\ge 0}\{\widetilde{M}_2 + \|x_n - w_n\|\} \le \widetilde{M}_3$ for some $\widetilde{M}_3 > 0$. By hypotheses (H0)-(H6) and Lemma 2.1 of [16], we obtain the claim. □

Proposition 3.3 *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that $\{x_n\}$ is asymptotically regular. Then $\|x_n - u_n\| = \|x_n - T_{r_n}x_n\| \to 0$ and $\|y_n - \tilde{y}_n\| \to 0$ as $n \to \infty$.*

Proof Take a fixed $p \in \Omega$ arbitrarily. We recall that, by the firm nonexpansivity of T_{r_n}, a standard calculation shows that for $p \in \mathrm{GMEP}(\Theta, h)$,

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.18}$$

Observe that

$$\begin{aligned} \|\Lambda_n^k y_n - p\|^2 &= \|P_C(I - \lambda_{k,n}A_k)\Lambda_n^{k-1}y_n - P_C(I - \lambda_{k,n}A_k)p\|^2 \\ &\leq \|(I - \lambda_{k,n}A_k)\Lambda_n^{k-1}y_n - (I - \lambda_{k,n}A_k)p\|^2 \\ &\leq \|\Lambda_n^{k-1}y_n - p\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|A_k\Lambda_n^{k-1}y_n - A_kp\|^2 \\ &\leq \|y_n - p\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|A_k\Lambda_n^{k-1}y_n - A_kp\|^2 \end{aligned} \tag{3.19}$$

for each $k \in \{1, 2, \ldots, M\}$.

Utilizing Lemma 2.1 and Lemma 3.1 of [16], we obtain from $0 \leq \gamma l < \tau$, (3.1), (3.4), and (3.18)

$$\begin{aligned} &\|y_n - p\|^2 \\ &\quad = \|\alpha_n\gamma(f(y_{n,N}) - f(p)) + (I - \alpha_n\mu F)z_n - (I - \alpha_n\mu F)p + \alpha_n(\gamma f - \mu F)p\|^2 \\ &\quad \leq \|\alpha_n\gamma(f(y_{n,N}) - f(p)) + (I - \alpha_n\mu F)z_n - (I - \alpha_n\mu F)p\|^2 + 2\alpha_n\langle(\gamma f - \mu F)p, y_n - p\rangle \\ &\quad \leq [\alpha_n\gamma\|f(y_{n,N}) - f(p)\| + \|(I - \alpha_n\mu F)z_n - (I - \alpha_n\mu F)p\|]^2 + 2\alpha_n\langle(\gamma f - \mu F)p, y_n - p\rangle \\ &\quad \leq [\alpha_n\gamma l\|y_{n,N} - p\| + (1 - \alpha_n\tau)\|z_n - p\|]^2 + 2\alpha_n\langle(\gamma f - \mu F)p, y_n - p\rangle \\ &\quad = \left[\alpha_n\tau\frac{\gamma l}{\tau}\|y_{n,N} - p\| + (1 - \alpha_n\tau)\|z_n - p\|\right]^2 + 2\alpha_n\langle(\gamma f - \mu F)p, y_n - p\rangle \\ &\quad \leq \alpha_n\tau\frac{(\gamma l)^2}{\tau^2}\|y_{n,N} - p\|^2 + (1 - \alpha_n\tau)\|z_n - p\|^2 + 2\alpha_n\langle(\gamma f - \mu F)p, y_n - p\rangle \\ &\quad \leq \alpha_n\tau\|y_{n,N} - p\|^2 + \|z_n - p\|^2 + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\ &\quad \leq \alpha_n\tau\|y_{n,N} - p\|^2 + \|y_{n,N} - p\|^2 - \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 \\ &\qquad - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\ &\quad \leq \alpha_n\tau\|y_{n,N} - p\|^2 + \|u_n - p\|^2 - \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 \\ &\qquad - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\ &\quad \leq \alpha_n\tau\|y_{n,N} - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2 - \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 \\ &\qquad - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\|. \end{aligned} \tag{3.20}$$

Since $(\gamma_n + \delta_n)\xi \leq \gamma_n$ for all $n \geq 0$, utilizing [17] we have from (3.19) and (3.20)

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\quad = \|\beta_n(x_n - p) + \gamma_n(\tilde{y}_n - p) + \delta_n(T\tilde{y}_n - p)\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \left\| \beta_n(x_n - p) + (\gamma_n + \delta_n)\frac{1}{\gamma_n + \delta_n}\left[\gamma_n(\tilde{y}_n - p) + \delta_n(T\tilde{y}_n - p)\right]\right\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\left\|\frac{1}{\gamma_n + \delta_n}\left[\gamma_n(\tilde{y}_n - p) + \delta_n(T\tilde{y}_n - p)\right]\right\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\|\tilde{y}_n - p\|^2 \\
&= \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|\tilde{y}_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\left\|\Lambda_n^k y_n - p\right\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\left[\|y_n - p\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\left\|A_k\Lambda_n^{k-1}y_n - A_kp\right\|^2\right] \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\left[\alpha_n\tau\|y_{n,N} - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\right. \\
&\quad - \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 \\
&\quad \left. + 2\alpha_n\left\|(\gamma f - \mu F)p\right\|\|y_n - p\| + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\left\|A_k\Lambda_n^{k-1}y_n - A_kp\right\|^2\right] \\
&\le \|x_n - p\|^2 - (1 - \beta_n)\left[\|x_n - u_n\|^2 + \lambda_{k,n}(2\eta_k - \lambda_{k,n})\left\|A_k\Lambda_n^{k-1}y_n - A_kp\right\|^2\right. \\
&\quad \left. + \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 + \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2\right] \\
&\quad + \alpha_n\tau\|y_{n,N} - p\|^2 + 2\alpha_n\left\|(\gamma f - \mu F)p\right\|\|y_n - p\|. \qquad (3.21)
\end{aligned}
$$

So, we deduce from $\{\beta_n\} \subset [c,d] \subset (0,1)$ and $\{\lambda_{k,n}\} \subset [a_k, b_k] \subset (0, 2\eta_k)$, $k = 1,\dots,M$, that

$$
\begin{aligned}
&(1 - d)\left[\|x_n - u_n\|^2 + \lambda_{k,n}(2\eta_k - \lambda_{k,n})\left\|A_k\Lambda_n^{k-1}y_n - A_kp\right\|^2\right. \\
&\qquad \left. + \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 + \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2\right] \\
&\quad \le (1 - \beta_n)\left[\|x_n - u_n\|^2 + \lambda_{k,n}(2\eta_k - \lambda_{k,n})\left\|A_k\Lambda_n^{k-1}y_n - A_kp\right\|^2\right. \\
&\qquad \left. + \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 + \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2\right] \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\tau\|y_{n,N} - p\|^2 + 2\alpha_n\left\|(\gamma f - \mu F)p\right\|\|y_n - p\| \\
&\quad \le \|x_n - x_{n+1}\|\left(\|x_n - p\| + \|x_{n+1} - p\|\right) + \alpha_n\tau\|y_{n,N} - p\|^2 + 2\alpha_n\left\|(\gamma f - \mu F)p\right\|\|y_n - p\|.
\end{aligned}
$$

By Propositions 3.1 and 3.2 we know that the sequences $\{x_n\}$, $\{y_n\}$, and $\{y_{n,N}\}$ are bounded, and that $\{x_n\}$ is asymptotically regular. Therefore, from $\alpha_n \to 0$ we obtain

$$
\begin{aligned}
\lim_{n\to\infty}\|x_n - u_n\| &= \lim_{n\to\infty}\|F_2y_{n,N} - F_2p\| = \lim_{n\to\infty}\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\| \\
&= \lim_{n\to\infty}\left\|A_k\Lambda_n^{k-1}y_n - A_kp\right\| = 0 \qquad (3.22)
\end{aligned}
$$

for each $k \in \{1,\dots,M\}$.

By Proposition 2.1(iii), we deduce that for each $k \in \{1,2,\dots,M\}$,

$$
\begin{aligned}
\left\|\Lambda_n^k y_n - p\right\|^2 &= \left\|P_C(I - \lambda_{k,n}A_k)\Lambda_n^{k-1}y_n - P_C(I - \lambda_{k,n}A_k)p\right\|^2 \\
&\le \left\langle (I - \lambda_{k,n}A_k)\Lambda_n^{k-1}y_n - (I - \lambda_{k,n}A_k)p, \Lambda_n^k y_n - p\right\rangle \\
&= \frac{1}{2}\left(\left\|(I - \lambda_{k,n}A_k)\Lambda_n^{k-1}y_n - (I - \lambda_{k,n}A_k)p\right\|^2 + \left\|\Lambda_n^k y_n - p\right\|^2\right. \\
&\quad \left. - \left\|(I - \lambda_{k,n}A_k)\Lambda_n^{k-1}y_n - (I - \lambda_{k,n}A_k)p - \left(\Lambda_n^k y_n - p\right)\right\|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{2}\big(\big\|\Lambda_n^{k-1}y_n - p\big\|^2 + \big\|\Lambda_n^k y_n - p\big\|^2 \\
&\quad - \big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n - \lambda_{k,n}\big(A_k\Lambda_n^{k-1}y_n - A_k p\big)\big\|^2\big) \\
&\le \frac{1}{2}\big(\|y_n - p\|^2 + \big\|\Lambda_n^k y_n - p\big\|^2 \\
&\quad - \big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n - \lambda_{k,n}\big(A_k\Lambda_n^{k-1}y_n - A_k p\big)\big\|^2\big),
\end{aligned}
$$

which immediately leads to

$$
\begin{aligned}
\big\|\Lambda_n^k y_n - p\big\|^2 &\le \|y_n - p\|^2 - \big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n - \lambda_{k,n}\big(A_k\Lambda_n^{k-1}y_n - A_k p\big)\big\|^2 \\
&= \|y_n - p\|^2 - \big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|^2 - \lambda_{k,n}^2\big\|A_k\Lambda_n^{k-1}y_n - A_k p\big\|^2 \\
&\quad + 2\lambda_{k,n}\big\langle\Lambda_n^{k-1}y_n - \Lambda_n^k y_n, A_k\Lambda_n^{k-1}y_n - A_k p\big\rangle \\
&\le \|y_n - p\|^2 - \big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|^2 \\
&\quad + 2\lambda_{k,n}\big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|\big\|A_k\Lambda_n^{k-1}y_n - A_k p\big\|.
\end{aligned}
\tag{3.23}
$$

From (3.4), (3.20), (3.21), and (3.23) we conclude that

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\tilde{y}_n - p\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big\|\Lambda_n^k y_n - p\big\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big[\|y_n - p\|^2 - \big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|^2 \\
&\qquad + 2\lambda_{k,n}\big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|\big\|A_k\Lambda_n^{k-1}y_n - A_k p\big\|\big] \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big[\alpha_n\tau\|y_{n,N} - p\|^2 \\
&\qquad + \|z_n - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu F)p\big\|\|y_n - p\| \\
&\qquad - \big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|^2 + 2\lambda_{k,n}\big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|\big\|A_k\Lambda_n^{k-1}y_n - A_k p\big\|\big] \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big[\alpha_n\tau\|y_{n,N} - p\|^2 \\
&\qquad + \|x_n - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu F)p\big\|\|y_n - p\| \\
&\qquad - \big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|^2 + 2\lambda_{k,n}\big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|\big\|A_k\Lambda_n^{k-1}y_n - A_k p\big\|\big] \\
&\quad \le \|x_n - p\|^2 - (1-\beta_n)\big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|^2 \\
&\qquad + 2\lambda_{k,n}\big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|\big\|A_k\Lambda_n^{k-1}y_n - A_k p\big\| \\
&\qquad + \alpha_n\tau\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu F)p\big\|\|y_n - p\|,
\end{aligned}
\tag{3.24}
$$

which together with $\{\beta_n\} \subset [c,d] \subset (0,1)$ and $\{\lambda_{k,n}\} \subset [a_k, b_k] \subset (0, 2\eta_k)$, $k = 1, \ldots, M$, yields

$$
\begin{aligned}
&(1-d)\big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|^2 \\
&\quad \le (1-\beta_n)\big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\lambda_{k,n}\big\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\big\|\big\|A_k\Lambda_n^{k-1}y_n - A_k p\big\|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \alpha_n \tau \|y_{n,N} - p\|^2 + 2\alpha_n \|(\gamma f - \mu F)p\| \|y_n - p\| \\
&\le \|x_n - x_{n+1}\| \big(\|x_n - p\| + \|x_{n+1} - p\| \big) \\
&\quad + 2b_k \| \Lambda_n^{k-1} y_n - \Lambda_n^k y_n \| \| A_k \Lambda_n^{k-1} y_n - A_k p \| \\
&\quad + \alpha_n \tau \|y_{n,N} - p\|^2 + 2\alpha_n \|(\gamma f - \mu F)p\| \|y_n - p\|.
\end{aligned}
$$

Since $\alpha_n \to 0$, and $\{x_n\}$, $\{y_n\}$, and $\{y_{n,N}\}$ are bounded, we obtain from (3.22) and the asymptotical regularity of $\{x_n\}$ (due to Proposition 3.2),

$$
\lim_{n\to\infty} \| \Lambda_n^{k-1} y_n - \Lambda_n^k y_n \| = 0, \quad \forall k \in \{1, \dots, M\}. \tag{3.25}
$$

Note that

$$
\begin{aligned}
\|y_n - \tilde{y}_n\| &= \| \Lambda_n^0 y_n - \Lambda_n^M y_n \| \\
&\le \| \Lambda_n^0 y_n - \Lambda_n^1 y_n \| + \| \Lambda_n^1 y_n - \Lambda_n^2 y_n \| + \cdots \| \Lambda_n^{M-1} y_n - \Lambda_n^M y_n \|.
\end{aligned}
$$

Thus,

$$
\lim_{n\to\infty} \|y_n - \tilde{y}_n\| = 0. \tag{3.26}
$$

□

Remark 3.1 By the last proposition we have $\omega_w(x_n) = \omega_w(u_n)$ and $\omega_s(x_n) = \omega_s(u_n)$, *i.e.*, the sets of strong/weak cluster points of $\{x_n\}$ and $\{u_n\}$ coincide.

Of course, if $\beta_{n,i} \to \beta_i \ne 0$ as $n \to \infty$, for all indices i, the assumptions of Proposition 3.2 are enough to assure that

$$
\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,i}} = 0, \quad \forall i \in \{1, \dots, N\}.
$$

In the next proposition, we estimate the case in which at least one sequence $\{\beta_{n,k_0}\}$ is a null sequence.

Proposition 3.4 *Let us suppose that $\Omega \ne \emptyset$. Let us suppose that* (H0) *holds. Moreover, for an index $k_0 \in \{1, \dots, N\}$, $\lim_{n\to\infty} \beta_{n,k_0} = 0$ and the following hold*:

(H7) *for each $i \in \{1, \dots, N\}$ and $k \in \{1, \dots, M\}$*,

$$
\begin{aligned}
\lim_{n\to\infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\alpha_n \beta_{n,k_0}} &= \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n \beta_{n,k_0}} = \lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n \beta_{n,k_0}} = \lim_{n\to\infty} \frac{|r_n - r_{n-1}|}{\alpha_n \beta_{n,k_0}} \\
&= \lim_{n\to\infty} \frac{1}{\alpha_n \beta_{n,k_0}} \left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right| \\
&= \lim_{n\to\infty} \frac{|\lambda_{k,n} - \lambda_{k,n-1}|}{\alpha_n \beta_{n,k_0}} = 0;
\end{aligned}
$$

(H8) *there exists a constant $b > 0$ such that $\frac{1}{\alpha_n} \left| \frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}} \right| < b$ for all $n \ge 1$.*
Then

$$
\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} = 0.
$$

Proof We start by (3.17). Dividing both terms by β_{n,k_0} we have

$$\begin{aligned}\frac{\|x_{n+1}-x_n\|}{\beta_{n,k_0}} &\le \bigl(1-(\tau-\gamma l)(1-d)\alpha_n\bigr)\frac{\|x_n-x_{n-1}\|}{\beta_{n,k_0}} \\ &\quad + \widetilde{M}_3\Biggl[\frac{|r_n-r_{n-1}|}{\epsilon\beta_{n,k_0}} + \frac{\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|}{\beta_{n,k_0}} \\ &\quad + \frac{\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|}{\beta_{n,k_0}} + \frac{|\alpha_n-\alpha_{n-1}|}{\beta_{n,k_0}} \\ &\quad + \frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\beta_{n,k_0}} + \frac{|\beta_n-\beta_{n-1}|}{\beta_{n,k_0}}\Biggr]. \end{aligned} \tag{3.27}$$

So, by (H8) we have

$$\begin{aligned}&\frac{\|x_{n+1}-x_n\|}{\beta_{n,k_0}} \\ &\quad\le \bigl[1-(\tau-\gamma l)(1-d)\alpha_n\bigr]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1,k_0}} \\ &\qquad + \bigl[1-(\tau-\gamma l)(1-d)\alpha_n\bigr]\|x_n-x_{n-1}\|\left|\frac{1}{\beta_{n,k_0}}-\frac{1}{\beta_{n-1,k_0}}\right| \\ &\qquad + \widetilde{M}_3\Biggl[\frac{|r_n-r_{n-1}|}{\epsilon\beta_{n,k_0}} + \frac{\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|}{\beta_{n,k_0}} + \frac{\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|}{\beta_{n,k_0}} \\ &\qquad + \frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\beta_{n,k_0}} + \frac{|\alpha_n-\alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n-\beta_{n-1}|}{\beta_{n,k_0}}\Biggr] \\ &\quad\le \bigl[1-(\tau-\gamma l)(1-d)\alpha_n\bigr]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1,k_0}} + \|x_n-x_{n-1}\|\left|\frac{1}{\beta_{n,k_0}}-\frac{1}{\beta_{n-1,k_0}}\right| \\ &\qquad + \widetilde{M}_3\Biggl[\frac{|r_n-r_{n-1}|}{\epsilon\beta_{n,k_0}} + \frac{\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|}{\beta_{n,k_0}} + \frac{\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|}{\beta_{n,k_0}} \\ &\qquad + \frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\beta_{n,k_0}} + \frac{|\alpha_n-\alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n-\beta_{n-1}|}{\beta_{n,k_0}}\Biggr] \\ &\quad\le \bigl[1-(\tau-\gamma l)(1-d)\alpha_n\bigr]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1,k_0}} + \alpha_n b\|x_n-x_{n-1}\| \\ &\qquad + \widetilde{M}_3\Biggl[\frac{|r_n-r_{n-1}|}{\epsilon\beta_{n,k_0}} + \frac{\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|}{\beta_{n,k_0}} + \frac{\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|}{\beta_{n,k_0}} \\ &\qquad + \frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\beta_{n,k_0}} + \frac{|\alpha_n-\alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n-\beta_{n-1}|}{\beta_{n,k_0}}\Biggr] \\ &\quad= \bigl[1-\alpha_n(\tau-\gamma l)(1-d)\bigr]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1,k_0}} \\ &\qquad + \alpha_n(\tau-\gamma l)(1-d)\cdot\frac{1}{(\tau-\gamma l)(1-d)}\Biggl\{b\|x_n-x_{n-1}\| \\ &\qquad + \widetilde{M}_3\Biggl[\frac{|r_n-r_{n-1}|}{\epsilon\alpha_n\beta_{n,k_0}} + \frac{\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|}{\alpha_n\beta_{n,k_0}} + \frac{\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|}{\alpha_n\beta_{n,k_0}} \\ &\qquad + \frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\alpha_n\beta_{n,k_0}} + \frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n\beta_{n,k_0}} + \frac{|\beta_n-\beta_{n-1}|}{\alpha_n\beta_{n,k_0}}\Biggr]\Biggr\}.\end{aligned}$$

Therefore, utilizing Lemma 2.1 of [16], from (H0), (H7), and the asymptotical regularity of $\{x_n\}$ (due to Proposition 3.2), we deduce that

$$\lim_{n\to\infty} \frac{\|x_{n+1}-x_n\|}{\beta_{n,k_0}} = 0. \qquad \square$$

Proposition 3.5 *Let us suppose that* $\Omega \neq \emptyset$. *Let us suppose that* (H0)-(H6) *hold. Then* $\|z_n - y_{n,N}\| \to 0$ *as* $n \to \infty$.

Proof Let $p \in \Omega$. In terms of the firm nonexpansivity of P_C and the ζ_j-inverse-strong monotonicity of F_j for $j = 1, 2$, we obtain from $\nu_j \in (0, 2\zeta_j)$, $j = 1, 2$, and (3.4)

$$\begin{aligned}
\|\tilde{y}_{n,N} - \tilde{p}\|^2 &= \big\|P_C(I-\nu_2F_2)y_{n,N} - P_C(I-\nu_2F_2)p\big\|^2 \\
&\leq \big\langle (I-\nu_2F_2)y_{n,N} - (I-\nu_2F_2)p, \tilde{y}_{n,N} - \tilde{p}\big\rangle \\
&= \frac{1}{2}\Big[\big\|(I-\nu_2F_2)y_{n,N} - (I-\nu_2F_2)p\big\|^2 + \|\tilde{y}_{n,N} - \tilde{p}\|^2 \\
&\quad - \big\|(I-\nu_2F_2)y_{n,N} - (I-\nu_2F_2)p - (\tilde{y}_{n,N} - \tilde{p})\big\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|y_{n,N} - p\|^2 + \|\tilde{y}_{n,N} - \tilde{p}\|^2 \\
&\quad - \big\|(y_{n,N} - \tilde{y}_{n,N}) - \nu_2(F_2y_{n,N} - F_2p) - (p - \tilde{p})\big\|^2\Big] \\
&= \frac{1}{2}\Big[\|y_{n,N} - p\|^2 + \|\tilde{y}_{n,N} - \tilde{p}\|^2 - \big\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\big\|^2 \\
&\quad + 2\nu_2\big\langle (y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p}), F_2y_{n,N} - F_2p\big\rangle - \nu_2^2\|F_2y_{n,N} - F_2p\|^2\Big]
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - p\|^2 &= \big\|P_C(I-\nu_1F_1)\tilde{y}_{n,N} - P_C(I-\nu_1F_1)\tilde{p}\big\|^2 \\
&\leq \big\langle (I-\nu_1F_1)\tilde{y}_{n,N} - (I-\nu_1F_1)\tilde{p}, z_n - p\big\rangle \\
&= \frac{1}{2}\Big[\big\|(I-\nu_1F_1)\tilde{y}_{n,N} - (I-\nu_1F_1)\tilde{p}\big\|^2 + \|z_n - p\|^2 \\
&\quad - \big\|(I-\nu_1F_1)\tilde{y}_{n,N} - (I-\nu_1F_1)\tilde{p} - (z_n - p)\big\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|\tilde{y}_{n,N} - \tilde{p}\|^2 + \|z_n - p\|^2 - \big\|(\tilde{y}_{n,N} - z_n) + (p - \tilde{p})\big\|^2 \\
&\quad + 2\nu_1\big\langle F_1\tilde{y}_{n,N} - F_1\tilde{p}, (\tilde{y}_{n,N} - z_n) + (p - \tilde{p})\big\rangle - \nu_1^2\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|y_{n,N} - p\|^2 + \|z_n - p\|^2 - \big\|(\tilde{y}_{n,N} - z_n) + (p - \tilde{p})\big\|^2 \\
&\quad + 2\nu_1\big\langle F_1\tilde{y}_{n,N} - F_1\tilde{p}, (\tilde{y}_{n,N} - z_n) + (p - \tilde{p})\big\rangle\Big].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\tilde{y}_{n,N} - \tilde{p}\|^2 &\leq \|y_{n,N} - p\|^2 - \big\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\big\|^2 \\
&\quad + 2\nu_2\big\langle (y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p}), F_2y_{n,N} - F_2p\big\rangle \\
&\quad - \nu_2^2\|F_2y_{n,N} - F_2p\|^2
\end{aligned} \tag{3.28}$$

and

$$\|z_n - p\|^2 \leq \|y_{n,N} - p\|^2 - \|(\tilde{y}_{n,N} - z_n) + (p - \tilde{p})\|^2 + 2\nu_1\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|\|(\tilde{y}_{n,N} - z_n) + (p - \tilde{p})\|. \tag{3.29}$$

Consequently, from (3.4), (3.24) and (3.28), it follows that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\bigl[\alpha_n\tau\|y_{n,N} - p\|^2 + \|z_n - p\|^2 + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\
&\quad - \|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\|^2 + 2\lambda_{k,n}\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\|\|A_k\Lambda_n^{k-1}y_n - A_k p\|\bigr] \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\bigl[\alpha_n\tau\|y_{n,N} - p\|^2 + \|\tilde{y}_{n,N} - \tilde{p}\|^2 + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\
&\quad + 2\lambda_{k,n}\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\|\|A_k\Lambda_n^{k-1}y_n - A_k p\|\bigr] \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\bigl[\alpha_n\tau\|y_{n,N} - p\|^2 + \|y_{n,N} - p\|^2 - \|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|^2 \\
&\quad + 2\nu_2\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|\|F_2 y_{n,N} - F_2 p\| + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\
&\quad + 2\lambda_{k,n}\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\|\|A_k\Lambda_n^{k-1}y_n - A_k p\|\bigr] \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\bigl[\alpha_n\tau\|y_{n,N} - p\|^2 + \|x_n - p\|^2 - \|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|^2 \\
&\quad + 2\nu_2\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|\|F_2 y_{n,N} - F_2 p\| + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\
&\quad + 2\lambda_{k,n}\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\|\|A_k\Lambda_n^{k-1}y_n - A_k p\|\bigr] \\
&\leq \|x_n - p\|^2 + \alpha_n\tau\|y_{n,N} - p\|^2 - (1-\beta_n)\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|^2 \\
&\quad + 2\nu_2\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|\|F_2 y_{n,N} - F_2 p\| + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\
&\quad + 2\lambda_{k,n}\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\|\|A_k\Lambda_n^{k-1}y_n - A_k p\|,
\end{aligned}$$

which yields

$$\begin{aligned}
&(1-d)\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|^2 \\
&\leq (1-\beta_n)\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|^2 \\
&\leq \|x_n - p\|^2 + \alpha_n\tau\|y_{n,N} - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\nu_2\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|\|F_2 y_{n,N} - F_2 p\| + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\
&\quad + 2\lambda_{k,n}\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\|\|A_k\Lambda_n^{k-1}y_n - A_k p\| \\
&\leq \|x_n - x_{n+1}\|\bigl(\|x_n - p\| + \|x_{n+1} - p\|\bigr) + \alpha_n\tau\|y_{n,N} - p\|^2 \\
&\quad + 2\nu_2\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\|\|F_2 y_{n,N} - F_2 p\| + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\
&\quad + 2\lambda_{k,n}\|\Lambda_n^{k-1}y_n - \Lambda_n^k y_n\|\|A_k\Lambda_n^{k-1}y_n - A_k p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, and $\{x_n\}$, $\{y_n\}$, $\{y_{n,N}\}$, and $\{\tilde{y}_{n,N}\}$ are bounded, we deduce from (3.22) that

$$\lim_{n\to\infty}\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\| = 0. \tag{3.30}$$

Furthermore, from (3.4), (3.24), and (3.29), it follows that

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\bigl[\alpha_n\tau\|y_{n,N}-p\|^2+\|z_n-p\|^2+2\alpha_n\bigl\|(\gamma f-\mu F)p\bigr\|\|y_n-p\|\\
&\qquad+2\lambda_{k,n}\bigl\|\Lambda_n^{k-1}y_n-\Lambda_n^k y_n\bigr\|\bigl\|A_k\Lambda_n^{k-1}y_n-A_kp\bigr\|\bigr]\\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\bigl[\alpha_n\tau\|y_{n,N}-p\|^2+\|y_{n,N}-p\|^2-\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|^2\\
&\qquad+2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|+2\alpha_n\bigl\|(\gamma f-\mu F)p\bigr\|\|y_n-p\|\\
&\qquad+2\lambda_{k,n}\bigl\|\Lambda_n^{k-1}y_n-\Lambda_n^k y_n\bigr\|\bigl\|A_k\Lambda_n^{k-1}y_n-A_kp\bigr\|\bigr]\\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\bigl[\alpha_n\tau\|y_{n,N}-p\|^2+\|x_n-p\|^2-\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|^2\\
&\qquad+2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|+2\alpha_n\bigl\|(\gamma f-\mu F)p\bigr\|\|y_n-p\|\\
&\qquad+2\lambda_{k,n}\bigl\|\Lambda_n^{k-1}y_n-\Lambda_n^k y_n\bigr\|\bigl\|A_k\Lambda_n^{k-1}y_n-A_kp\bigr\|\bigr]\\
&\quad\le \|x_n-p\|^2+\alpha_n\tau\|y_{n,N}-p\|^2-(1-\beta_n)\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|^2\\
&\qquad+2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|+2\alpha_n\bigl\|(\gamma f-\mu F)p\bigr\|\|y_n-p\|\\
&\qquad+2\lambda_{k,n}\bigl\|\Lambda_n^{k-1}y_n-\Lambda_n^k y_n\bigr\|\bigl\|A_k\Lambda_n^{k-1}y_n-A_kp\bigr\|,
\end{aligned}$$

which leads to

$$\begin{aligned}
&(1-d)\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|^2\\
&\quad\le (1-\beta_n)\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|^2\\
&\quad\le \|x_n-p\|^2+\alpha_n\tau\|y_{n,N}-p\|^2-\|x_{n+1}-p\|^2\\
&\qquad+2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|+2\alpha_n\bigl\|(\gamma f-\mu F)p\bigr\|\|y_n-p\|\\
&\qquad+2\lambda_{k,n}\bigl\|\Lambda_n^{k-1}y_n-\Lambda_n^k y_n\bigr\|\bigl\|A_k\Lambda_n^{k-1}y_n-A_kp\bigr\|\\
&\quad\le \|x_n-x_{n+1}\|\bigl(\|x_n-p\|+\|x_{n+1}-p\|\bigr)+\alpha_n\tau\|y_{n,N}-p\|^2\\
&\qquad+2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|+2\alpha_n\bigl\|(\gamma f-\mu F)p\bigr\|\|y_n-p\|\\
&\qquad+2\lambda_{k,n}\bigl\|\Lambda_n^{k-1}y_n-\Lambda_n^k y_n\bigr\|\bigl\|A_k\Lambda_n^{k-1}y_n-A_kp\bigr\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, and $\{x_n\}$, $\{z_n\}$, $\{y_n\}$, $\{y_{n,N}\}$, and $\{\tilde{y}_{n,N}\}$ are bounded, we deduce from (3.22) that

$$\lim_{n\to\infty}\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|=0. \tag{3.31}$$

Note that

$$\|y_{n,N}-z_n\|\le\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|+\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|.$$

Hence from (3.30) and (3.31) we get

$$\lim_{n\to\infty}\|y_{n,N}-z_n\|=\lim_{n\to\infty}\|y_{n,N}-Gy_{n,N}\|=0. \tag{3.32}$$

□

Proposition 3.6 *Let us suppose that* $\Omega \neq \emptyset$. *Let us suppose that* $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ *for each* $i = 1, \dots, N$. *Moreover, suppose that* (H0)-(H6) *are satisfied. Then,* $\lim_{n\to\infty} \|S_i u_n - u_n\| = 0$ *for each* $i = 1, \dots, N$ *provided* $\|Ty_n - y_n\| \to$ *as* $n \to \infty$.

Proof First of all, observe that

$$\begin{aligned} x_{n+1} - x_n &= \gamma_n(\tilde{y}_n - x_n) + \delta_n(T\tilde{y}_n - x_n) \\ &= \gamma_n(\tilde{y}_n - y_n) + \gamma_n(y_n - x_n) + \delta_n(T\tilde{y}_n - Ty_n) \\ &\quad + \delta_n(Ty_n - y_n) + \delta_n(y_n - x_n) \\ &= (\gamma_n + \delta_n)(y_n - x_n) + \gamma_n(\tilde{y}_n - y_n) + \delta_n(T\tilde{y}_n - Ty_n) + \delta_n(Ty_n - y_n) \\ &= (1 - \beta_n)(y_n - x_n) + \gamma_n(\tilde{y}_n - y_n) + \delta_n(T\tilde{y}_n - Ty_n) + \delta_n(Ty_n - y_n). \end{aligned}$$

By Proposition 3.2 we know that $\{x_n\}$ is asymptotically regular. Utilizing [17] we have from $(\gamma_n + \delta_n)\xi \leq \gamma_n$,

$$\begin{aligned} (1 - d)\|y_n - x_n\| &\leq (1 - \beta_n)\|y_n - x_n\| \\ &= \big\|x_{n+1} - x_n - \gamma_n(\tilde{y}_n - y_n) - \delta_n(T\tilde{y}_n - Ty_n) - \delta_n(Ty_n - y_n)\big\| \\ &\leq \|x_{n+1} - x_n\| + \big\|\gamma_n(\tilde{y}_n - y_n) + \delta_n(T\tilde{y}_n - Ty_n)\big\| + \delta_n\|Ty_n - y_n\| \\ &\leq \|x_{n+1} - x_n\| + (\gamma_n + \delta_n)\|\tilde{y}_n - y_n\| + \delta_n\|Ty_n - y_n\| \\ &\leq \|x_{n+1} - x_n\| + \|\tilde{y}_n - y_n\| + \|Ty_n - y_n\|, \end{aligned}$$

which together with (3.26) and $\|Ty_n - y_n\| \to 0$, implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.33}$$

Let us show that for each $i \in \{1, \dots, N\}$, one has $\|S_i u_n - y_{n,i-1}\| \to 0$ as $n \to \infty$. Let $p \in \Omega$. When $i = N$, by Lemma 2.2(b) we have from (3.2), (3.4), and (3.20)

$$\begin{aligned} \|y_n - p\|^2 &\leq \alpha_n\tau\|y_{n,N} - p\|^2 + \|z_n - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu F)p\big\|\|y_n - p\| \\ &\leq \alpha_n\tau\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu F)p\big\|\|y_n - p\| + \|y_{n,N} - p\|^2 \\ &= \alpha_n\tau\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu F)p\big\|\|y_n - p\| + \beta_{n,N}\|S_N u_n - p\|^2 \\ &\quad + (1 - \beta_{n,N})\|y_{n,N-1} - p\|^2 - \beta_{n,N}(1 - \beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 \\ &\leq \alpha_n\tau\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu F)p\big\|\|y_n - p\| + \beta_{n,N}\|u_n - p\|^2 \\ &\quad + (1 - \beta_{n,N})\|u_n - p\|^2 - \beta_{n,N}(1 - \beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 \\ &= \alpha_n\tau\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu F)p\big\|\|y_n - p\| + \|u_n - p\|^2 \\ &\quad - \beta_{n,N}(1 - \beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 \\ &\leq \alpha_n\tau\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu F)p\big\|\|y_n - p\| + \|x_n - p\|^2 \\ &\quad - \beta_{n,N}(1 - \beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2. \end{aligned}$$

So, we have

$$\begin{aligned}
&\beta_{n,N}(1-\beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 \\
&\quad \le \alpha_n \tau \|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \\
&\qquad + \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\quad \le \alpha_n \tau \|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \\
&\qquad + \|x_n - y_n\| \big(\|x_n - p\| + \|y_n - p\|\big).
\end{aligned}$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_{n,N} \le \limsup_{n\to\infty} \beta_{n,N} < 1$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ (due to (3.33)), it is known that $\{\|S_N u_n - y_{n,N-1}\|\}$ is a null sequence.

Let $i \in \{1,\ldots,N-1\}$. Then one has

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n \tau \|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| + \|y_{n,N} - p\|^2 \\
&\le \alpha_n \tau \|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| + \beta_{n,N}\|S_N u_n - p\|^2 \\
&\quad + (1-\beta_{n,N})\|y_{n,N-1} - p\|^2 \\
&\le \alpha_n \tau \|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| + \beta_{n,N}\|x_n - p\|^2 \\
&\quad + (1-\beta_{n,N})\|y_{n,N-1} - p\|^2 \\
&\le \alpha_n \tau \|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| + \beta_{n,N}\|x_n - p\|^2 \\
&\quad + (1-\beta_{n,N})\big[\beta_{n,N-1}\|S_{N-1}u_n - p\|^2 + (1-\beta_{n,N-1})\|y_{n,N-2} - p\|^2\big] \\
&\le \alpha_n \tau \|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \\
&\quad + \big(\beta_{n,N} + (1-\beta_{n,N})\beta_{n,N-1}\big)\|x_n - p\|^2 \\
&\quad + \prod_{k=N-1}^{N} (1-\beta_{n,k})\|y_{n,N-2} - p\|^2,
\end{aligned}$$

and so, after $(N-i+1)$ iterations,

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n \tau \|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \\
&\quad + \left(\beta_{n,N} + \sum_{j=i+2}^{N}\left(\prod_{l=j}^{N}(1-\beta_{n,l})\right)\beta_{n,j-1}\right)\|x_n - p\|^2 \\
&\quad + \prod_{k=i+1}^{N}(1-\beta_{n,k})\|y_{n,i} - p\|^2 \\
&\le \alpha_n \tau \|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \\
&\quad + \left(\beta_{n,N} + \sum_{j=i+2}^{N}\left(\prod_{l=j}^{N}(1-\beta_{n,l})\right)\beta_{n,j-1}\right)\|x_n - p\|^2 \\
&\quad + \prod_{k=i+1}^{N}(1-\beta_{n,k})\big[\beta_{n,i}\|S_i u_n - p\|^2 \\
&\quad + (1-\beta_{n,i})\|y_{n,i-1} - p\|^2 - \beta_{n,i}(1-\beta_{n,i})\|S_i u_n - y_{n,i-1}\|^2\big]
\end{aligned}$$

$$\begin{aligned} &\le \alpha_n \tau \|y_{n,N} - p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| + \|x_n - p\|^2 \\ &\quad - \beta_{n,i} \prod_{k=i}^{N} (1 - \beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2. \end{aligned} \tag{3.34}$$

Again we obtain

$$\begin{aligned} &\beta_{n,i} \prod_{k=i}^{N} (1 - \beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2 \\ &\quad \le \alpha_n \tau \|y_{n,N} - p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \\ &\qquad + \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\quad \le \alpha_n \tau \|y_{n,N} - p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \\ &\qquad + \|x_n - y_n\| \big(\|x_n - p\| + \|y_n - p\|\big). \end{aligned}$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_{n,i} \le \limsup_{n\to\infty} \beta_{n,i} < 1$ for each $i = 1, \dots, N - 1$, and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ (due to (3.33)), it is known that

$$\lim_{n\to\infty} \|S_i u_n - y_{n,i-1}\| = 0.$$

Obviously for $i = 1$, we have $\|S_1 u_n - u_n\| \to 0$.

To conclude, we have

$$\|S_2 u_n - u_n\| \le \|S_2 u_n - y_{n,1}\| + \|y_{n,1} - u_n\| = \|S_2 u_n - y_{n,1}\| + \beta_{n,1} \|S_1 u_n - u_n\|,$$

from which $\|S_2 u_n - u_n\| \to 0$. Thus by induction $\|S_i u_n - u_n\| \to 0$ for all $i = 2, \dots, N$ since it is enough to observe that

$$\begin{aligned} \|S_i u_n - u_n\| &\le \|S_i u_n - y_{n,i-1}\| + \|y_{n,i-1} - S_{i-1} u_n\| + \|S_{i-1} u_n - u_n\| \\ &\le \|S_i u_n - y_{n,i-1}\| + (1 - \beta_{n,i-1}) \|S_{i-1} u_n - y_{n,i-2}\| + \|S_{i-1} u_n - u_n\|. \end{aligned}$$ □

Remark 3.2 As an example, we consider $M = 1$, $N = 2$ and the sequences:

(a) $\lambda_{1,n} = \eta_1 - \frac{1}{n}$, $\forall n > \frac{1}{\eta_1}$;

(b) $\alpha_n = \frac{1}{\sqrt{n}}$, $r_n = 2 - \frac{1}{n}$, $\forall n > 1$;

(c) $\beta_n = \beta_{n,1} = \frac{1}{2} - \frac{1}{n}$, $\beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}$, $\forall n > 2$.

Then they satisfy the hypotheses on the parameter sequences in Proposition 3.6.

Proposition 3.7 *Let us suppose that $\Omega \neq \emptyset$ and $\beta_{n,i} \to \beta_i$ for all i as $n \to \infty$. Suppose there exists $k \in \{1, \dots, N\}$ such that $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1, \dots, N\}$ be the largest index such that $\beta_{n,k_0} \to 0$ as $n \to \infty$. Suppose that*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;

(ii) *if* $i \le k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;

(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0, 1)$.

Moreover, suppose that (H0), (H7), *and* (H8) *hold. Then,* $\lim_{n\to\infty} \|S_i u_n - u_n\| = 0$ *for each* $i = 1, \dots, N$ *provided* $\|Ty_n - y_n\| \to 0$ *as* $n \to \infty$.

Proof First of all we note that if (H7) holds then also (H1)-(H6) are satisfied. So $\{x_n\}$ is asymptotically regular.

Let k_0 be as in the hypotheses. As in Proposition 3.6, for every index $i \in \{1, \dots, N\}$ such that $\beta_{n,i} \to \beta_i \neq 0$ (which leads to $0 < \liminf_{n\to\infty} \beta_{n,i} \le \limsup_{n\to\infty} \beta_{n,i} < 1$), one has $\|S_i u_n - y_{n,i-1}\| \to 0$ as $n \to \infty$.

For all the other indices $i \le k_0$, we can prove that $\|S_i u_n - y_{n,i-1}\| \to 0$ as $n \to \infty$ in a similar manner. By the relation (due to (3.21) and (3.34))

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \beta_n \|x_n - p\|^2 + (1-\beta_n)\|\tilde{y}_n - p\|^2 \\ &\le \beta_n \|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\ &\le \beta_n \|x_n - p\|^2 + (1-\beta_n) \\ &\quad \times \Bigg[\alpha_n \tau \|y_{n,N} - p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \\ &\quad + \|x_n - p\|^2 - \beta_{n,i} \prod_{k=i}^{N} (1-\beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2 \Bigg] \\ &\le \|x_n - p\|^2 + \alpha_n \tau \|y_{n,N} - p\|^2 + 2\alpha_n \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \\ &\quad - (1-\beta_n)\beta_{n,i} \prod_{k=i}^{N} (1-\beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2, \end{aligned}$$

we immediately obtain that

$$\begin{aligned} &(1-d) \prod_{k=i}^{N} (1-\beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2 \\ &\quad \le (1-\beta_n) \prod_{k=i}^{N} (1-\beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2 \\ &\quad \le \frac{\alpha_n}{\beta_{n,i}} \big[\tau \|y_{n,N} - p\|^2 + 2 \big\|(\gamma f - \mu F)p\big\| \|y_n - p\| \big] \\ &\qquad + \frac{\|x_n - x_{n+1}\|}{\beta_{n,i}} \big(\|x_n - p\| + \|x_{n+1} - p\| \big). \end{aligned}$$

By Proposition 3.4 or by hypothesis (ii) on the sequences, we have

$$\frac{\|x_n - x_{n+1}\|}{\beta_{n,i}} = \frac{\|x_n - x_{n+1}\|}{\beta_{n,k_0}} \cdot \frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0.$$

So, the conclusion follows. □

Remark 3.3 Let us consider $M = 1$, $N = 3$ and the following sequences:

(a) $\alpha_n = \frac{1}{n^{1/2}}$, $r_n = 2 - \frac{1}{n^2}$, $\forall n > 1$;

(b) $\lambda_{1,n} = \eta_1 - \frac{1}{n^2}$, $\forall n > \frac{1}{\eta_1^{1/2}}$;

(c) $\beta_{n,1} = \frac{1}{n^{1/4}}$, $\beta_n = \beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}$, $\beta_{n,3} = \frac{1}{n^{1/3}}$, $\forall n > 1$.

It is easy to see that all hypotheses (i)-(iii), (H0), (H7), and (H8) of Proposition 3.7 are satisfied.

Remark 3.4 Under the hypotheses of Proposition 3.7, analogously to Proposition 3.6, one can see that

$$\lim_{n\to\infty} \|S_i u_n - y_{n,i-1}\| = 0, \quad \forall i \in \{2, \dots, N\}.$$

Corollary 3.1 *Let us suppose that the hypotheses of either Proposition* 3.6 *or Proposition* 3.7 *are satisfied. Then* $\omega_w(x_n) = \omega_w(u_n) = \omega_w(y_{n,1})$, $\omega_s(x_n) = \omega_s(u_n) = \omega_s(y_{n,1})$, *and* $\omega_w(x_n) \subset \Omega$.

Proof By Remark 3.1, we have $\omega_w(x_n) = \omega_w(u_n)$ and $\omega_s(x_n) = \omega_s(u_n)$. Note that by Remark 3.4,

$$\lim_{n\to\infty} \|S_N u_n - y_{n,N-1}\| = 0.$$

In the meantime, it is known that

$$\lim_{n\to\infty} \|S_N u_n - u_n\| = \lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Hence we have

$$\lim_{n\to\infty} \|S_N u_n - y_n\| = 0. \tag{3.35}$$

Furthermore, it follows from (3.1) that

$$\lim_{n\to\infty} \|y_{n,N} - y_{n,N-1}\| = \lim_{n\to\infty} \beta_{n,N} \|S_N u_n - y_{n,N-1}\| = 0,$$

which, together with $\lim_{n\to\infty} \|S_N u_n - y_{n,N-1}\| = 0$, yields

$$\lim_{n\to\infty} \|S_N u_n - y_{n,N}\| = 0. \tag{3.36}$$

Combining (3.35) and (3.36), we conclude that

$$\lim_{n\to\infty} \|y_n - y_{n,N}\| = 0, \tag{3.37}$$

which, together with $\lim_{n\to\infty} \|x_n - y_n\| = 0$, leads to

$$\lim_{n\to\infty} \|x_n - y_{n,N}\| = 0. \tag{3.38}$$

Now we observe that

$$\|x_n - y_{n,1}\| \le \|x_n - u_n\| + \|y_{n,1} - u_n\| = \|x_n - u_n\| + \beta_{n,1} \|S_1 u_n - u_n\|.$$

By Propositions 3.3 and 3.6, $\|x_n - u_n\| \to 0$ and $\|S_1 u_n - u_n\| \to 0$ as $n \to \infty$, and hence

$$\lim_{n\to\infty} \|x_n - y_{n,1}\| = 0.$$

So we get $\omega_w(x_n) = \omega_w(y_{n,1})$ and $\omega_s(x_n) = \omega_s(y_{n,1})$.

Let $p \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$. Since $p \in \omega_w(u_n)$, by Proposition 3.6 and [18] (demiclosedness principle), we have $p \in \mathrm{Fix}(S_i)$ for each $i = 1, \ldots, N$, *i.e.*, $p \in \bigcap_{i=1}^N \mathrm{Fix}(S_i)$. Taking into account $p \in\in \omega_w(y_{n,N})$ (due to (3.38)) and $\|y_{n,N} - Gy_{n,N}\| \to 0$ (due to (3.32)), by [18] we know that $p \in \mathrm{Fix}(G) =: \Xi$. Also, since $p \in \omega_w(y_n)$ (due to (3.33)), in terms of $\|Ty_n - y_n\| \to 0$ and Proposition 2.1 of [19], we get $p \in \mathrm{Fix}(T)$. Moreover, by [20] and Proposition 3.3 we know that $p \in \mathrm{GMEP}(\Theta, h)$. Next we prove that $p \in \bigcap_{m=1}^M \mathrm{VI}(C, A_m)$. As a matter of fact, from (3.25) and (3.33) we know that $y_{n_i} \rightharpoonup p$ and $\Lambda_{n_i}^m y_{n_i} \rightharpoonup p$ for each $m = 1, \ldots, M$. Let

$$\widetilde{T}_m v = \begin{cases} A_m v + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases}$$

where $m \in \{1, 2, \ldots, M\}$. Let $(v, u) \in G(\widetilde{T}_m)$. Since $u - A_m v \in N_C v$ and $\Lambda_n^m y_n \in C$, we have

$$\langle v - \Lambda_n^m y_n, u - A_m v \rangle \ge 0.$$

On the other hand, from $\Lambda_n^m y_n = P_C(I - \lambda_{m,n} A_m) \Lambda_n^{m-1} y_n$ and $v \in C$, we have

$$\langle v - \Lambda_n^m y_n, \Lambda_n^m y_n - (\Lambda_n^{m-1} y_n - \lambda_{m,n} A_m \Lambda_n^{m-1} y_n) \rangle \ge 0,$$

and hence

$$\left\langle v - \Lambda_n^m y_n, \frac{\Lambda_n^m y_n - \Lambda_n^{m-1} y_n}{\lambda_{m,n}} + A_m \Lambda_n^{m-1} y_n \right\rangle \ge 0.$$

Therefore, we have

$$\begin{aligned}
&\langle v - \Lambda_{n_i}^m y_{n_i}, u \rangle \\
&\quad \ge \langle v - \Lambda_{n_i}^m y_{n_i}, A_m v \rangle \\
&\quad \ge \langle v - \Lambda_{n_i}^m y_{n_i}, A_m v \rangle - \left\langle v - \Lambda_{n_i}^m y_{n_i}, \frac{\Lambda_{n_i}^m y_{n_i} - \Lambda_{n_i}^{m-1} y_{n_i}}{\lambda_{m,n_i}} + A_m \Lambda_{n_i}^{m-1} y_{n_i} \right\rangle \\
&\quad = \langle v - \Lambda_{n_i}^m y_{n_i}, A_m v - A_m \Lambda_{n_i}^m y_{n_i} \rangle + \langle v - \Lambda_{n_i}^m y_{n_i}, A_m \Lambda_{n_i}^m y_{n_i} - A_m \Lambda_{n_i}^{m-1} y_{n_i} \rangle \\
&\qquad - \left\langle v - \Lambda_{n_i}^m y_{n_i}, \frac{\Lambda_{n_i}^m y_{n_i} - \Lambda_{n_i}^{m-1} y_{n_i}}{\lambda_{m,n_i}} \right\rangle \\
&\quad \ge \langle v - \Lambda_{n_i}^m y_{n_i}, A_m \Lambda_{n_i}^m y_{n_i} - A_m \Lambda_{n_i}^{m-1} y_{n_i} \rangle - \left\langle v - \Lambda_{n_i}^m y_{n_i}, \frac{\Lambda_{n_i}^m y_{n_i} - \Lambda_{n_i}^{m-1} y_{n_i}}{\lambda_{m,n_i}} \right\rangle.
\end{aligned}$$

From (3.25) and since A_m is Lipschitz continuous, we obtain

$$\lim_{n \to \infty} \|A_m \Lambda_n^m y_n - A_m \Lambda_n^{m-1} y_n\| = 0.$$

From $\Lambda_{n_i}^m y_{n_i} \rightharpoonup p$, $\{\lambda_{m,n}\} \subset [a_m, b_m] \subset (0, 2\eta_m)$, $\forall m \in \{1, 2, \ldots, M\}$ and (3.25), we have

$$\langle v - p, u \rangle \ge 0.$$

Since $\widetilde{T}_m$ is maximal monotone, we have $p \in \widetilde{T}_m^{-1}0$ and hence $p \in \mathrm{VI}(C, A_m)$, $m = 1, 2, \dots, M$, which implies $p \in \bigcap_{m=1}^{M} \mathrm{VI}(C, A_m)$. Consequently, it is known that $p \in \mathrm{Fix}(T) \cap \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \bigcap_{m=1}^{M} \mathrm{VI}(C, A_m) \cap \mathrm{GMEP}(\Theta, h) \cap \Xi =: \Omega$. □

Theorem 3.1 *Let us suppose that $\Omega \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $i = 1, \dots, N$, be sequences in $(0,1)$ such that $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ for each index i. Moreover, let us suppose that (H0)-(H6) hold. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ defined by scheme (3.1), all converge strongly to $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ if and only if $\|y_n - Ty_n\| \to 0$ as $n \to \infty$, where $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ is the unique solution of the hierarchical VIP*

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \Omega. \tag{3.39}$$

Proof First of all, we note that $F : C \to H$ is η-strongly monotone and κ-Lipschitzian on C and $f : C \to C$ is an l-Lipschitz continuous mapping with $0 \leq \gamma l < \tau$. Observe that

$$\mu\eta \geq \tau \quad \Leftrightarrow \quad \kappa \geq \eta.$$

It is clear that

$$\langle (\mu F - \gamma f)x - (\mu F - \gamma f)y, x - y \rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x, y \in C.$$

Hence we deduce that $\mu F - \gamma f$ is $(\mu\eta - \gamma l)$-strongly monotone. In the meantime, it is easy to see that $\mu F - \gamma f$ is $(\mu\kappa + \gamma l)$-Lipschitz continuous with constant $\mu\kappa + \gamma l > 0$. Thus, there exists a unique solution x^* in Ω to the VIP (3.39).

Now, observe that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - \mu F)x^*, x_n - x^* \rangle = \lim_{i\to\infty} \langle (\gamma f - \mu F)x^*, x_{n_i} - x^* \rangle. \tag{3.40}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some $p \in H$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup p$. Then by Corollary 3.1, we get $p \in \omega_w(x_n) \subset \Omega$. Hence, from (3.39) and (3.40), we have

$$\limsup_{n\to\infty} \langle (\gamma f - \mu F)x^*, x_n - x^* \rangle = \langle (\gamma f - \mu F)x^*, p - x^* \rangle \leq 0. \tag{3.41}$$

Since (H1)-(H6) hold, the sequence $\{x_n\}$ is asymptotically regular (according to Proposition 3.2). In terms of (3.33) and Proposition 3.3, $\|x_n - y_n\| \to 0$ and $\|x_n - u_n\| \to 0$ as $n \to \infty$.

Let us show that $\|x_n - x^*\| \to 0$ as $n \to \infty$. Indeed, putting $p = x^*$, we deduce from (3.3), (3.4), (3.20), and (3.21) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n)\|\tilde{y}_n - x^*\|^2 \\
&\quad \leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n)\|y_n - x^*\|^2 \\
&\quad \leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n)\bigg[\alpha_n \tau \frac{(\gamma l)^2}{\tau^2} \|y_{n,N} - x^*\|^2 + (1 - \alpha_n \tau)\|z_n - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\alpha_n\big\langle(\gamma f - \mu F)x^*, y_n - x^*\big\rangle\Big] \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\bigg[\alpha_n\frac{(\gamma l)^2}{\tau}\|x_n - x^*\|^2 + (1-\alpha_n\tau)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\big\langle(\gamma f - \mu F)x^*, y_n - x^*\big\rangle\bigg] \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\bigg[\bigg(1 - \alpha_n\frac{\tau^2 - (\gamma l)^2}{\tau}\bigg)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\big\langle(\gamma f - \mu F)x^*, y_n - x^*\big\rangle\bigg] \\
&= \bigg(1 - \alpha_n(1-\beta_n)\frac{\tau^2 - (\gamma l)^2}{\tau}\bigg)\|x_n - x^*\|^2 + 2\alpha_n(1-\beta_n)\big\langle(\gamma f - \mu F)x^*, y_n - x^*\big\rangle \\
&\le \bigg(1 - \alpha_n(1-\beta_n)\frac{\tau^2 - (\gamma l)^2}{\tau}\bigg)\|x_n - x^*\|^2 \\
&\quad + \alpha_n(1-\beta_n)\frac{\tau^2 - (\gamma l)^2}{\tau}\cdot\frac{2\tau}{\tau^2 - (\gamma l)^2}\big\langle(\gamma f - \mu F)x^*, y_n - x^*\big\rangle. \qquad (3.42)
\end{aligned}
$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\{\beta_n\} \subset [c,d] \subset (0,1)$ and $\|x_n - y_n\| \to 0$, we obtain $\sum_{n=0}^{\infty}\alpha_n(1 - \beta_n)\frac{\tau^2-(\gamma l)^2}{\tau} \ge \sum_{n=0}^{\infty}\alpha_n(1-d)\frac{\tau^2-(\gamma l)^2}{\tau} = \infty$ and

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{2\tau}{\tau^2 - (\gamma l)^2}\big\langle(\gamma f - \mu F)x^*, y_n - x^*\big\rangle \\
&\quad = \limsup_{n\to\infty}\frac{2\tau}{\tau^2 - (\gamma l)^2}\big(\big\langle(\gamma f - \mu F)x^*, x_n - x^*\big\rangle + \big\langle(\gamma f - \mu F)x^*, y_n - x_n\big\rangle\big) \\
&\quad = \limsup_{n\to\infty}\frac{2\tau}{\tau^2 - (\gamma l)^2}\big\langle(\gamma f - \mu F)x^*, x_n - x^*\big\rangle \le 0
\end{aligned}
$$

(due to (3.41)). Applying Lemma 2.1 of [16] to (3.42), we infer that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

In a similar way, we can conclude another theorem as follows.

Theorem 3.2 *Let us suppose that* $\Omega \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $i = 1,\dots,N$, *be sequences in* $(0,1)$ *such that* $\beta_{n,i} \to \beta_i$ *for each index* i *as* $n \to \infty$. *Suppose that there exists* $k \in \{1,\dots,N\}$ *for which* $\beta_{n,k} \to 0$ *as* $n \to \infty$. *Let* $k_0 \in \{1,\dots,N\}$ *the largest index for which* $\beta_{n,k_0} \to 0$. *Moreover, let us suppose that* (H0), (H7), *and* (H8) *hold and*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;

(ii) *if* $i \le k_0$ *and* $\beta_{n,i} \to \beta_i$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;

(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0,1)$.

Then the sequences $\{x_n\}$, $\{y_n\}$, *and* $\{u_n\}$ *defined by scheme* (3.1) *all converge strongly to* $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ *if and only if* $\|y_n - Ty_n\| \to 0$ *as* $n \to \infty$, *where* $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ *is the unique solution of the hierarchical VIP*

$$\big\langle(\gamma f - \mu F)x^*, x - x^*\big\rangle \le 0, \quad \forall x \in \Omega.$$

Remark 3.5 According to the above argument for Theorems 3.1 and 3.2, we can readily see that if, in scheme (3.1), the iterative step $y_n = P_C[\alpha_n\gamma f(y_{n,N}) + (I - \alpha_n\mu F)Gy_{n,N}]$ is replaced by the iterative one, $y_n = P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)Gy_{n,N}]$, then Theorems 3.1 and 3.2 remain valid.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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