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Weak convergence of a hybrid type method with errors for a maximal monotone mapping in Banach spaces

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Abstract

In this paper, we propose a hybrid type method which consists of a resolvent operator technique and a generalized projection onto a moving half-space for approximating a zero of a maximal monotone mapping in Banach spaces. The weak convergence of the iterative sequence generated by the algorithm is also proved. Our results extend and improve the recent ones announced by Zhang (Oper. Res. Lett. 40:564-567, 2012) and Wei and Zhou (Nonlinear Anal. 71:341-346, 2009).

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1 Introduction

Let E be a Banach space with norm $\|\cdot\|$, and E^* be the dual space of E. $\langle \cdot, \cdot \rangle$ denotes the duality pairing of E and E^*. Let $M : E \to 2^{E^*}$ be a maximal monotone mapping. We consider the following problem: Find $x \in E$ such that

$$0 \in Mx. \tag{1.1}$$

This is the zero point problem of a maximal monotone mapping. We denote the set of solutions of Problem (1.1) by $VI(E, M)$ and suppose $VI(E, M) \neq \emptyset$.

Problem (1.1) plays an important role in optimizations. This is because it can be reduced to a convex minimization problem and a variational inequality problem. Many authors have constructed many iterative algorithms to approximate a solution of Problem (1.1) in several settings (see [1–11] and the references therein).

Recently, in [9], Wei and Zhou proposed the following iterative algorithm.

Algorithm 1.1

$$\begin{cases} x_0 \in E, \quad r_0 > 0, \\ y_n = (J + r_n M)^{-1} J(x_n + e_n), \\ Jz_n = \alpha_n Jx_n + (1 - \alpha_n)Jy_n, \\ H_n = \{v \in E : \phi(v, z_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, x_n + e_n)\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{cases} \tag{1.2}$$

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where $\{\alpha_n\} \subset [0,1]$ with $\alpha_n \leq 1 - \delta$ for some $\delta \in (0,1)$, $\{r_n\} \subset (0, +\infty)$ with $\inf_{n\geq 0} r_n > 0$ and the error sequence $\{e_n\} \subset E$ such that $\|e_n\| \to 0$, as $n \to \infty$. They proved the iterative sequence (1.2) converges strongly to $\Pi_{VI(E,M)} x_0$.

We note that, in Algorithm 1.1, we want to compute the generalized projection of x_0 onto $H_n \cap W_n$ to obtain the next iterate. If the set $H_n \cap W_n$ is a specific half-space, then the generalized projection onto it is easily executed. But $H_n \cap W_n$ may not be a half-space, although both of H_n and W_n are half-spaces. If $H_n \cap W_n$ is a general closed and convex set, then it is not easy to compute a minimal general distance. This might seriously affect the efficiency of Algorithm 1.1.

In order to make up the defect, we will construct a new iterative algorithm in this paper by referring to the idea of [5] as follows.

In [5], Zhang proposed a modified proximal point algorithm with errors for approximating a solution of Problem (1.1) in Hilbert spaces. More precisely, he proposed Algorithm 1.2 and proved Theorems 1.1 and 1.2 as follows.

Algorithm 1.2 (*i.e.* Algorithm 2.1 of [5])

Step 0. Select an initial $x_0 \in \mathscr{H}$ (a Hilbert space) and set $k = 0$.

Step 1. Find $y_k \in \mathscr{H}$ such that

$$y_k = J_k(x_k + e_k), \tag{1.3}$$

where $J_k = (I + \lambda_k M)^{-1}$ is the resolvent operator, the positive sequence $\{\lambda_k\}$ satisfies $\alpha := \inf_{k\geq 0} \lambda_k > 0$ and $\{e_k\}$ is an error sequence.

Step 2. Set $K = \{z \in \mathscr{H}, \langle x_k - y_k + e_k, z - y_k\rangle \leq 0\}$ and

$$x_{k+1} = (1 - \beta_k)x_k + \beta_k P_K\big(x_k - \rho_k(x_k - y_k)\big), \tag{1.4}$$

where P_K denotes the metric projection from $\mathscr{H}$ onto K and $\{\beta_k\}_{k=0}^{+\infty} \subset (0,1]$ and $\{\rho_k\}_{k=0}^{+\infty} \subset [0,2)$ are real sequences.

Theorem 1.1 (*i.e.* Theorem 2.1 of [5]) *Let $\{x_k\}$ be the sequence generated by Algorithm* 1.2. *If*

(i) $\|e_k\| \leq \eta_k \|x_k - y_k\|$ *for* $\eta_k \geq 0$ *with* $\sum_{k=0}^{\infty} \eta_k^2 < +\infty$,

(ii) $\{\beta_k\}_{k=0}^{+\infty} \subset [c,d]$ *for some* $c, d \in (0,1)$,

(iii) $0 < \inf_{k\geq 0} \rho_k \leq \sup_{k\geq 0} \rho_k < 2$,

then the sequence $\{x_k\}$ converges weakly to a solution of Problem (1.1) *in Hilbert spaces.*

Theorem 1.2 (*i.e.* Theorem 2.3 of [5]) *Let $\{x_k\}$ be the sequence generated by Algorithm* 1.2 *for* $\rho_k = 0$. *If*

(i) $\lim_{k\to\infty} \|e_k\| = 0$,

(ii) $\{\beta_k\}_{k=0}^{+\infty} \subset [c,1]$ *for some* $c > 0$,

then the sequence $\{x_k\}$ converges weakly to a solution of Problem (1.1) *in Hilbert spaces.*

We note that the set K is a half-space, and hence Algorithm 1.2 is easier to execute than Algorithm 1.1. But, since the metric projection strictly depends on the inner product

properties of Hilbert spaces, Algorithm 1.2 can no longer be applied for Problem (1.1) in Banach spaces.

However, many important problems related to practical problems are generally defined in Banach spaces. For example, the maximal monotone operator related to an elliptic boundary value problem has a Sobolev space $W^{m,p}(\Omega)$ as its natural domain of definition [12]. Therefore, it is meaningful to consider Problem (1.1) in Banach spaces. Motivated and inspired by Algorithms 1.1 and 1.2, the purpose of this paper is to construct a new iterative algorithm for approximating a solution of Problem (1.1) in Banach spaces. In the algorithm, we will replace the generalized projection onto $H_n \cap W_n$ constructed in Algorithm 1.1 by a generalized projection onto a specific constructible half-space by using the idea of Algorithm 1.2. This will make up the defect of Algorithm 1.1 mentioned above.

2 Preliminaries

In the sequel, we use $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong convergence and weak convergence of the sequence $\{x_n\}$ in E to x, respectively.

Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) := \{ v \in E^* : \langle x, v \rangle = \|v\|^2 = \|x\|^2 \}, \quad \forall x \in E. \tag{2.1}$$

It is well known that if E is smooth then J is single-valued and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We shall still denote the single-valued duality mapping by J. Recall that if E is smooth, strictly convex and reflexive, then the duality mapping J is strictly monotone, single-valued, one-to-one and onto, for more details, refer to [7].

The duality mapping J from a smooth Banach space E into E^* is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup Jx$; see [6] and the references therein.

Definition 2.1 ([13]) A multi-valued operator $M : E \to 2^{E^*}$ with domain $D(M) = \{z \in E : Mz \neq \emptyset\}$ and range $R(M) = \bigcup \{Mz \in E^* : z \in D(M)\}$ is said to be

(i) monotone if $\langle x_1 - x_2, u_1 - u_2 \rangle \geq 0$ for each $x_i \in D(M)$ and $u_i \in M(x_i)$, $i = 1, 2$;

(ii) maximal monotone, if M is monotone and its graph $G(M) = \{(x, u) : u \in Mx\}$ is not properly contained in the graph of any other monotone operator. It is well known that a monotone mapping M is maximal if and only if for $(x, u) \in E \times E^*$, $\langle x - y, u - v \rangle \geq 0$ for every $(y, v) \in G(M)$ implies $u \in Mx$.

Let E be a smooth Banach space. Define

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Clearly, from the definition of ϕ we have

(A1) $(\|x\| - \|y\|)^2 \leq \phi(y, x) \leq (\|x\| + \|y\|)^2$,

(A2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,

(A3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\| \|Jx - Jy\| + \|y - x\| \|y\|$.

Let E be a reflexive, strictly convex, and smooth Banach space. K denotes a nonempty, closed, and convex subset of E. By Alber [14], for each $x \in E$, there exists a unique element $x_0 \in K$ (denoted by $\Pi_K(x)$) such that

$$\phi(x_0, x) = \min_{y \in K} \phi(y, x).$$

The mapping $\Pi_K : E \to K$ defined by $\Pi_K(x) = x_0$ is called the generalized projection operator from E onto K. Moreover, x_0 is called the generalized projection of x. See [15] for some properties of Π_K. If E is a Hilbert space, then Π_K is coincident with the metric projection P_K from E onto K.

Lemma 2.1 ([7]) *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty, closed and convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.2 ([16]) *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and let $x \in E$. Then $x_0 = \Pi_C(x)$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.3 ([16]) *Let E be a uniformly convex and smooth Banach space. Let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$, or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 2.4 ([13]) *Let E be a reflexive Banach space and λ be a positive number. If $T : E \to 2^{E^*}$ is a maximal monotone mapping, then $R(J + \lambda T) = E^*$ and $(J + \lambda T)^{-1} : E^* \to E$ is a demi-continuous single-valued maximal monotone mapping.*

Lemma 2.5 ([9]) *Let E be a real reflexive, strictly convex, and smooth Banach space, $T : E \to 2^{E^*}$ be a maximal monotone operator with $T^{-1}0 \neq \emptyset$, then for $\forall x \in E$, $y \in T^{-1}0$ and $r > 0$, we have*

$$\phi\big(y, Q_r^T x\big) + \phi\big(Q_r^T x, x\big) \leq \phi(y, x),$$

where $Q_r^T x = (J + rT)^{-1} Jx$.

Lemma 2.6 ([17]) *Let $\{a_n\}$ and $\{t_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + t_n \quad \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \to \infty} a_n$ exists.

Lemma 2.7 *Let S be a nonempty, closed, and convex subset of a uniformly convex, smooth Banach space E. Let $\{x_n\}$ be a bounded sequence in E. Suppose that, for all $u \in S$,*

$$\phi(u, x_{n+1}) \leq (1 + \theta_n)\phi(u, x_n) \tag{2.2}$$

for every $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} \theta_n < \infty$. Then $\{\Pi_S(x_n)\}$ is a Cauchy sequence.

Proof Put $u_n := \Pi_S x_n$ for all $n \geq 1$. For $\omega \in S$, we have $\phi(u_n, x_n) \leq \phi(\omega, x_n)$. Thus, $\{u_n\}$ is bounded. By $u_{n+1} = \Pi_S x_{n+1}$ and $u_n = \Pi_S x_n \in S$, we infer that

$$\phi(u_{n+1}, x_{n+1}) \leq \phi(u_n, x_{n+1}) \leq \phi(u_n, x_n) + \theta_n M^*,$$

where $M^* = \sup\{\phi(u_n, x_n), n \geq 1\}$. Since $\sum_{n=1}^{\infty} \theta_n < \infty$, it follows from Lemma 2.6 that $\lim_{n \to \infty} \phi(u_n, x_n)$ exists. Using (2.2), for all $m \geq 1$, we have $\phi(u_n, x_{n+m}) \leq \prod_{i=0}^{m-1}(1 + \theta_{n+i})\phi(u_n, x_n)$. Then we infer from $u_{n+m} = \Pi_S x_{n+m}$ and $u_n = \Pi_S x_n \in S$ that

$$\phi(u_n, u_{n+m}) \leq \phi(u_n, x_{n+m}) - \phi(u_{n+m}, x_{n+m})$$

$$\leq \prod_{i=0}^{m-1}(1 + \theta_{n+i})\phi(u_n, x_n) - \phi(u_{n+m}, x_{n+m})$$

$$\leq e^{\sum_{i=0}^{m-1} \theta_{n+i}} \phi(u_n, x_n) - \phi(u_{n+m}, x_{n+m}).$$

Therefore, $\lim_{n \to \infty} \phi(u_n, u_{n+m}) = 0$, and hence we have, from Lemma 2.3, $\lim_{n \to \infty} \|u_n - u_{n+m}\| = 0$, for all $m \geq 1$. Consequently, $\{u_n\}$ is a Cauchy sequence. $\square$

3 Main results

In this section, we construct a new iterative algorithm and prove two convergence theorems for two different iterative sequences generated by the new iterative algorithm for solving Problem (1.1) in Banach spaces.

Algorithm 3.1

Step 0. (Initiation) Arbitrarily select initial $x_0 \in E$ and set $k = 0$, where E is a reflexive, strictly convex, and smooth Banach space.

Step 1. (Resolvent step) Find $y_k \in E$ such that

$$y_k = Q_{\lambda_k}^M(x_k + e_k), \tag{3.1}$$

where $Q_{\lambda_k}^M = (J + \lambda_k M)^{-1} J$, the positive sequence $\{\lambda_k\}$ satisfies $\alpha_1 := \inf_{k \geq 0} \lambda_k > 0$ and $\{e_k\}$ is an error sequence.

Step 2. (Projection step) Set $C_k = \{z \in E : \langle z - y_k, J(x_k + e_k) - J(y_k) \rangle \leq 0\}$ and

$$x_{k+1} = J^{-1}\big((1 - \beta_k)Jx_k + \beta_k J \Pi_{C_k} J^{-1}\big(Jx_k - \rho_k(Jx_k - Jy_k)\big)\big), \tag{3.2}$$

where $\{\beta_k\} \subset (0, 1]$, and $\{\rho_k\} \subset (0, 1]$.

Step 3. Let $k = k + 1$ and return to Step 1.

Now we show the convergence of the iterative sequence generated by Algorithm 3.1 in the Banach space E.

Theorem 3.1 *Let E be a uniformly convex, uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous and $M : E \to 2^{E^*}$ be a maximal monotone mapping such that $VI(E, M) \neq \emptyset$. If*

$$\max\big(\|x_k + e_k\|^2 - \|x_k\|^2, \|J(x_k + e_k) - Jx_k\|\big) \leq \eta_k, \qquad \sum_{k=0}^{\infty} \eta_k < \infty, \tag{3.3}$$

and $\liminf_{k \to \infty} \beta_k \rho_k > 0$, then the iterative sequence $\{x_k\}$ generated by Algorithm 3.1 converges weakly to an element $\hat{x} \in VI(E, M)$. Further, $\hat{x} = \lim_{k \to \infty} \Pi_{VI(E,M)}(x_k)$.

Proof We split the proof into four steps.

  Step 1. Show that $\{x_k\}$ is bounded.

  Suppose $x^* \in VI(E, M)$, then we have $0 \in M(x^*)$. From (3.1), it follows that

$$\frac{1}{\lambda_k}\big(J(x_k + e_k) - Jy_k\big) \in M(y_k).$$

By the monotonicity of M, we deduce that

$$\left\langle x^* - y_k, -\frac{1}{\lambda_k}\big(J(x_k + e_k) - Jy_k\big) \right\rangle \geq 0, \tag{3.4}$$

which leads to

$$x^* \in C_k = \big\{z \in E : \langle z - y_k, J(x_k + e_k) - J(y_k) \rangle \leq 0\big\}.$$

Let $t_k = \Pi_{C_k} J^{-1}(Jx_k - \rho_k(Jx_k - Jy_k))$. It follows from (3.2) that

$$\begin{aligned}
\phi\big(x^*, x_{k+1}\big) &\leq \|x^*\|^2 - 2\langle x^*, (1 - \beta_k)Jx_k + \beta_k Jt_k \rangle + (1 - \beta_k)\|Jx_k\|^2 + \beta_k \|Jt_k\|^2 \\
&\leq (1 - \beta_k)\phi\big(x^*, x_k\big) + \beta_k \phi\big(x^*, t_k\big).
\end{aligned} \tag{3.5}$$

By Lemmas 2.1 and 2.5, we deduce that

$$\begin{aligned}
\phi\big(x^*, t_k\big) &\leq \phi\big(x^*, J^{-1}\big(Jx_k - \rho_k(Jx_k - Jy_k)\big)\big) \\
&\leq \|x^*\|^2 - 2\langle x^*, Jx_k - \rho_k(Jx_k - Jy_k) \rangle + (1 - \rho_k)\|Jx_k\|^2 + \rho_k \|Jy_k\|^2 \\
&= (1 - \rho_k)\phi\big(x^*, x_k\big) + \rho_k \phi\big(x^*, y_k\big) \\
&\leq (1 - \rho_k)\phi\big(x^*, x_k\big) + \rho_k\big(\phi\big(x^*, x_k + e_k\big) - \phi(y_k, x_k + e_k)\big) \\
&= \phi\big(x^*, x_k\big) + \rho_k\big(\phi\big(x^*, x_k + e_k\big) - \phi\big(x^*, x_k\big)\big) - \rho_k \phi(y_k, x_k + e_k) \\
&\leq \phi\big(x^*, x_k\big) + \rho_k \eta_k\big(2\|x^*\| + 1\big).
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned}
\phi\big(x^*, x_{k+1}\big) &\leq \phi\big(x^*, x_k\big) + \beta_k \rho_k \eta_k\big(2\|x^*\| + 1\big) - \beta_k \rho_k \phi(y_k, x_k + e_k) \\
&\leq \phi\big(x^*, x_k\big) + \beta_k \rho_k \eta_k\big(2\|x^*\| + 1\big) \leq \phi\big(x^*, x_k\big) + M\eta_k,
\end{aligned} \tag{3.7}$$

where $M = \sup_{k \geq 0}(\beta_k \rho_k)(2\|x^*\| + 1)$. Since $\sum_{k=0}^{\infty} \eta_k < \infty$, (3.7) implies that $\lim_{k \to \infty} \phi(x^*, x_k)$ exists by Lemma 2.6. Hence, $\{x_k\}$ is bounded. From (3.3), we have $\|x_k + e_k\|^2 \leq \|x_k\|^2 + \eta_k$, and hence $\{x_k + e_k\}$ is also bounded.

  Step 2. Show that $\{x_k\}$ and $\{y_k\}$ have the same weak accumulation points.

  It follows from (3.7) that

$$\beta_k \rho_k \phi(y_k, x_k + e_k) \leq \phi\big(x^*, x_k\big) - \phi\big(x^*, x_{k+1}\big) + \beta_k \rho_k \eta_k\big(2\|x^*\| + 1\big),$$

and hence

$$\phi(y_k, x_k + e_k) \leq \frac{1}{\beta_k \rho_k}\big(\phi\big(x^*, x_k\big) - \phi\big(x^*, x_{k+1}\big)\big) + \eta_k\big(2\|x^*\| + 1\big).$$

Since $\liminf_{k \to \infty} \beta_k \rho_k > 0$, $\lim_{k \to \infty} \phi(x^*, x_k)$ exists and $\sum_{k=0}^{\infty} \eta_k < \infty$, we have

$$\lim_{k \to \infty} \phi(y_k, x_k + e_k) = 0. \tag{3.8}$$

By Lemma 2.3, we deduce that

$$\lim_{k \to \infty} \left\| y_k - (x_k + e_k) \right\| = 0. \tag{3.9}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{k \to \infty} \left\| Jy_k - J(x_k + e_k) \right\| = 0. \tag{3.10}$$

It follows from (3.3) that

$$\lim_{k \to \infty} \left\| Jx_k - J(x_k + e_k) \right\| = 0. \tag{3.11}$$

Since $\| Jy_k - Jx_k \| \leq \| Jy_k - J(x_k + e_k) \| + \| J(x_k + e_k) - Jx_k \|$, it follows from (3.10) and (3.11) that

$$\lim_{k \to \infty} \| Jy_k - Jx_k \| = 0. \tag{3.12}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{k \to \infty} \| y_k - x_k \| = 0. \tag{3.13}$$

Consequently, we see that $\{x_k\}$ and $\{y_k\}$ have the same weak accumulation points.

Step 3. Show that each weak accumulation point of the sequence $\{x_k\}$ is a solution of Problem (1.1).

Since $\{x_k\}$ is bounded, let us suppose $\hat{x}$ is a weak accumulation point of $\{x_k\}$. Hence, we can extract a subsequence that weakly converges to $\hat{x}$. Without loss of generality, let us suppose that $x_k \rightharpoonup \hat{x}$ as $k \to \infty$. Then from (3.13), we have $y_k \rightharpoonup \hat{x}$ as $k \to \infty$.

For any $(v, u) \in G(M)$, it follows from the monotonicity of M and (3.1) that

$$\left\langle y_k - v, \frac{1}{\lambda_k} \left(J(x_k + e_k) - Jy_k \right) - u \right\rangle \geq 0,$$

which implies that

$$\langle x_k - v, -u \rangle \geq \langle x_k - y_k, -u \rangle - \left\langle y_k - v, \frac{1}{\lambda_k} \left(J(x_k + e_k) - Jy_k \right) \right\rangle$$

$$\geq \langle x_k - y_k, -u \rangle - \frac{1}{\alpha_1} \| y_k - v \| \| J(x_k + e_k) - Jy_k \|. \tag{3.14}$$

Taking the limits in (3.14), by (3.10), (3.13), and the boundedness of $\{y_k\}$, we have

$$\langle \hat{x} - v, -u \rangle = \lim_{k \to \infty} \langle x_k - v, -u \rangle \geq 0.$$

Since M is maximal monotone, by the arbitrariness of $(v, u) \in G(M)$, we conclude that $(\hat{x}, 0) \in G(M)$ and hence $\hat{x}$ is a solution of Problem (1.1), *i.e.*, $\hat{x} \in VI(E, M)$.

Step 4. Show that $x_k \rightharpoonup \hat{x}$, as $k \to \infty$ and $\hat{x} = \lim_{k \to \infty} \Pi_{VI(E,M)}(x_k)$.

Put $u_k = \Pi_{VI(E,M)}(x_k)$. Since $\hat{x} \in VI(E, M)$, we have $\phi(u_k, x_k) \leq \phi(\hat{x}, x_k)$, which implies that $\{u_k\}$ is bounded. Since $u_k \in VI(E, M)$, we have from (3.7)

$$\phi(u_k, x_{k+1}) \leq \phi(u_k, x_k) + M\eta_k = \left(1 + \frac{M\eta_k}{\phi(u_k, x_k)}\right)\phi(u_k, x_k)$$

$$\leq \left(1 + M^*\eta_k\right)\phi(u_k, x_k), \tag{3.15}$$

where $M^* = \sup_{k \geq 0}\left(\frac{M}{\phi(u_k, x_k)}\right)$. Since $\sum_{k=0}^{\infty} \eta_k < \infty$, it follows from Lemma 2.7 that $\{u_k\}$ is a Cauchy sequence. Since $VI(E, M)$ is closed, we see that $\{u_k\}$ converges strongly to $z \in VI(E, M)$. By the uniform smoothness of E, we also have

$$\lim_{k \to \infty} \|Ju_k - Jz\| = 0. \tag{3.16}$$

On the other hand, it follows from $\hat{x} \in VI(E, M)$, $u_k = \Pi_{VI(E,M)}x_k$, and Lemma 2.2 that

$$\langle u_k - \hat{x}, Jx_k - Ju_k \rangle \geq 0.$$

Let $k \to \infty$, it follows from the weakly sequential continuity of J and (3.16) that $\langle z - \hat{x}, J\hat{x} - Jz \rangle \geq 0$. Since E is strictly convex, we have $z = \hat{x}$. Therefore, $\{x_k\}$ converges weakly to $\hat{x} \in VI(E, M)$, where $\hat{x} = \lim_{k \to \infty} \Pi_{VI(E,M)}x_k$. □

Next, we show the convergence of the iterative sequence when $\rho_k = 0$.

Theorem 3.2 *Let E be a uniformly convex, uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous and $M : E \to 2^{E^*}$ be a maximal monotone mapping such that $VI(E, M) \neq \emptyset$. Let $\{x_k\}$ be the sequence generated by Algorithm 3.1 for $\rho_k = 0$.*

If $\{\lambda_k\}$, $\{\beta_k\}$, and $\{e_k\}$ satisfy $\inf_{k \geq 0} \lambda_k = \alpha_1 > 0$, $0 < \beta_k \leq 1$, $\liminf_{k \to \infty} \beta_k > 0$, and $\lim_{k \to \infty} \|e_k\| = 0$, then the iterative sequence $\{x_k\}$ converges weakly to an element $\hat{x} \in VI(E, M)$. Further, $\hat{x} = \lim_{k \to \infty} \Pi_{VI(E,M)}(x_k)$.

Proof Suppose $x^* \in VI(E, M)$, then we have $0 \in M(x^*)$. It is similar to the proof of Theorem 3.1, we have

$$x^* \in C_k = \left\{z \in E : \langle z - y_k, J(x_k + e_k) - J(y_k) \rangle \leq 0\right\}.$$

Let $t_k = \Pi_{C_k}(x_k)$. It follows from (3.2) for $\rho_k = 0$ that

$$\phi(x^*, x_{k+1}) \leq \|x^*\|^2 - 2\langle x^*, (1 - \beta_k)Jx_k + \beta_k Jt_k \rangle + (1 - \beta_k)\|Jx_k\|^2 + \beta_k\|Jt_k\|^2$$

$$\leq (1 - \beta_k)\phi(x^*, x_k) + \beta_k\phi(x^*, t_k). \tag{3.17}$$

By Lemma 2.1, we deduce that

$$\phi(x^*, t_k) = \phi(x^*, \Pi_{C_k}x_k) \leq \phi(x^*, x_k) - \phi(t_k, x_k). \tag{3.18}$$

From (3.17) and (3.18), we have

$$\phi\big(x^*, x_{k+1}\big) \leq \phi\big(x^*, x_k\big) - \beta_k \phi(t_k, x_k) \leq \phi\big(x^*, x_k\big), \tag{3.19}$$

which implies that $\lim_{k\to\infty} \phi(x^*, x_k)$ exists and hence $\{x_k\}$ is bounded. Consequently, $\{x_k + e_k\}$ is also bounded. Since $\liminf_{k\to\infty} \beta_k > 0$, from (3.19), we have

$$\lim_{k\to\infty} \phi(t_k, x_k) = 0. \tag{3.20}$$

Thus, it follows from Lemma 2.3 that

$$\lim_{k\to\infty} \|t_k - x_k\| = 0, \tag{3.21}$$

and hence $\{t_k\}$ is bounded. Since $\|(x_k + e_k) - t_k\| \leq \|x_k - t_k\| + \|e_k\|$, we have from $\lim_{k\to\infty} \|e_k\| = 0$ and (3.21)

$$\lim_{k\to\infty} \big\|(x_k + e_k) - t_k\big\| = 0.$$

Hence, it follows from (A3) that

$$\lim_{k\to\infty} \phi(t_k, x_k + e_k) = 0. \tag{3.22}$$

From the definition of C_k, we have $y_k = \Pi_{C_k}(x_k + e_k)$. Hence, we obtain from Lemma 2.1 and $t_k = \Pi_{C_k}(x_k) \in C_k$

$$\phi(t_k, y_k) \leq \phi(t_k, x_k + e_k) - \phi(y_k, x_k + e_k) \leq \phi(t_k, x_k + e_k). \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\lim_{k\to\infty} \phi(t_k, y_k) = 0. \tag{3.24}$$

From Lemma 2.3, we obtain

$$\lim_{k\to\infty} \|t_k - y_k\| = 0. \tag{3.25}$$

Since $\|y_k - x_k\| \leq \|y_k - t_k\| + \|t_k - x_k\|$, we have from (3.25) and (3.21)

$$\lim_{k\to\infty} \|y_k - x_k\| = 0. \tag{3.26}$$

Since $\|x_k + e_k - y_k\| \leq \|x_k + e_k - x_k\| + \|x_k - y_k\|$, it follows from $\lim_{k\to\infty} \|e_k\| = 0$ and (3.26) that

$$\lim_{k\to\infty} \|x_k + e_k - y_k\| = 0. \tag{3.27}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{k\to\infty} \big\|J(x_k + e_k) - Jy_k\big\| = 0. \tag{3.28}$$

By a similar proof to Step 3 and Step 4 in the proof of Theorem 3.1, we can easily obtain the desired conclusion. Therefore, we omit it. □

Remark 3.1 There are the following differences between Theorem 3.2 and the recent results announced by [5, 9] and [18]:

(i) When $E = \mathscr{H}$ (a Hilbert space), Theorem 3.2 reduces to Theorem 2.3 of Zhang [5] (*i.e.* Theorem 1.2 of this paper). That is to say, Theorem 3.2 extends Theorem 2.3 of Zhang [5] from Hilbert spaces to more general Banach spaces. Furthermore, we see that the convergence point of $\{x_k\}$ is $\lim_{k\to\infty} \Pi_{VI(E,M)}(x_k)$, which is more concrete than that of Theorem 2.3 of Zhang [5].

(ii) In Algorithm 3.1, the set C_k is a half-space, and hence it is easier to compute the generalized projection of the current iterate onto it than that onto a general closed convex set $H_n \cap W_n$ or H_n constructed in [9, 18] to obtain the next iterate. Hence, Algorithm 3.1 improves Algorithm 1.1 and those algorithms in [18] from a numerical point of view.

In the following, we give a simple example to compare Algorithm 1.1 constructed in [9] with Algorithm 3.1 for $\rho_k = 0$.

Example 3.1 Let $E = \mathbb{R}$, $M : \mathbb{R} \to \mathbb{R}$ and $M(x) = x$. It is obvious that M is maximal monotone and $VI(E,M) = \{0\} \neq \emptyset$.

The numerical experiment result of Algorithm 1.1 Take $r_k = 1 + \frac{1}{k+2}$, $\alpha_k = \frac{1}{2} - \frac{1}{k+2}$, $e_k = 0$, for all $k \geq 0$, and initial point $x_0 = -\frac{1}{3} \in \mathbb{R}$. Then $\{x_k\}$ generated by Algorithm 1.1 is the following sequence:

$$\begin{cases} x_0 = -\frac{1}{3} \in \mathbb{R}, \\ x_{k+1} = \frac{7k^2+29k+28}{8k^2+36k+40}x_k, \quad k \geq 0, \end{cases} \tag{3.29}$$

and $x_k \to 0$ as $k \to \infty$, where $0 \in VI(E,M)$.

Proof By Algorithm 1.1, we have $y_0 = \frac{1}{1+r_0}x_0 = -\frac{2}{15}$, $z_0 = -\frac{2}{15} > x_0$, $H_0 = \{v \in \mathbb{R}, \|v - z_0\| \leq \|v - x_0\|\} = [z_0 - (\frac{z_0-x_0}{2}), +\infty) = [-\frac{7}{30}, +\infty)$, $W_0 = \{v \in \mathbb{R}, \langle v - x_0, x_0 - x_0 \rangle \leq 0\} = \mathbb{R}$. Therefore, $H_0 \cap W_0 = H_0 = [-\frac{7}{30}, +\infty)$ and $x_1 = P_{[-\frac{7}{30},+\infty)}(-\frac{1}{3}) = -\frac{7}{30} = \frac{7\cdot0^2+29\cdot0+28}{8\cdot0^2+36\cdot0+40}x_0$. Suppose that $x_{k+1} = \frac{7k^2+29k+28}{8k^2+36k+40}x_k$. By Algorithm 1.1, $y_{k+1} = \frac{k+3}{2k+7}x_{k+1}$, and hence

$$0 > z_{k+1} = \alpha_{k+1}x_{k+1} + (1 - \alpha_{k+1})\frac{k+3}{2k+7}x_{k+1} > x_{k+1}, \tag{3.30}$$

$H_{k+1} = \{v \in \mathbb{R} : \|v - z_{k+1}\| \leq \|v - x_{k+1}\|\} = [z_{k+1} - \frac{z_{k+1}-x_{k+1}}{2}, +\infty) \subset [x_{k+1}, +\infty)$, $W_{k+1} = \{v \in \mathbb{R} : \langle v - x_{k+1}, x_0 - x_{k+1} \rangle \leq 0\} = [x_{k+1}, +\infty)$. Therefore, $H_{k+1} \cap W_{k+1} = H_{k+1} = [z_{k+1} - \frac{z_{k+1}-x_{k+1}}{2}, +\infty)$ and

$$x_{k+2} = P_{[z_{k+1}-\frac{z_{k+1}-x_{k+1}}{2},+\infty)}(x_0) = z_{k+1} - \frac{z_{k+1}-x_{k+1}}{2}. \tag{3.31}$$

Combine (3.30) with (3.31), we obtain $x_{k+2} = \frac{7(k+1)^2+29(k+1)+28}{8(k+1)^2+36(k+1)+40}x_{k+1}$. By induction, (3.29) holds. □

Table 1 The numerical experiment result of Algorithm 1.1

k	0	1	2	3	4	5	6	7	$\cdots$
x_k	$-\frac{1}{3}$	$-\frac{7}{30}$	$-\frac{8}{45}$	$-\frac{9}{135}$	$-\frac{89}{1,650}$	$-\frac{1,424}{32,175}$	$-\frac{127,448}{3,378,375}$	$-\frac{3,616,337}{114,864,750}$	$\cdots$

Next, we give the numerical experiment results by using Table 1, which shows that the iteration process of the sequence $\{x_k\}$ as initial point $x_0 = -\frac{1}{3}$. From the figures, we can see that $\{x_k\}$ converges to 0.

The numerical experiment result of Algorithm 3.1 for $\rho_k = 0$ Take $\lambda_k = 1 + \frac{1}{k+2}$, $k \geq 0$, $\beta_k = 0$, $e_k = 0$, for all $k \geq 0$, and initial point $x_0 = -\frac{1}{3} \in \mathbb{R}$. Then $\{x_k\}$ generated by Algorithm 3.1 is the following sequence:

$$\begin{cases} x_0 = -\frac{1}{3}, \\ x_{k+1} = \frac{k+2}{2k+5}x_k, \quad k \geq 0, \end{cases} \tag{3.32}$$

and $x_k \to 0$ as $k \to \infty$, where $0 \in VI(E, M)$.

Proof By (3.1),

$$0 > y_0 = \frac{1}{(1 + \lambda_0)}x_0 = \frac{1}{2 + \frac{1}{0+2}}x_0 = \frac{0+2}{2 \cdot 0 + 5}x_0 = \frac{2}{5}x_0 = -\frac{2}{15} > x_0.$$

By Algorithm 3.1, we have $C_0 = \{z \in \mathbb{R} : \langle z - y_0, x_0 - y_0 \rangle \leq 0\} = [y_0, +\infty)$. By (3.2), $x_1 = P_{C_0}(x_0) = y_0 = -\frac{2}{15} > x_0$. This is

$$\begin{cases} x_0 = -\frac{1}{3}, \\ x_1 = y_0 = \frac{0+2}{2 \cdot 0 + 5}x_0 > x_0. \end{cases}$$

Suppose that

$$\begin{cases} x_0 = -\frac{1}{3}, \\ x_{k+1} = y_k = \frac{k+2}{2k+5}x_k > x_k. \end{cases} \tag{3.33}$$

By (3.1),

$$0 > y_{k+1} = \frac{1}{1 + \lambda_{k+1}}x_{k+1} = \frac{1}{2 + \frac{1}{(k+1)+2}}x_{k+1} = \frac{(k+1)+2}{2(k+1)+5}x_{k+1} > x_{k+1}. \tag{3.34}$$

It follows from Algorithm 3.1 and (3.34) that $C_{k+1} = \{z \in E : \langle z - y_{k+1}, x_{k+1} - y_{k+1} \rangle \leq 0\} = [y_{k+1}, +\infty)$, and hence

$$x_{k+2} = P_{C_{k+1}}x_{k+1} = P_{[y_{k+1}, +\infty)}x_{k+1} = y_{k+1} = \frac{(k+1)+2}{2(k+1)+5}x_{k+1}.$$

By induction, (3.32) holds. □

Next, we give the numerical experiment results by using Table 2, which shows that the iteration process of the sequence $\{x_k\}$ as initial point $x_0 = -\frac{1}{3}$. From the figures, we can see that $\{x_k\}$ converges to 0.

Table 2 The numerical experiment result of Algorithm 3.1

k	0	1	2	3	4	5	6	7	$\cdots$
x_k	$-\frac{1}{3}$	$-\frac{2}{15}$	$-\frac{6}{105}$	$-\frac{8}{315}$	$-\frac{8}{693}$	$-\frac{16}{3{,}003}$	$-\frac{16}{6{,}435}$	$-\frac{128}{109{,}395}$	$\cdots$

Remark 3.2 Comparing Table 1 with Table 2, we can intuitively see that the convergence speed of Algorithm 3.1 for $\rho_k = 0$ is faster than that of Algorithm 1.1 constructed in [9].

Remark 3.3 In [19, 20], the authors proposed several different iterative algorithms for approximating zeros of m-accretive operators in Banach spaces. The nonexpansiveness of the resolvent operator of the m-accretive operator is employed in theses algorithms. Since the resolvent operator of a maximal monotone operator is not nonexpansive in Banach spaces, these algorithms cannot be applied to Problem (1.1).

Remark 3.4 In [21], the authors established viscosity iterative algorithms for approximating a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in Hilbert spaces by using the nonexpansiveness of the metric projection operator. However, the metric projection operator is not nonexpansive in Banach spaces. Therefore, the algorithms of [21] cannot be applied to Problem (1.1) of this paper in Banach spaces.

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