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Clark's fixed point theorem on a complete random normed module

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Abstract

Motivated by the recent results in random metric theory, we establish Clark's fixed point theorem on complete random normed modules under two kinds of topologies. When the base space of the random normed module is trivial, our results automatically degenerate to the classical case.

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1 Introduction

Random metric theory is based on the idea of randomizing the classical space theory of functional analysis. All the basic notions such as random normed modules, random inner product modules and random locally convex modules, together with their random conjugate spaces, were naturally presented by Guo in the course of the development of random functional analysis [1–4]. In the last ten years, random metric theory and its applications in the theory of conditional risk measures have undergone a systematic and deep development [5–13]. Especially after 2009, in [5] Guo gives the relations between the basic results currently available derived from two kinds of topologies, namely the (ε,λ)-topology and the locally L^0-convex topology. In [6], Guo gives some basic results on L^0-convex analysis together with some applications to conditional risk measures and studies the relations among three kinds of conditional convex risk measures. These results pave the way for further research of the random metric theory and conditional convex risk measures.

In 1978, Clark presented Clark's fixed pointed theorem [14]. It has been applied in many fields such as optimization theory, different equations, and fixed point theory. Based on the recent work of random metric theory, in this paper, we establish Clark's fixed pointed theorem on complete random normed modules under two kinds of topologies. A random normed module is a random generalization of an ordinary normed space. Different from ordinary normed spaces, random normed modules possess the rich stratification structure, which is introduced in this paper. It is this kind of rich stratification structure that makes the theory of random normed modules deeply developed and also becomes the most useful part of random metric theory. When the probability $(\Omega,\mathcal{F},P)$ is trivial, namely $\mathcal{F}=\{\emptyset,\Omega\}$, our results reduce to the classical Clark's fixed pointed theorem. So the extension of our results is nontrivial.

The reminder of this article is organized as follows. In Section 2 we briefly recall some necessary notions and facts. In Section 3 we present and prove our main results.

2 Preliminary

Throughout this paper, $(\Omega, \mathcal{F}, P)$ denotes a probability space, K the real number field R or the complex number field C, N the set of positive integers, $\bar{L}^0(\mathcal{F})$ the set of equivalence classes of extended real-valued random variables on Ω and $L^0(\mathcal{F}, K)$ the algebra of equivalence classes of K-valued $\mathcal{F}$-measurable random variables on Ω under the ordinary scalar multiplication, addition and multiplication operations on equivalence classes, denoted by $L^0(\mathcal{F})$ when $K = R$.

The pleasant properties of the complete lattice $\bar{L}^0(\mathcal{F})$ (see the introduction for the notation $\bar{L}^0(\mathcal{F})$) are summarized as follows.

Proposition 2.1 ([15]) *For every subset A of $\bar{L}^0(\mathcal{F})$, there exist countable subsets $\{a_n \mid n \in N\}$ and $\{b_n \mid n \in N\}$ of A such that $\bigvee_{n\geq 1} a_n = \vee A$ and $\bigwedge_{n\geq 1} b_n = \wedge A$. Further, if A is directed (dually directed) with respect to $\leq$, then the above $\{a_n \mid n \in N\}$ (accordingly, $\{b_n \mid n \in N\}$) can be chosen as nondecreasing (correspondingly, nonincreasing) with respect to $\leq$.*

Specially, $L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi \geq 0\}$, $L^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\}$.

As usual, $\xi > \eta$ means $\xi \geq \eta$ and $\xi \neq \eta$, whereas $\xi > \eta$ on A means $\xi^0(\omega) > \eta^0(\omega)$ a.s. on A for any $A \in \mathcal{F}$ and ξ and η in $\bar{L}^0(\mathcal{F})$, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively.

For any $A \in \mathcal{F}$, A^c denotes the complement of A, $\tilde{A} = \{B \in \mathcal{F} \mid P(A \Delta B) = 0\}$ denotes the equivalence class of A, where Δ is the symmetric difference operation, I_A the characteristic function of A, and $\tilde{I}_A$ is used to denote the equivalence class of I_A; given two ξ and η in $\bar{L}^0(\mathcal{F})$, and $A = \{\omega \in \Omega : \xi^0 \neq \eta^0\}$, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η respectively, then we always write $[\xi \neq \eta]$ for the equivalence class of A and $I_{[\xi\neq\eta]}$ for $\tilde{I}_A$. One can also understand the implication of such notations as $I_{[\xi\leq\eta]}$, $I_{[\xi<\eta]}$ and $I_{[\xi=\eta]}$.

For an arbitrarily chosen representative ξ^0 of $\xi \in L^0(\mathcal{F}, K)$, define two $\mathcal{F}$-measurable random variables $(\xi^0)^{-1}$ and $|\xi^0|$ by $(\xi^0)^{-1}(\omega) = \frac{1}{\xi^0(\omega)}$ if $\xi^0(\omega) \neq 0$, and $(\xi^0)^{-1}(\omega) = 0$ otherwise, and by $|\xi^0|(\omega) = |\xi^0(\omega)|$, $\forall \omega \in \Omega$. Then the equivalence class ξ^{-1} of $(\xi^0)^{-1}$ is called the generalized inverse of ξ and the equivalence class $|\xi|$ of $|\xi^0|$ is called the absolute value of ξ. It is clear that $\xi \cdot \xi^{-1} = I_{[\xi\neq 0]}$.

Definition 2.2 ([16]) An ordered pair $(E, \|\cdot\|)$ is called a random normed space (briefly, an *RN* space) over K with base $(\Omega, \mathcal{F}, P)$ if E is a linear space and $\|\cdot\|$ is a mapping from E to $L^0_+(\mathcal{F})$ such that the following three axioms are satisfied:

(1) $\|x\| = 0$ if and only if $x = \theta$ (the null vector of E);

(2) $\|\alpha x\| = |\alpha|\|x\|$, $\forall \alpha \in K$ and $x \in E$;

(3) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in E$,

where the mapping $\|\cdot\|$ is called the random norm on E and $\|x\|$ is called the random norm of a vector $x \in E$.

In addition, if E is left module over the algebra $L^0(\mathcal{F}, K)$ such that the following is also satisfied:

(4) $\|\xi x\| = |\xi|\|x\|$, $\forall \xi \in L^0(\mathcal{F}, K)$ and $x \in E$,

then such an *RN* space is called an *RN* module over K with base $(\Omega, \mathcal{F}, P)$ and such a random norm $\|\cdot\|$ is called an L^0-norm on E.

There are two important topologies in random metric theory as follows.

Proposition 2.3 ([4]) *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. For any real numbers $\varepsilon > 0$, $0 < \lambda < 1$, let $N_\theta(\varepsilon, \lambda) = \{x \in E \mid P\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\} > 1 - \lambda\}$ and $\mathcal{U}_\theta = \{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$, then $\mathcal{U}_\theta$ is a local base at θ of some Hausdorff linear topology, called the (ε, λ)-topology induced by $\|\cdot\|$. Further, we have the following statements*:

1. *$L^0(\mathcal{F}, K)$ is a topological algebra over K endowed with its (ε, λ)-topology, which is exactly the topology of convergence in probability P*;
2. *E is a topological module over the topological algebra $L^0(\mathcal{F}, K)$ when E and $L^0(\mathcal{F}, K)$ are endowed with their respective (ε, λ)-topologies*;
3. *A net $\{x_\alpha, \alpha \in \wedge\}$ in E converges in the (ε, λ)-topology to $x \in E$ iff $\{\|x_\alpha - x\|, \alpha \in \wedge\}$ converges in probability P to* 0.

From now on, for all *RN* modules, (ε, λ)-topology is denoted by $\mathcal{T}_{\varepsilon,\lambda}$.

Proposition 2.4 ([17]) *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. For any $\varepsilon \in L^0_{++}(\mathcal{F})$, let $B(\varepsilon) = \{x \in E \mid \|x\| \leq \varepsilon\}$ and $\mathcal{U}_\theta = \{B(\varepsilon) \mid \varepsilon \in L^0_{++}(\mathcal{F})\}$. A set $G \subset E$ is called $\mathcal{T}_c$-open if for every $x \in G$ there exists some $B(\varepsilon) \in \mathcal{U}_\theta$ such that $x + B(\varepsilon) \subset G$. Let $\mathcal{T}_c$ be the family of $\mathcal{T}_c$-open subsets, then $\mathcal{T}_c$ is a Hausdorff topology on E, called the locally L^0-convex topology induced by $\|\cdot\|$. Further, the following statements are true*:

1. *$L^0(\mathcal{F}, K)$ is a topological ring endowed with its locally L^0-convex topology*;
2. *E is a topological module over the topological ring $L^0(\mathcal{F}, K)$ when E and $L^0(\mathcal{F}, K)$ are endowed with their respective locally L^0-convex topologies*;
3. *A net $\{x_\alpha, \alpha \in \wedge\}$ in E converges in the locally L^0-convex topology to $x \in E$ iff $\{\|x_\alpha - x\|, \alpha \in \wedge\}$ converges in the locally L^0-convex topology of $L^0(\mathcal{F}, K)$ to θ.*

From now on, for all *RN* modules, locally L^0-convex topology is denoted by $\mathcal{T}_c$. Since $\mathcal{T}_c$ is not necessarily a linear topology as proved in [4], but $(E, \mathcal{T}_c)$ is always a topological group with respect to the addition operation for any *RN* module $(E, \|\cdot\|)$, and hence $\mathcal{T}_c$-Cauchy nets and $\mathcal{T}_c$-completeness are still well defined.

Let $(E, \|\cdot\|)$ be an *RN* module over K with base $(\Omega, \mathcal{F}, P)$, $p_A = \tilde{I}_A \cdot p$ is called the A-stratification of p for each given $A \in \mathcal{F}$ and p in E. The so-called stratification structure of E means that E includes every stratification of an element in E. Clearly, $p_A = \theta$ when $P(A) = 0$ and $p_A = p$ when $P(\Omega \setminus A) = 0$, which are both called trivial stratifications of p. Further, when $(\Omega, \mathcal{F}, P)$ is a trivial probability space, every element in E has merely two trivial stratifications since $\mathcal{F} = \{\Omega, \emptyset\}$; when $(\Omega, \mathcal{F}, P)$ is arbitrary, every element in E can possess arbitrarily many nontrivial intermediate stratifications. It is this kind of rich stratification structure of *RN* modules that makes the theory of *RN* modules deeply developed and also becomes the most useful part of random metric theory.

To introduce the main results of this paper, let us first recall the definition of the countable concatenation property as follows.

Definition 2.5 ([5]) Let E be a left module over the algebra $L^0(\mathcal{F}, K)$. A formal sum $\sum_{n\in N} \tilde{I}_{A_n} x_n$ is called a *countable concatenation* of a sequence $\{x_n \mid n \in N\}$ in E with respect to a countable partition $\{A_n \mid n \in N\}$ of Ω to $\mathcal{F}$. Moreover, a countable concatenation $\sum_{n\in N} \tilde{I}_{A_n} x_n$ is well defined or $\sum_{n\in N} \tilde{I}_{A_n} x_n \in E$ if there is $x \in E$ such that $\tilde{I}_{A_n} x = \tilde{I}_{A_n} x_n$, $\forall n \in N$. A subset G of E is said to *have the countable concatenation property* if every countable concatenation $\sum_{n\in N}\tilde{I}_{A_n}x_n$ with $x_n \in G$ for each $n \in N$ still belongs to G, namely $\sum_{n\in N}\tilde{I}_{A_n}x_n$ is well defined and there exists $x \in G$ such that $x = \sum_{n\in N}\tilde{I}_{A_n}x_n$.

Definition 2.6 ([17]) Let E be a left module over the algebra $L^0(\mathcal{F})$ and f be a function from E to $\bar{L}^0(\mathcal{F})$, then

(1) f is $L^0(\mathcal{F})$-convex if $f(\xi x + (1-\xi)y) \le \xi f(x) + (1-\xi)f(y)$ for all x and y in E and $\xi \in L^0_+(\mathcal{F})$ such that $0 \le \xi \le 1$. (Here, we make the convention that $0\cdot(\pm\infty) = 0$ and $\infty - \infty = \infty$!)

(2) f is said to have the local property if $\tilde{I}_A f(x) = \tilde{I}_A f(\tilde{I}_A x)$ for all $x \in E$ and $A \in \mathcal{F}$.

Now, we introduce a kind of lower semicontinuity for $\bar{L}^0$-valued functions, which is very suitable for the study of conditional risk measures [6].

Definition 2.7 ([17]) Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$. A function $f: E \to \bar{L}^0(\mathcal{F})$ is called $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous if $\operatorname{epi}(f) := \{(x,r) \in E \times L^0(\mathcal{F}) \mid f(x) \le r\}$ is closed in $(E, \mathcal{T}_{\varepsilon,\lambda}) \times (L^0(\mathcal{F}), \mathcal{T}_{\varepsilon,\lambda})$. A function $f: E \to \bar{L}^0(\mathcal{F})$ is called $\mathcal{T}_c$-lower semicontinuous if $\operatorname{epi}(f)$ is closed in $(E, \mathcal{T}_c) \times (L^0(\mathcal{F}), \mathcal{T}_c)$.

Let E be a left module over the algebra $L^0(\mathcal{F}, K)$, a nonempty subset M of E is called $L^0(\mathcal{F})$-convex if $\xi x + \eta y \in M$ for any x and $y \in M$ and ξ and $\eta \in L^0_+(\mathcal{F})$ such that $\xi + \eta = 1$.

It is well known from [17] that $f: E \to \bar{L}^0(\mathcal{F})$ is $L^0(\mathcal{F})$-convex iff f has the local property and $\operatorname{epi}(f)$ is $L^0(\mathcal{F})$-convex.

3 Main results

The main results in this section are Theorem 3.10 and 3.13 below. To introduce them, we first give some necessary notions and terminology.

Definition 3.1 Let E be a left module over the algebra $L^0(\mathcal{F})$. A nonempty subset K of E is called a random cone of E if $\xi\cdot x \in K$, $\forall \xi \in L^0_+(\mathcal{F})$, $x \in K$.

Definition 3.2 Let K be a random cone of an $L^0(\mathcal{F})$-module E. K is called

(1) $L^0(\mathcal{F})$-convex if $x_1 + x_2 \in K$, $\forall x_1, x_2 \in K$;

(2) pointed if $x \in K$, $-x \in K \Rightarrow x = 0$.

Lemma 3.3 *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $\alpha \in L^0_{++}(\mathcal{F})$ with $0 < \alpha < 1$. Define $K_\alpha := \{(x,r) \in E \times L^0(\mathcal{F}) : \alpha\|x\| \le -r\}$. Then K_α is a $\mathcal{T}_{\varepsilon,\lambda}$-closed $L^0(\mathcal{F})$-convex random cone.*

Further, if E has the countable concatenation property, then K_α has the countable concatenation property.

Proof It is easy to see that K_α is $L^0(\mathcal{F})$-convex and $\mathcal{T}_{\varepsilon,\lambda}$-closed.

Let $\{(x_n, r_n), n \in N\}$ be in K_α, namely $\alpha\|x_n\| \le -r_n$, $\forall n \ge 1$. For any countable partition $\{A_n, n \ge 1\}$ of Ω to $\mathcal{F}$, it follows that $\alpha\|\sum_{n=1}^k \tilde{I}_{A_n}\cdot x_n\| \le \alpha\cdot\sum_{n=1}^k \|\tilde{I}_{A_n}\cdot x_n\| \le \sum_{n=1}^k(-\tilde{I}_{A_n}\cdot r_n)$. By the countable concatenation properties of E and $L^0(\mathcal{F})$, one can have $\alpha\|\sum_{n=1}^\infty \tilde{I}_{A_n}\cdot x_n\| \le \sum_{n=1}^\infty(-\tilde{I}_{A_n}\cdot r_n)$, namely $\sum_{n=1}^\infty \tilde{I}_{A_n}(x_n, r_n) \in K_\alpha$. □

Remark 3.4 If K is a pointed and $L^0(\mathcal{F})$- convex random cone of an $L^0(\mathcal{F})$-module E, then we can define a partial ordering '$\le_K$', namely $x \le_K y \Leftrightarrow y - x \in K$. And the partial ordering satisfies:

(1) $x \leq_K y \Leftrightarrow \lambda x \leq_K \lambda y$, $\forall \lambda \in L^0_+(\mathcal{F})$;
(2) $x \leq_K y \Rightarrow x + z \leq_K y + z$, $\forall x, y, z \in E$.

In classical metric spaces, the following result Lemma 3.5 is clear. But when we come to *RN* modules, it is not easy to be proved.

Lemma 3.5 ([9]) *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$, $G \subset E$ be a subset with the countable concatenation property and $f : E \to \bar{L}^0(\mathcal{F})$ have the local property. If $f|_G$ is proper and bounded from below on G (resp., bounded from above on G), then, for each $\varepsilon \in L^0_{++}(\mathcal{F})$, there exists $x_\varepsilon \in G$ such that $f(x_\varepsilon) \leq \wedge f(G) + \varepsilon$ (accordingly, $f(x_\varepsilon) \geq \vee f(G) - \varepsilon$).*

Lemma 3.6 *Let $(E, \|\cdot\|)$ be a $\mathcal{T}_{\varepsilon,\lambda}$-complete RN module over R with base $(\Omega, \mathcal{F}, P)$, $A \subset E \times L^0(\mathcal{F})$ be a nonempty $\mathcal{T}_{\varepsilon,\lambda}$-closed subset such that A has the countable concatenation property. If $\alpha \in L^0_{++}(\mathcal{F})$ with $0 < \alpha < 1$ and $\bigwedge\{r \in L^0(\mathcal{F}) : (x, r) \in A\} = 0$, then for each $(x_0, r_0) \in A$, there exists $(\bar{x}, \bar{r}) \in A$ such that*:
(1) $(\bar{x}, \bar{r}) \in A \cap [K_\alpha + (x_0, r_0)]$;
(2) $(\bar{x}, \bar{r}) = A \cap [K_\alpha + (\bar{x}, \bar{r})]$.

Proof Define a function $f : E \times L^0(\mathcal{F}) \to L^0(\mathcal{F})$ by $f(x, r) = r$, $\forall (x, r) \in E \times L^0(\mathcal{F})$. And define a mapping $\|\cdot\|_* : E \times L^0(\mathcal{F}) \to L^0_+(\mathcal{F})$ by $\|x\|_* = \|\xi\| \vee |r|$, $\forall x = (\xi, r) \in E \times L^0(\mathcal{F})$.

It is easy to check that $(E \times L^0(\mathcal{F}), \|\cdot\|_*)$ is an RN space.

Furthermore, $E \times L^0(\mathcal{F})$ is an RN module if we define a module multiplication $\cdot : L^0(\mathcal{F}) \cdot (E \times L^0(\mathcal{F})) \to E \times L^0(\mathcal{F})$ by $\gamma \cdot (\xi, r) = (\gamma\xi, \gamma r)$, $\forall \gamma \in L^0(\mathcal{F})$, $\forall (\xi, r) \in E \times L^0(\mathcal{F})$.

For each $x = (\xi, r) \in E \times L^0(\mathcal{F})$, it follows that $I_B \cdot f(I_B \cdot x) = I_B \cdot f(I_B \cdot (\xi, r)) = I_B \cdot f(I_B \cdot \xi, I_B \cdot r) = I_B \cdot (I_B \cdot r) = I_B \cdot r = I_B \cdot f(x)$, $\forall B \in \mathcal{F}$, which means f has the local property.

Since E has the countable concatenation property, by Lemma 3.3 K_α has the countable concatenation property. Further, since A has the countable concatenation property, it implies that $A \cap [K_\alpha + (x, r)]$, $\forall (x, r) \in E \times L^0(\mathcal{F})$ has the countable concatenation property.

Obviously, $\bigwedge\{r \in L^0(\mathcal{F}) : (x, r) \in A\} = 0$ implies that f is bounded from below on A.

Lemma 3.5 yields a sequence $\{(x_n, r_n) : n \geq 0\}$ such that $(x_{n+1}, r_{n+1}) \in A_n = A \cap [K_\alpha + (x_n, r_n)]$ and $r_{n+1} = f(x_{n+1}, r_{n+1}) < \wedge f(A_n) + \frac{1}{n+1}$, $\forall n \geq 0$.

One can have $A_{n+1} \subset A_n$, $\forall n \geq 0$, which follows from $K_\alpha + (x_{n+1}, r_{n+1}) \subset K_\alpha + [K_\alpha + (x_n, r_n)] = K_\alpha + (x_n, r_n)$.

We now prove that $\{x_n, n \geq 0\}$ and $\{r_n, n \geq 0\}$ are $\mathcal{T}_{\varepsilon,\lambda}$-Cauchy sequences of E and $L^0(\mathcal{F})$, respectively.

For each $(y, s) \in A_n$, it is easy to see $s \geq \wedge f(A_n)$ and $(y, s) \in K_\alpha + (x_n, r_n)$, namely $\alpha\|y - x_n\| \leq r_n - s$. Thus one can have $\alpha\|y - x_n\| \leq r_n - s \leq \wedge f(A_{n-1}) + \frac{1}{n} - s \leq \wedge f(A_n) + \frac{1}{n} - s \leq \frac{1}{n}$, $\forall (x_n, r_n) \in A_{n-1}$, $(y, s) \in A_n$. Then it follows that $\alpha\|y_1 - y_2\| \leq \alpha\|y_1 - x_n\| + \alpha\|y_2 - x_n\| \leq \frac{2}{n}$ and $|s_2 - s_1| \leq |s_2 - r_n| + |r_n - s_1| \leq \frac{2}{n}$, $\forall (y_1, s_1), (y_2, s_2) \in A_n$. Hence $\operatorname{diam}(A_n) := \bigvee_{(y_1,s_1),(y_2,s_2) \in A_n} \|(y_1, s_1) - (y_2, s_2)\|_* = \bigvee_{(y_1,s_1),(y_2,s_2) \in A_n} (\|y_1 - y_2\| \vee |s_1 - s_2|) \leq \frac{2}{n\alpha} \to 0$ in the (ε, λ)-topology as $n \to \infty$. Since for each $\varepsilon \in R$ with $\varepsilon > 0$ and each $\lambda \in R$ with $0 < \lambda < 1$, there exists N such that $P\{w : \frac{2}{n\alpha}(w) \leq \varepsilon\} > 1 - \lambda$, $\forall n \geq N$, then $\|(x_n, r_n) - (x_m, r_m)\|_* = \|x_n - x_m\| \vee |r_n - r_m| \leq \operatorname{diam}(A_N) \leq \frac{2}{N\alpha}$, $\forall m, n \geq N$. Therefore $P\{w : \|x_n - x_m\|(w) \leq \varepsilon\} \geq P\{w : \frac{2}{N\alpha}(w) \leq \varepsilon\} > 1 - \lambda$, $\forall m, n \geq N$, which verifies that $\{x_n, n \geq 0\}$ is a $\mathcal{T}_{\varepsilon,\lambda}$-Cauchy sequence of E. Similarly, $\{r_n, n \geq 0\}$ is a $\mathcal{T}_{\varepsilon,\lambda}$-Cauchy sequence of $L^0(\mathcal{F})$.

Since E and $L^0(\mathcal{F})$ are both $\mathcal{T}_{\varepsilon,\lambda}$-complete, there exist $\bar{x} \in E$ and $\bar{r} \in L^0(\mathcal{F})$ such that $\{x_n : n \in N\}$ converges in the (ε, λ)-topology to $\bar{x}$ and $\{r_n : n \in N\}$ converges in the (ε, λ)-

topology to $\bar{r}$, which implies $\{(x_n, r_n) : n \in N\}$ converges in the (ε, λ)-topology to $(\bar{x}, \bar{r})$. Both A and K_α are $\mathcal{T}_{\varepsilon,\lambda}$-closed, so $A_n = A \cap [K_\alpha + (x_n, r_n)]$, $\forall n \geq 0$ is also $\mathcal{T}_{\varepsilon,\lambda}$-closed. From $A_{n+1} \subset A_n$, $\forall n \geq 0$, one can have $(\bar{x}, \bar{r}) \in \bigcap_{n=0}^{\infty} A_n$.

For each $(\hat{x}, \hat{r}) \in \bigcap_{n=0}^{\infty} A_n$, $(\hat{x}, \hat{r}) \in A_n$ implies $\|\hat{x} - x_n\| \leq \frac{1}{\alpha n}$, $\forall (x_n, r_n) \in A_{n-1}$. Then it follows that $\{x_n : n \in N\}$ converges in the (ε, λ)-topology to $\hat{x}$. Similarly, we can prove that $\{r_n : n \in N\}$ converges in the (ε, λ)-topology to $\hat{r}$. Since $\mathcal{T}_{\varepsilon,\lambda}$ is Hausdorff on E and $L^0(\mathcal{F})$, one can have $(\bar{x}, \bar{r}) = (\hat{x}, \hat{r})$. Thus we have $\{(\bar{x}, \bar{r})\} = \bigcap_{n=0}^{\infty} A_n$.

From $(\bar{x}, \bar{r}) \in A_0$, (1) is proved.

If $(y, s) \in A \cap [K_\alpha + (\bar{x}, \bar{r})]$, then $(y, s) \in \bigcap_{n=0}^{\infty} A_n$, which follows from $K_\alpha + (\bar{x}, \bar{r}) \subset K_\alpha + [K_\alpha + (x_n, r_n)] = K_\alpha + (x_n, r_n)$, $\forall n \geq 0$. Hence $(y, s) = (\bar{x}, \bar{r})$, namely $(\bar{x}, \bar{r}) = A \cap [K_\alpha + (\bar{x}, \bar{r})]$. Thus (2) is proved. □

To prove Theorem 3.10 below, we need Lemma 3.7, which is very easy and thus its proof is omitted.

Lemma 3.7 *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property, and let $f : E \to L^0(\mathcal{F})$ have the local property. Then* epi(f) *has the countable concatenation property.*

Theorem 3.8 *Let $(E, \|\cdot\|)$ be a $\mathcal{T}_{\varepsilon,\lambda}$-complete RN module over R with base $(\Omega, \mathcal{F}, P)$, G be a $\mathcal{T}_{\varepsilon,\lambda}$-closed subset of E, $\varepsilon \in L^0_{++}(\mathcal{F})$ and $\varphi : G \to \bar{L}^0(\mathcal{F})$ be proper, $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous and bounded from below on G. Then for each point $x_0 \in G$ satisfying $\varphi(x_0) \leq \wedge\varphi(G) + \varepsilon$ and each $\alpha \in L^0_{++}(\mathcal{F})$, there exists $z \in G$ such that the following are satisfied*:

(1) $\varphi(z) \leq \varphi(x_0) - \alpha\|z - x_0\|$;

(2) $\|z - x_0\| \leq \alpha^{-1} \cdot \varepsilon$;

(3) *for each $x \in G$ such that $x \neq z$, $\varphi(x) \nleq \varphi(z) - \alpha\|x - z\|$.*

Proof We can, without loss of generality, suppose $\bigwedge\{\varphi(x), x \in E\} = 0$. Then $\varphi(x_0) \leq \varepsilon$, $\forall \varepsilon \in L^0_{++}(\mathcal{F})$.

Since φ is $T_{\varepsilon,\lambda}$-l.s.c , it follows that epi(φ) is closed in $(E, T_{\varepsilon,\lambda}) \times (L^0(\mathcal{F}), T_{\varepsilon,\lambda})$.

Take $A = \text{epi}(\varphi)$ and $(x_0, r_0) = (x_0, \varphi(x_0)) \in A$.

By Lemma 3.7, A has the countable concatenation property.

According to Lemma 3.6, there exists $(z, r) \in A$ such that:

(a) $(z, r) \in A \cap [K_\alpha + (x_0, \varphi(x_0))]$;

(b) $(z, r) = A \cap [K_\alpha + (z, r)]$.

From (a), one can have $\alpha\|z - x_0\| \leq \varphi(x_0) - r \leq \varphi(x_0) - \varphi(z) \leq \varphi(x_0) \leq \varepsilon$, which yields (1) and (2).

Now, we prove (3) as follows.

We can deduce $\varphi(z) = r$. Otherwise, one can have $(z, r) \neq (z, \varphi(z))$. By (b), we have $(z, \varphi(z)) \bar{\in} K_\alpha + (z, r)$, namely $0 \leq r - \varphi(z)$ does not hold. That is in contradiction to $(z, r) \in A$.

(3) is obvious, when $\varphi(x) = \infty$. If $\varphi(x) < \infty$ and $x \neq z$, then $(x, \varphi(x)) \neq (z, \varphi(z)) = (z, r)$. From (b), one can have $(x, \varphi(x)) \bar{\in} K_\alpha + (z, \varphi(z))$, namely $\alpha\|x - z\| \nleq \varphi(z) - \varphi(x)$. □

Theorem 3.9 *Let $(E, \|\cdot\|)$ be a $\mathcal{T}_{\varepsilon,\lambda}$-complete RN module over R with base $(\Omega, \mathcal{F}, P)$, $\phi : E \to \bar{L}^0(\mathcal{F})$ be a proper $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous function which is bounded from below, and $T : E \to E$ be a mapping such that $\phi(Tu) + \|Tu - u\| \leq \phi(u)$, $\forall u \in E$. Then T has a fixed point.*

Proof It follows from Theorem 3.8 that there exists some point $z \in E$, for every $x \neq z$, there exists $A_x \in \mathcal{F}$ such that $P(A_x) > 0$ and $\alpha \cdot \|x - z\| > \psi(z) - \psi(x)$ on A_x. We deduce that $Tz = z$. If $Tz \neq z$ holds, it follows from Theorem 3.8 that there exists $A_{Tz} \in \mathcal{F}$ such that $P(A_{Tz}) > 0$ and $\alpha \cdot \|Tz - z\| > \psi(z) - \psi(Tz)$ on A_{Tz}, which is in contradiction with $\alpha \cdot \|u - Tu\| \leq \psi(u) - \psi(Tu), \forall u \in E$. □

In 1978, Clark presented Clark's fixed point theorem [14], which means that in a complete metric space, 'directional contraction' admits a fixed point. Now we establish random version of Clark's fixed point theorem on a $\mathcal{T}_{\varepsilon,\lambda}$-complete *RN* module, namely Theorem 3.10 below.

Theorem 3.10 *Let $(E, \|\cdot\|)$ be a $\mathcal{T}_{\varepsilon,\lambda}$-complete RN module over R with base $(\Omega, \mathcal{F}, P)$, $\lambda \in L^0_{++}(\mathcal{F})$ with $0 < \lambda < 1$ on Ω and $f : E \to E$ be a $\mathcal{T}_{\varepsilon,\lambda}$-continuous function with the local property. If for each $v \in E$, there exists $x_0 \in E$ satisfying $x_0 \neq v$ and*

(1) $\|v - x_0\| + \|f(v) - x_0\| = \|v - f(v)\|$;

(2) $\|f(v) - f(x_0)\| \leq \lambda \|v - x_0\|$.

Then f has a fixed point.

Proof Define a function $g : E \to E$ by $g(v) = x_0$, when $f(v) \neq v$ and $g(v) = v$, when $f(v) = v$. It is obvious that f and g have the same fixed points.

Define a function $\phi : E \to L^0(\mathcal{F})$ by $\phi(v) = (1-\lambda)^{-1} \cdot \|v - f(v)\|$. Since f is $\mathcal{T}_{\varepsilon,\lambda}$-continuous, ϕ is $\mathcal{T}_{\varepsilon,\lambda}$-continuous. Thus we have ϕ is $\mathcal{T}_{\varepsilon,\lambda}$-l.s.c. From the local property of f, it follows that $I_A \cdot \phi(I_A \cdot v) = I_A \cdot (1-\lambda)^{-1} \cdot \|I_A v - f(I_A v)\| = (1-\lambda)^{-1} \cdot \|I_A v - I_A f(v)\| = I_A \cdot \phi(v), \forall v \in E$, $A \in \mathcal{F}$, which means ϕ has the local property. Clearly, 0 is the lower bound of ϕ.

In order to prove that g has a fixed point, we only need to prove $\|v - g(v)\| \leq \phi(v) - \phi(g(v))$ by Theorem 3.9.

If $v = g(v)$, it is obvious that $\|v - g(v)\| \leq \phi(v) - \phi(g(v))$.

If $v \neq g(v)$, then $g(v) = x_0$. By (1) and (2), one can have $0 \leq \lambda\|v - x_0\| - \|f(v) - f(x_0)\| \leq \lambda\|v - x_0\| - \|f(x_0) - x_0\| + \|x_0 - f(v)\| \leq (\lambda - 1)\|v - x_0\| - \|f(x_0) - x_0\| + \|v - f(v)\|$, which means $\|v - g(v)\| \leq \phi(v) - \phi(g(v))$. □

To obtain Clark's fixed point theorem under the locally L^0-convex topology, we need the following key results obtained in [5, 6].

Proposition 3.11 ([5]) *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then E is $\mathcal{T}_{\varepsilon,\lambda}$-complete if and only if E is $\mathcal{T}_c$-complete and has the countable concatenation property.*

Proposition 3.12 ([6]) *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property, let $f : E \to \bar{L}^0(\mathcal{F})$ be a function with the local property. Then f is $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous iff f is $\mathcal{T}_c$-lower semicontinuous.*

From both the relations of completeness of a random normed module and lower semicontinuity of a function under $\mathcal{T}_c$ and $\mathcal{T}_{\varepsilon,\lambda}$, we can obtain Clark's fixed point theorem under the topology $\mathcal{T}_c$ as follows.

Theorem 3.13 *Let $(E, \|\cdot\|)$ be a $\mathcal{T}_c$-complete RN module over R with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property, $\lambda \in L^0_{++}(\mathcal{F})$ with $0 < \lambda < 1$ on Ω and f :*

$E \to E$ be a $\mathcal{T}_c$-continuous function with the local property. If for each $v \in E$, there exists $x_0 \in E$ satisfying $x_0 \neq v$ and

(1) $\|v - x_0\| + \|f(v) - x_0\| = \|v - f(v)\|$;

(2) $\|f(v) - f(x_0)\| \leq \lambda\|v - x_0\|$.

Then f has a fixed point.

Remark 3.14 From the results above, it is easy to see that Clark's fixed point theorem on complete RN modules is essentially independent of a special choice of $\mathcal{T}_c$ and $\mathcal{T}_{\varepsilon,\lambda}$. It is an algebra result.

When the base space $(\Omega, \mathcal{F}, P)$ of the *RN* module is trivial, namely $\mathcal{F} = \{\emptyset, \Omega\}$, our result automatically degenerates to the classical Daneš theorem. So our result is a nontrivial random extension.

Competing interests
The author declares that they have no competing interests.

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References

1. Guo, TX, Xiao, HX, Chen, XX: A basic strict separation theorem in random locally convex modules. Nonlinear Anal. **71**, 3794-3804 (2009)
2. Guo, TX, Li, SB: The James theorem in complete random normed modules. J. Math. Anal. Appl. **308**, 257-265 (2005)
3. Guo, TX: The Radon-Nikodým property of conjugate spaces and the w*-equivalence theorem for w*-measurable functions. Sci. China Ser. A **39**, 1034-1041 (1996)
4. Guo, TX: Survey of recent developments of random metric theory and its applications in China (II). Acta Anal. Funct. Appl. **3**, 129-158 (2001)
5. Guo, TX: Relations between some basic results derived from two kinds of topologies for a random locally convex module. J. Funct. Anal. **258**, 3024-3047 (2010)
6. Guo, TX: Recent progress in random metric theory and its applications to conditional risk measures. Sci. China Ser. A **54**, 633-660 (2011)
7. Guo, TX, Shi, G: The algebraic structure of finitely generated $L^0(\mathcal{F}, K)$-modules and the Helly theorem in random normed modules. J. Math. Anal. Appl. **381**, 833-842 (2011)
8. Tang, YH: The Wintner theorem in unital complete random normed algebras. Bull. Korean Math. Soc. **50**(6), 1973-1979 (2013)
9. Guo, TX, Yang, YJ: Ekeland's variational principle for an $\bar{L}^0$-valued function on a complete random metric space. J. Math. Anal. Appl. **389**, 1-14 (2012)
10. Wu, MZ: Farkas's lemma in random locally convex modules and Minkowski-Weyl type results in $L^0(\mathcal{F}, R^n)$. J. Math. Anal. Appl. **404**, 300-309 (2013)
11. Zhao, SE, Zhao, Y: A minimax theorem for $\bar{L}^0$-valued functions on random normed modules. J. Funct. Spaces Appl. (2013). doi:10.1155/2013/704251
12. Guo, TX, Zhao, SE, Zeng, XL: The relations among the three kinds of conditional risk measures. Sci. China Ser. A **57**, 1753-1764 (2014)
13. Zhao, SE, Zhao, Y, Yao, MH: On non-L^0-linear perturbations of random isometries in random normed modules. J. Inequal. Appl. **2014**, 496 (2014)
14. Clark, FH: Pointwise contraction criteria for the existence of fixed points. Can. Math. Bull. **21**(1), 7-11 (1978)
15. Dunford, N, Schwartz, JT: Linear Operators (I). Interscience, New York (1957)
16. Guo, TX: Some basic theories of random normed linear spaces and random inner product spaces. Acta Anal. Funct. Appl. **1**(2), 160-184 (1999)
17. Filipović, D, Kupper, M, Vogelpoth, N: Separation and duality in locally L^0-convex modules. J. Funct. Anal. **256**, 3996-4029 (2009)