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Mann's type extragradient for solving split feasibility and fixed point problems of Lipschitz asymptotically quasi-nonexpansive mappings

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Abstract

The purpose of this paper is to introduce and analyze Mann's type extragradient for finding a common solution set Γ of the split feasibility problem and the set Fix(T) of fixed points of Lipschitz asymptotically quasi-nonexpansive mappings T in the setting of infinite-dimensional Hilbert spaces. Consequently, we prove that the sequence generated by the proposed algorithm converges weakly to an element of Fix(T) $\cap\, \Gamma$ under mild assumption. The result presented in the paper also improves and extends some result of Xu (Inverse Probl. 26:105018, 2010; Inverse Probl. 22:2021-2034, 2006) and some others.

MSC: 49J40; 47H05

Keywords: split feasibility problems; fixed point problems; extragradient methods; asymptotically quasi-nonexpansive mappings; maximal monotone mappings

1 Introduction

The split feasibility problem (SFP) in finite dimensional spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning [3–5]. The split feasibility problem in an infinite-dimensional Hilbert space can be found in [2, 4, 6–10] and references therein.

Throughout this paper, we always assume that H_1, H_2 are real Hilbert spaces, '$\rightarrow$', '$\rightharpoonup$' denote strong and weak convergence, respectively, and $F(T)$ is the fixed point set of a mapping T.

Let C and Q be nonempty closed convex subsets of infinite-dimensional real Hilbert spaces H_1 and H_2, respectively, and let $A \in B(H_1, H_2)$, where $B(H_1, H_2)$ denotes the class of all bounded linear operators from H_1 to H_2. The split feasibility problem (SFP) is finding a point $\hat{x}$ with the property

$$\hat{x} \in C, \qquad A\hat{x} \in Q. \tag{1.1}$$

In the sequel, we use Γ to denote the set of solutions of SFP (1.1), *i.e.*,

$$\Gamma = \{\hat{x} \in C : A\hat{x} \in Q\}.$$

Assuming that the SFP is consistent (*i.e.*, (1.1) has a solution), it is not hard to see that $x \in C$ solves (1.1) if and only if it solves the fixed-point equation

$$x = P_C\bigl(I - \gamma A^*(I - P_Q)A\bigr)x, \quad x \in C, \tag{1.2}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant, and A^* denotes the adjoint of A.

To solve (1.2), Byrne [2] proposed his *CQ* algorithm, which generates a sequence (x_k) by

$$x_{k+1} = P_C\bigl(I - \gamma A^*(I - P_Q)A\bigr)x_k, \quad k \in \mathbb{N}, \tag{1.3}$$

where $\gamma \in (0, 2/\lambda)$, and again λ is the spectral radius of the operator A^*A.

The *CQ* algorithm (1.3) is a special case of the Krasnonsel'skii-Mann (K-M) algorithm. The K-M algorithm generates a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,$$

where $\{\alpha_n\}$ is a sequence in the interval $(0, 1)$ and the initial guess $x_0 \in C$ is chosen arbitrarily. Due to the fixed point for formulation (1.2) of the SFP, we can apply the K-M algorithm to the operator $P_C(I - \gamma A^*(I - P_Q)A)$ to obtain a sequence given by

$$x_{k+1} = (1 - \alpha_k)x_k + \alpha_k P_C\bigl(I - \gamma A^*(I - P_Q)A\bigr)x_k, \quad k \in \mathbb{N}, \tag{1.4}$$

where $\gamma \in (0, 2/\lambda)$, and again λ is the spectral radius of the operator A^*A.

Then, as long as (x_k) satisfies the condition $\sum_{k=1}^{\infty} \alpha_k(1 - \alpha_k) = +\infty$, we have weak convergence of the sequence generated by (1.4).

Very recently, Xu [8] gave a continuation of the study on the *CQ* algorithm and its convergence. He applied Mann's algorithm to the SFP and proposed an averaged *CQ* algorithm which was proved to be weakly convergent to a solution of the SFP. He derived a weak convergence result, which shows that for suitable choices of iterative parameters (including the regularization), the sequence of iterative solutions can converge weakly to an exact solution of the SFP. He also established the strong convergence result, which shows that the minimum-norm solution can be obtained.

On the other hand, Korpelevich [11] introduced an iterative method, the so-called extragradient method, for finding the solution of a saddle point problem. He proved that the sequences generated by the proposed iterative algorithm converge to a solution of a saddle point problem.

Motivated by the idea of an extragradient method in [12], Ceng [13] introduced and analyzed an extragradient method with regularization for finding a common element of the solution set Γ of the split feasibility problem and the set $\mathrm{Fix}(T)$ of a nonexpansive mapping T in the setting of infinite-dimensional Hilbert spaces. Chang [14] introduced an algorithm for solving the split feasibility problems for total quasi-asymptotically nonexpansive mappings in infinite-dimensional Hilbert spaces.

The purpose of this paper is to study and analyze a Mann's type extragradient method for finding a common element of the solution set Γ of the SFP and the set $\mathrm{Fix}(T)$ of asymptotically quasi-nonexpansive mappings and Lipshitz continuous mappings in a real Hilbert space. We prove that the sequence generated by the proposed method converges weakly to an element $\hat{x}$ in $\mathrm{Fix}(T) \cap \Gamma$.

2 Preliminaries

We first recall some definitions, notations and conclusions which will be needed in proving our main results.

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, and let C be a nonempty closed and convex subset of H.

Let E be a Banach space. A mapping $T : E \to E$ is said to be *demi-closed at origin* if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x^*$ and $\|(I - T)x_n\| \to 0$, then $x^* = Tx^*$.

A Banach space E is said to have *the Opial property* if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x^*$, then

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \text{ with } y \neq x^*.$$

Remark 2.1 It is well known that each Hilbert space possesses the Opial property.

Proposition 2.2 *For given $x \in H$ and $z \in C$:*

(i) *$z = P_C x$ if and only if $\langle x - z, y - z \rangle \leq 0$ for all $y \in C$.*
(ii) *$z = P_C x$ if and only if $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$ for all $y \in C$.*
(iii) *For all $x, y \in H$, $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2$.*

Definition 2.3 Let C be a nonempty, closed and convex subset of a real Hilbert space H. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. Then T is said to be

(1) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
(2) *asymptotically nonexpansive* if there exists a sequence $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{2.1}$$

for all $x, y \in C$ and $n \geq 1$;
(3) *asymptotically quasi-nonexpansive* if there exists a sequence $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \leq k_n \|x - p\| \tag{2.2}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$;
(4) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \tag{2.3}$$

for all $x, y \in C$ and $n \geq 1$.

Remark 2.4 By the above definitions, it is clear that:

(i) a nonexpansive mapping is an asymptotically quasi-nonexpansive mapping;
(ii) a quasi-nonexpansive mapping is an asymptotically-quasi nonexpansive mapping;
(iii) an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive mapping.

Proposition 2.5 (see [15]) *We have the following assertions.*

(1) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*
(2) *If T is ν-ism and $\gamma > 0$, then γT is $\frac{\nu}{\gamma}$-ism.*
(3) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$.*

Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.

Proposition 2.6 (see [15, 16]) *Let $S, T, V : H \to H$ be given operators. We have the following assertions.*

(1) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is averaged and V is nonexpansive, then T is averaged.*
(2) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*
(3) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is firmly nonexpansive and V is nonexpansive, then T is averaged.*
(4) *The composite of finite many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^n$ is averaged, then so is the composite $T_1 \circ T_2 \circ \cdots \circ T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*
(5) *If the mappings $\{T_i\}_{i=1}^n$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{n} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N).$$

The notation $\mathrm{Fix}(T)$ *denotes the set of all fixed points of the mapping T, that is,* $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$.

Lemma 2.7 (see [17], demiclosedness principle) *Let C be a nonempty closed and convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(S) \neq \emptyset$. *If the sequence $\{x_n\} \subseteq C$ converges weakly to x and the sequence $\{(I - S)x_n\}$ converges strongly to y, then $(I - S)x = y$; in particular, if $y = 0$, then $x \in$* $\mathrm{Fix}(S)$.

Lemma 2.8 (see [18]) *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be two sequences of nonnegative numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 0,$$

if $\sum_{n=0}^\infty b_n$ converges, then $\lim_{n\to\infty} a_n$ *exists.*

The following lemma gives some characterizations and useful properties of the metric projection P_C in a Hilbert space.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C, \tag{2.4}$$

where P_C is called the *metric projection of* H onto C. We know that P_C is a nonexpansive mapping of H onto C.

Lemma 2.9 (see [19]) *Let C be a nonempty closed and convex subset of a real Hilbert space H, and let P_C be the metric projection from H onto C. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if the following holds*:

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \tag{2.5}$$

Lemma 2.10 (see [20]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $P_C : H \to C$ be the metric projection from H onto C. Then the following inequality holds*:

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \forall y \in C. \tag{2.6}$$

Lemma 2.11 (see [19]) *Let H be a real Hilbert space. Then the following equations hold*:

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$ *for all* $x, y \in H$;

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$ *for all* $t \in [0,1]$ *and* $x, y \in H$.

Throughout this paper, we assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty. Let $f : H_1 \to \mathbb{R}$ be a continuous differentiable function. The minimization problem

$$\min_{x \in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 \tag{2.7}$$

is ill-posed. Therefore (see [8]) consider the following Tikhonov regularized problem:

$$\min_{x \in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{2.8}$$

where $\alpha > 0$ is the regularization parameter.

We observe that the gradient

$$\nabla f_\alpha = \nabla f + \alpha I = A^*(I - P_Q)A + \alpha I \tag{2.9}$$

is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone.

Let C be a nonempty closed convex subset of a real Hilbert space H, and let $F : C \to H$ be a monotone mapping. The variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle Fx, y - x \rangle \geq 0, \quad \forall y \in C.$$

The solution set of the VIP is denoted by $VIP(C, F)$. It is well known that

$$x \in VI(C, F) \quad \Leftrightarrow \quad x = P_C(x - \lambda Fx), \quad \forall \lambda > 0.$$

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply

$$\langle x - y, f - g\rangle \geq 0.$$

A monotone mapping $T : H \to 2^H$ is called *maximal* if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x,f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$. Let $F : C \to H$ be a monotone and k-Lipschitz continuous mapping, and let $N_C v$ be the normal cone to C at $v \in C$, that is,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Fv + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,F)$; see [18] for more details.

We can use fixed point algorithms to solve the SFP on the basis of the following observation.

Let $\lambda > 0$ and assume that $x^* \in \Gamma$. Then $Ax^* \in Q$, which implies that $(I - P_Q)Ax^* = 0$, and thus $\lambda A^*(I - P_Q)Ax^* = 0$. Hence, we have the fixed point equation $(I - \lambda A^*(I - P_Q)A)x^* = x^*$. Requiring that $x^* \in C$, we consider the fixed point equation

$$P_C(I - \lambda\nabla f)x^* = P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*. \tag{2.10}$$

It is proved in [8, Proposition 3.2] that the solutions of fixed point equation (2.10) are exactly the solutions of the SFP; namely, for given $x^* \in H_1$, x^* solves the SFP if and only if x^* solves fixed point equation (2.10).

Proposition 2.12 (see [13]) *Given $x^* \in H_1$, the following statements are equivalent.*
(i) *x^* solves the SFP*;
(ii) *x^* solves fixed point equation* (2.10);
(iii) *x^* solves the variational inequality problem* (*VIP*) *of finding $x^* \in C$ such that*

$$\langle \nabla f(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C, \tag{2.11}$$

where $\nabla f = A^*(I - P_Q)A$ and A^* is the adjoint of A.

Proof (i) ⇔ (ii). See the proof in ([8], Proposition 3.2).
(ii) ⇔ (iii). Observe that

$$\begin{aligned} & P_C(I - \lambda A^*(I - P_Q)A)x^* = x^* \\ & \quad\Leftrightarrow \quad \langle (I - \lambda A^*(I - P_Q)A)x^* - x^*, x - x^*\rangle \leq 0, \quad \forall x \in C \end{aligned}$$

$$\Leftrightarrow \quad -\lambda\langle A^*(I-P_Q)Ax^*, x-x^*\rangle \leq 0, \quad \forall x \in C$$
$$\Leftrightarrow \quad \langle \nabla f(x^*), x-x^*\rangle \geq 0, \quad \forall x \in C,$$

where $\nabla f = A^*(I-P_Q)A$. □

Remark 2.13 It is clear from Proposition 2.12 that

$$\Gamma = \mathrm{Fix}(P_C(I-\lambda\nabla f)) = VI(C,\nabla f),$$

for any $\lambda > 0$, where $\mathrm{Fix}(P_C(I-\lambda\nabla f))$ and $VI(C,\nabla f)$ denote the set of fixed points of $P_C(I-\lambda\nabla f)$ and the solution set of VIP.

Proposition 2.14 (see [13]) *There hold the following statements*:

(i) *the gradient*

$$\nabla f_\alpha = \nabla f + \alpha I = A^*(I-P_Q)A + \alpha I$$

is $(\alpha + \|A\|^2)$*-Lipschitz continuous and* α*-strongly monotone*;

(ii) *the mapping* $P_C(I-\lambda\nabla f_\alpha)$ *is a contraction with coefficient*

$$\sqrt{1-\lambda\big(2\alpha-\lambda(\|A\|^2+\alpha)^2\big)}\left(\leq \sqrt{1-\alpha\lambda} \leq 1-\frac{1}{2}\alpha\lambda\right),$$

where $0 < \lambda \leq \frac{\alpha}{(\|A\|^2+\alpha)^2}$;

(iii) *if the SFP is consistent, then the strong* $\lim_{\alpha\to 0} x_\alpha$ *exists and is the minimum norm solution of the SFP.*

3 Main result

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $T : C \to C$ be an uniformly L-Lipschitzian and asymptotically quasi-nonexpansive mapping with* $\mathrm{Fix}(T) \cap \Gamma \neq \emptyset$ *and* $\{k_n\} \subset [1,\infty)$ *for all* $n \in \mathbb{N}$ *such that* $\lim_{n\to\infty} k_n = 1$. *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *be the sequences in C generated by the following algorithm*:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} x_n), \\ u_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n), \\ x_{n+1} = \beta_n u_n + (1-\beta_n) T^n u_n, \end{cases} \tag{3.1}$$

where $\nabla f_{\alpha_n} = \nabla f + \alpha_n I = A^*(I-P_Q)A + \alpha_n I$, *and the sequences* $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions*:

(i) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(ii) $\{\lambda_n\} \in (0, \frac{1}{\|A\|^2})$ *and* $\sum_{n=1}^\infty \lambda_n < \infty$,

(iii) $\sum_{n=1}^\infty \alpha_n < \infty$.

Then the sequence $\{x_n\}$ *converges weakly to an element* $\hat{x} \in \mathrm{Fix}(T) \cap \Gamma$.

Proof We first show that $P_C(I-\lambda\nabla f_\alpha)$ is ζ-averaged for each $\lambda_n \in (0, \frac{2}{\alpha+\|A\|^2})$, where

$$\zeta = \frac{2+\lambda(\alpha+\|A\|^2)}{4}.$$

Indeed, it is easy to see that $\nabla f = A^*(I - P_Q)A$ is $\frac{1}{\|A\|^2}$-ism, that is,

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \frac{1}{\|A\|^2}\|\nabla f(x) - \nabla f(y)\|^2.$$

Observe that

$$\begin{aligned}
&(\alpha + \|A\|^2)\langle \nabla f_\alpha(x) - \nabla f_\alpha(y), x - y\rangle \\
&\quad= (\alpha + \|A\|^2)\bigl[\alpha\|x-y\|^2 + \langle \nabla f(x) - \nabla f(y), x - y\rangle\bigr] \\
&\quad= \alpha^2\|x-y\|^2 + \alpha\langle \nabla f(x) - \nabla f(y), x - y\rangle \\
&\qquad+ \alpha\|A\|^2\|x-y\|^2 + \|A\|^2\langle \nabla f(x) - \nabla f(y), x - y\rangle \\
&\quad\geq \alpha^2\|x-y\|^2 + 2\alpha\langle \nabla f(x) - \nabla f(y), x - y\rangle + \|\nabla f(x) - \nabla f(y)\|^2 \\
&\quad= \|\alpha(x-y) + \nabla f(x) - \nabla f(y)\|^2 \\
&\quad= \|\nabla f(x) - \nabla f(y)\|^2.
\end{aligned}$$

Hence, it follows that $\nabla f_\alpha = \alpha I + A^*(I - P_Q)A$ is $\frac{1}{\alpha+\|A\|^2}$-ism. Thus, $\lambda\nabla f_\alpha$ is $\frac{1}{\lambda(\alpha+\|A\|^2)}$-ism. By Proposition 2.5(iii) the composite $(I - \lambda\nabla f_\alpha)$ is $\frac{\lambda(\alpha+\|A\|^2)}{2}$-averaged. Therefore, noting that P_C is $\frac{1}{2}$-averaged and utilizing Proposition 2.6(iv), we know that for each $\lambda \in (0, \frac{2}{\alpha+\|A\|^2})$, $P_C(I - \lambda\nabla f_\alpha)$ is ζ-averaged with

$$\zeta = \frac{1}{2} + \frac{\lambda(\alpha + \|A\|^2)}{2} - \frac{1}{2}\cdot\frac{\lambda(\alpha + \|A\|^2)}{2} = \frac{2 + \lambda(\alpha + \|A\|^2)}{4} \in (0,1).$$

This shows that $P_C(I - \lambda\nabla f_\alpha)$ is nonexpansive. Furthermore, for $\{\lambda_n\} \in [a,b]$ with $a, b \in (0, \frac{1}{\|A\|^2})$, utilizing the fact that $\lim_{n\to\infty}\frac{1}{\alpha_n+\|A\|^2} = \frac{1}{\|A\|^2}$, we may assume that

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2} = \lim_{n\to\infty}\frac{1}{\alpha_n + \|A\|^2}, \quad \forall n \geq 0.$$

Without loss of generality, we may assume that

$$0 < a \leq \lambda_n \leq b < \frac{1}{\alpha_n + \|A\|^2}, \quad \forall n \geq 0.$$

Consequently, it follows that for each integer $n \geq 0$, $P_C(I - \lambda_n\nabla f_{\alpha_n})$ is ζ_n-averaged with

$$\zeta_n = \frac{1}{2} + \frac{\lambda_n(\alpha_n + \|A\|^2)}{2} - \frac{1}{2}\cdot\frac{\lambda_n(\alpha_n + \|A\|^2)}{2} = \frac{2 + \lambda_n(\alpha_n + \|A\|^2)}{4} \in (0,1).$$

This immediately implies that $P_C(I - \lambda_n\nabla f_{\alpha_n})$ is nonexpansive for all $n \geq 0$.

We divide the remainder of the proof into several steps.

Step 1. We will prove that $\{x_n\}$ is bounded. Indeed, we take fixed $p \in \mathrm{Fix}(T) \cap \Gamma$ arbitrarily. Then we get $P_C(I - \lambda_n\nabla f)p = p$ for $\lambda_n \in (0, \frac{1}{\|A\|^2})$. Since P_C and $(I - \lambda_n\nabla f_{\alpha_n})$ are nonexpansive mappings, then we have

$$\begin{aligned}
\|y_n - p\| &= \big\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f)p\big\| \\
&\le \big\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\big\| \\
&\quad + \big\|P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\big\| \\
&\le \|x_n - p\| + \big\|(I - \lambda_n \nabla f_{\alpha_n})p - (I - \lambda_n \nabla f)p\big\| \\
&= \|x_n - p\| + \big\|p - \lambda_n \nabla f_{\alpha_n} p - (p - \lambda_n \nabla f p)\big\| \\
&= \|x_n - p\| + \|\lambda_n \nabla f p - \lambda_n \nabla f_{\alpha_n} p\| \\
&= \|x_n - p\| + \lambda_n \|\nabla f p - \nabla f_{\alpha_n} p\| \\
&= \|x_n - p\| + \lambda_n \|\nabla f p - \nabla f p - \alpha_n p\| \\
&= \|x_n - p\| + \lambda_n \alpha_n \|p\|
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\|u_n - p\| &= \big\|P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n) - p\big\| \\
&= \big\|P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n) - P_C(I - \lambda_n \nabla f)p\big\| \\
&\le \big\|(x_n - \lambda_n \nabla f_{\alpha_n} y_n) - (p - \lambda_n \nabla f)p\big\| \\
&= \big\|(x_n - p) + (\lambda_n \nabla f p - \lambda_n \nabla f_{\alpha_n} y_n)\big\| \\
&= \big\|(x_n - p) + \lambda_n(\nabla f_p - \nabla f_{\alpha_n} y_n)\big\| \\
&= \big\|(x_n - p) + \lambda_n(\nabla f p - \nabla f_{\alpha_n} p + \nabla f_{\alpha_n} p - \nabla f_{\alpha_n} y_n)\big\| \\
&= \big\|(x_n - p) + \lambda_n\big(\nabla f p - (\nabla f p + \alpha_n p)\big) + \lambda_n(\nabla f_{\alpha_n} p - \nabla f_{\alpha_n} y_n)\big\| \\
&\le \|x_n - p\| + \lambda_n \alpha_n \|p\| + \lambda_n \big\|\nabla f_{\alpha_n}(p) - \nabla f_{\alpha_n}(y_n)\big\| \\
&\le \|x_n - p\| + \lambda_n \alpha_n \|p\| + \lambda_n\big(\alpha_n + \|A\|^2\big)\|p - y_n\|.
\end{aligned} \tag{3.3}$$

Substituting (3.2) into (3.3) and simplifying, we have

$$\begin{aligned}
\|u_n - p\| &\le \|x_n - p\| + \lambda_n \alpha_n \|p\| + \lambda_n\big(\alpha_n + \|A\|^2\big)\|p - y_n\| \\
&= \|x_n - p\| + \lambda_n \alpha_n \|p\| + \lambda_n\big(\alpha_n + \|A\|^2\big)\big[\|x_n - p\| + \lambda_n \alpha_n \|p\|\big] \\
&= \|x_n - p\| + \lambda_n \alpha_n \|p\| + \lambda_n\big(\alpha_n + \|A\|^2\big)\|x_n - p\| + \lambda_n^2 \alpha_n\big(\alpha_n + \|A\|^2\big)\|p\| \\
&= \|x_n - p\| + \lambda_n \alpha_n \|p\| + \lambda_n \alpha_n \|x_n - p\| + \lambda_n \|A\|^2 \|x_n - p\| + \lambda_n^2 \alpha_n^2 \|p\| \\
&\quad + \lambda_n^2 \alpha_n \|A\|^2 \|p\| \\
&= \big(1 + \lambda_n \alpha_n + \lambda_n \|A\|^2\big)\|x_n - p\| + \lambda_n \alpha_n \|p\|\big(1 + \lambda_n \alpha_n + \lambda_n \|A\|^2\big).
\end{aligned} \tag{3.4}$$

Since $u_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n)$ for each $n \ge 0$, then by Proposition 2.2(ii) we have

$$\begin{aligned}
\|u_n - p\|^2 &\le \big\|x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - p\big\|^2 - \big\|x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - u_n\big\|^2 \\
&= \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\big\langle \nabla f_{\alpha_n}(y_n), p - u_n\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\big(\big\langle \nabla f_{\alpha_n}(y_n) - \nabla f_{\alpha_n}(p), p - y_n\big\rangle \\
&\quad + \big\langle \nabla f_{\alpha_n}(p), p - y_n\big\rangle + \big\langle \nabla f_{\alpha_n}(y_n), y_n - u_n\big\rangle\big) \\
&\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\big(\big\langle \nabla f_{\alpha_n}(p), p - y_n\big\rangle + \big\langle \nabla f_{\alpha_n}(y_n), y_n - u_n\big\rangle\big) \\
&= \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\big[\big\langle (\alpha_n I + \nabla f)p, p - y_n\big\rangle + \big\langle \nabla f_{\alpha_n}(y_n), y_n - u_n\big\rangle\big] \\
&\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\big[\alpha_n\langle p, p - u_n\rangle + \big\langle \nabla f_{\alpha_n}(y_n), y_n - u_n\big\rangle\big] \\
&= \|x_n - p\|^2 - \|x_n - y_n + y_n - u_n\|^2 + 2\lambda_n\big[\alpha_n\langle p, p - u_n\rangle + \big\langle \nabla f_{\alpha_n}(y_n), y_n - u_n\big\rangle\big] \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - u_n\rangle - \|y_n - u_n\|^2 \\
&\quad + 2\lambda_n\big[\alpha_n\langle p, p - u_n\rangle + \big\langle \nabla f_{\alpha_n}(y_n), y_n - u_n\big\rangle\big] \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\big\langle x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - y_n, u_n - y_n\big\rangle \\
&\quad + 2\lambda_n\alpha_n\langle p, p - u_n\rangle.
\end{aligned}
$$

Furthermore, by Proposition 2.2(i) we have

$$
\begin{aligned}
&\big\langle x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - y_n, u_n - y_n\big\rangle \\
&\quad = \big\langle x_n - \lambda_n\nabla f_{\alpha_n}(x_n) - y_n, u_n - y_n\big\rangle \\
&\qquad + \big\langle \lambda_n\nabla f_{\alpha_n}(x_n) - \lambda_n\nabla f_{\alpha_n}(y_n), u_n - y_n\big\rangle \\
&\quad \le \big\langle \lambda_n\nabla f_{\alpha_n}(x_n) - \lambda_n\nabla f_{\alpha_n}(y_n), u_n - y_n\big\rangle \\
&\quad \le \lambda_n\big\|\nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(y_n)\big\|\|u_n - y_n\| \\
&\quad \le \lambda_n\big(\alpha_n + \|A\|^2\big)\|x_n - y_n\|\|u_n - y_n\|.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
\|u_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\lambda_n\big(\alpha_n + \|A\|^2\big)\|x_n - y_n\|\|u_n - y_n\| \\
&\quad + 2\lambda_n\alpha_n\|p\|\|p - u_n\|.
\end{aligned}
\tag{3.5}
$$

Consider

$$
\begin{aligned}
&\big[\lambda_n\big(\alpha_n + \|A\|^2\big)\|x_n - y_n\| - \|u_n - y_n\|\big]^2 \\
&\quad = \lambda_n^2\big(\alpha_n + \|A\|^2\big)^2\|x_n - y_n\|^2 \\
&\qquad - 2\lambda_n\big(\alpha_n + \|A\|^2\big)\|x_n - y_n\|\|u_n - y_n\| + \|u_n - y_n\|^2,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&2\lambda_n\big(\alpha_n + \|A\|^2\big)\|x_n - y_n\|\|u_n - y_n\| \\
&\quad = \lambda_n^2\big(\alpha_n + \|A\|^2\big)^2\|x_n - y_n\|^2 + \|u_n - y_n\|^2 \\
&\qquad - \big[\lambda_n\big(\alpha_n + \|A\|^2\big)\|x_n - y_n\| - \|u_n - y_n\|\big]^2 \\
&\quad \le \lambda_n^2\big(\alpha_n + \|A\|^2\big)^2\|x_n - y_n\|^2 + \|u_n - y_n\|^2.
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) into (3.5) and simplifying, we have

$$\begin{aligned}\|u_n-p\|^2 &\le \|x_n-p\|^2-\|x_n-y_n\|^2-\|y_n-u_n\|^2+\lambda_n^2\big(\alpha_n+\|A\|^2\big)^2\|x_n-y_n\|^2\\&\quad+\|u_n-y_n\|^2+2\lambda_n\alpha_n\|p\|\|p-u_n\|\\&=\|x_n-p\|^2-\|x_n-y_n\|^2+\lambda_n^2\big(\alpha_n+\|A\|^2\big)^2\|x_n-y_n\|^2\\&\quad+2\lambda_n\alpha_n\|p\|\|p-u_n\|\\&=\|x_n-p\|^2+\big(\lambda_n^2\big(\alpha_n+\|A\|^2\big)^2-1\big)\|x_n-y_n\|^2+2\lambda_n\alpha_n\|p\|\|p-u_n\|.\end{aligned}\tag{3.7}$$

Substituting (3.4) into (3.7) and simplifying, we have

$$\begin{aligned}\|u_n-p\|^2 &\le \|x_n-p\|^2+\big(\lambda_n^2\big(\alpha_n+\|A\|^2\big)^2-1\big)\|x_n-y_n\|^2\\&\quad+2\lambda_n\alpha_n\|p\|\big[\big(1+\lambda_n\alpha_n+\lambda_n\|A\|^2\big)\|x_n-p\|\\&\quad+\lambda_n\alpha_n\|p\|\big(1+\lambda_n\alpha_n+\lambda_n\|A\|^2\big)\big]\\&=\|x_n-p\|^2+\big(\lambda_n^2\big(\alpha_n+\|A\|^2\big)^2-1\big)\|x_n-y_n\|^2\\&\quad+2\lambda_n\alpha_n\|p\|\big(1+\lambda_n\alpha_n+\lambda_n\|A\|^2\big)\|x_n-p\|\\&\quad+2\lambda_n^2\alpha_n^2\|p\|^2\big(1+\lambda_n\alpha_n+\lambda_n\|A\|^2\big).\end{aligned}\tag{3.8}$$

Consequently, utilizing Lemma 2.11(ii) and the last relations, we conclude that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \big\|\beta_n u_n+(1-\beta_n)T^n u_n-\big(\beta_n+(1-\beta_n)\big)p\big\|^2\\&=\big\|\beta_n u_n-\beta_n p+(1-\beta_n)T^n u_n-(1-\beta_n)p\big\|^2\\&=\big\|\beta_n(u_n-p)+(1-\beta_n)\big(T^n u_n-p\big)\big\|^2\\&=\beta_n\|u_n-p\|^2+(1-\beta_n)\big\|T^n u_n-p\big\|^2-\beta_n(1-\beta_n)\big\|u_n-T^n u_n\big\|^2\\&\le\beta_n\|u_n-p\|^2+(1-\beta_n)k_n^2\|u_n-p\|^2-\beta_n(1-\beta_n)\big\|u_n-T^n u_n\big\|^2\\&=\big(\beta_n+(1-\beta_n)k_n^2\big)\|u_n-p\|^2-\beta_n(1-\beta_n)\big\|u_n-T^n u_n\big\|^2\\&=\big(\beta_n+(1-\beta_n)k_n^2\big)\big\{\|x_n-p\|^2+\big(\lambda_n^2\big(\alpha_n+\|A\|^2\big)^2-1\big)\|x_n-y_n\|^2\\&\quad+2\lambda_n\alpha_n\|p\|\big(1+\lambda_n\alpha_n+\lambda_n\|A\|^2\big)\|x_n-p\|\\&\quad+2\lambda_n^2\alpha_n^2\|p\|^2\big(1+\lambda_n\alpha_n+\lambda_n\|A\|^2\big)\big\}-\beta_n(1-\beta_n)\big\|u_n-T^n u_n\big\|^2\\&=\big(\beta_n+(1-\beta_n)k_n^2\big)\|x_n-p\|^2\\&\quad+\big(\beta_n+(1-\beta_n)k_n^2\big)\big(\lambda_n^2\big(\alpha_n+\|A\|^2\big)^2-1\big)\|x_n-y_n\|^2\\&\quad+2\lambda_n\alpha_n\big(\beta_n+(1-\beta_n)k_n^2\big)\big(1+\lambda_n\alpha_n+\alpha_n\|A\|^2\big)\|p\|\|x_n-p\|^2\\&\quad+2\big(\beta_n+(1-\beta_n)k_n^2\big)\lambda_n^2\alpha_n^2\|p\|^2\big(1+\lambda_n\alpha_n+\lambda_n\|A\|^2\big)\\&\quad-\beta_n(1-\beta_n)\big\|u_n-T^n u_n\big\|^2\\&=\big(k_n^2-\beta_n\big(k_n^2-1\big)\big)\|x_n-p\|^2\\&\quad+\big(k_n^2-\beta_n\big(k_n^2-1\big)\big)\big(\lambda_n^2\big(\alpha_n+\|A\|^2\big)^2-1\big)\|x_n-y_n\|^2\\&\quad+2\lambda_n\alpha_n\big(k_n^2-\beta_n(k_n^2-1)\big)\big(1+\lambda_n\alpha_n+\alpha_n\|A\|^2\big)\|p\|\|x_n-p\|^2\end{aligned}$$

$$+ 2\big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\lambda_n^2\alpha_n^2\|p\|^2\big(1 + \lambda_n\alpha_n + \lambda_n\|A\|^2\big)$$
$$- \beta_n(1 - \beta_n)\big\|u_n - T^n u_n\big\|^2. \tag{3.9}$$

Since $\lim_{n\to\infty} k_n = 1$, (i)-(iii) and by Corollary 2.8, we deduce that

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists for each } p \in \mathrm{Fix}(T) \cap \Gamma, \tag{3.10}$$

and the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are bounded. It follows that

$$\big\|T^n x_n - p\big\| \le k_n\|x_n - p\|.$$

Hence $\{T^n x_n - p\}$ is bounded.

Step 2. We will prove that

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0.$$

From (3.9) we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\|x_n - p\|^2 \\
&\quad + \big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\big(\lambda_n^2\big(\alpha_n + \|A\|^2\big)^2 - 1\big)\|x_n - y_n\|^2 \\
&\quad + 2\lambda_n\alpha_n\big(k_n^2 - \beta_n(k_n^2 - 1)\big)\big(1 + \lambda_n\alpha_n + \alpha_n\|A\|^2\big)\|p\|\|x_n - p\|^2 \\
&\quad + 2\big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\lambda_n^2\alpha_n^2\|p\|^2\big(1 + \lambda_n\alpha_n + \lambda_n\|A\|^2\big) \\
&\quad - \beta_n(1 - \beta_n)\big\|u_n - T^n u_n\big\|^2 \\
&= \big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\|x_n - p\|^2 \\
&\quad + \big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\big(\lambda_n^2\big(\alpha_n + \|A\|^2\big)^2 - 1\big)\|x_n - y_n\|^2 \\
&\quad + \alpha_n\big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)M_1 + \alpha_n\big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)M_2 \\
&\quad - \beta_n(1 - \beta_n)\big\|u_n - T^n u_n\big\|^2 \\
&= \big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\|x_n - p\|^2 \\
&\quad - \big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\big(1 - \lambda_n^2\big(\alpha_n + \|A\|^2\big)^2\big)\|x_n - y_n\|^2 \\
&\quad + \alpha_n\big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)(M_1 + M_2) - \beta_n(1 - \beta_n)\big\|u_n - T^n u_n\big\|^2,
\end{aligned}$$

where $M_1 = \sup_{n\ge 0}\{2\lambda_n(1 + \lambda_n\alpha_n + \alpha_n\|A\|^2)\|p\|\|x_n - p\|^2\} < \infty$ and

$$M_2 = \sup_{n\ge 0}\big\{2\lambda_n^2\alpha_n\|p\|^2\big(1 + \lambda_n\alpha_n + \lambda_n\|A\|^2\big)\big\} < \infty.$$

So,

$$\begin{aligned}
&\big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\big(1 - \lambda_n^2\big(\alpha_n + \|A\|^2\big)^2\big)\|x_n - y_n\|^2 + \beta_n(1 - \beta_n)\big\|u_n - T^n u_n\big\|^2 \\
&\quad \le \big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\big(k_n^2 - \beta_n\big(k_n^2 - 1\big)\big)(M_1 + M_2).
\end{aligned}$$

Since $\lim_{n\to\infty} k_n = 1$, $\alpha_n \to 0$, (i) and from (3.10), we have

$$\lim_{n\to 0}\|x_n - y_n\| = \lim_{n\to 0}\|u_n - T^n u_n\| = 0. \tag{3.11}$$

Furthermore, we obtain

$$\begin{aligned}\|y_n - u_n\| &= \|P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)) - P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n))\| \\ &\le \|(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)) - (x_n - \lambda_n \nabla f_{\alpha_n}(y_n))\| \\ &= \lambda_n \|\nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(y_n)\| \\ &\le \lambda_n(\alpha_n + \|A\|^2)\|x_n - y_n\|.\end{aligned}$$

This together with (3.11) implies that

$$\lim_{n\to 0}\|y_n - u_n\| = 0. \tag{3.12}$$

Also,

$$\|x_n - u_n\| \le \|x_n - y_n\| + \|y_n - u_n\|$$

together with (3.11) and (3.12) implies that

$$\lim_{n\to 0}\|x_n - u_n\| = 0. \tag{3.13}$$

We can rewrite (3.11) from (3.13) by

$$\lim_{n\to 0}\|x_n - T^n u_n\| = 0. \tag{3.14}$$

Consider

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\beta_n u_n + (1-\beta_n)T^n u_n - x_n\| \\ &\le \beta_n\|u_n - x_n\| + (1-\beta_n)\|T^n u_n - x_n\|.\end{aligned}$$

From (3.13) and (3.14), we obtain

$$\|x_{n+1} - x_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.15}$$

Next, we will show that (3.11) implies that

$$\lim_{n\to 0}\|u_n - Tu_n\| = 0. \tag{3.16}$$

We compute that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|P_C(x_{n+1} - \lambda_{n+1}\nabla f_{\alpha_{n+1}} x_{n+1}) - P_C(x_n - \lambda_n \nabla f_{\alpha_n} x_n)\| \\ &= \|P_C(I - \lambda_{n+1}\nabla f_{\alpha_{n+1}})x_{n+1} - P_C(I - \lambda_n \nabla f_{\alpha_n})x_n\|\end{aligned}$$

$$
\begin{aligned}
&\le \left\| P_C(I-\lambda_{n+1}\nabla f_{\alpha_{n+1}})x_{n+1} - P_C(I-\lambda_{n+1}\nabla f_{\alpha_{n+1}})x_n \right\| \\
&\quad + \left\| P_C(I-\lambda_{n+1}\nabla f_{\alpha_{n+1}})x_n - P_C(I-\lambda_n\nabla f_{\alpha_n})x_n \right\| \\
&\le \|x_{n+1}-x_n\| + \left\| (I-\lambda_{n+1}\nabla f_{\alpha_{n+1}})x_n - (I-\lambda_n\nabla f_{\alpha_n})x_n \right\| \\
&= \|x_{n+1}-x_n\| + \left\| x_n - \lambda_{n+1}\nabla f_{\alpha_{n+1}}x_n - (x_n - \lambda_n\nabla f_{\alpha_n}x_n) \right\| \\
&= \|x_{n+1}-x_n\| + \left\| \lambda_n\nabla f_{\alpha_n}x_n - \lambda_{n+1}\nabla f_{\alpha_{n+1}}x_n \right\| \\
&= \|x_{n+1}-x_n\| + \left\| \lambda_n(\nabla f + \alpha_n)x_n - \lambda_{n+1}(\nabla f + \alpha_{n+1})x_n \right\| \\
&= \|x_{n+1}-x_n\| + \left\| \lambda_n\nabla f x_n + \lambda_n\alpha_n x_n - (\lambda_{n+1}\nabla f x_n + \lambda_{n+1}\alpha_{n+1}x_n) \right\| \\
&= \|x_{n+1}-x_n\| + \left\| (\lambda_n - \lambda_{n+1})\nabla f x_n + \lambda_n\alpha_n x_n - \lambda_{n+1}\alpha_{n+1}x_n \right\| \\
&= \|x_{n+1}-x_n\| + \left\| (\lambda_n - \lambda_{n+1})\nabla f x_n + \lambda_n\alpha_n x_n - \lambda_n\alpha_{n+1}x_n \right. \\
&\quad \left. + \lambda_n\alpha_{n+1}x_n - \lambda_{n+1}\alpha_{n+1}x_n \right\| \\
&= \|x_{n+1}-x_n\| + \left\| (\lambda_n - \lambda_{n+1})\nabla f x_n + \lambda_n(\alpha_n - \alpha_{n+1})x_n + (\lambda_n - \lambda_{n+1})\alpha_{n+1}x_n \right\| \\
&\le \|x_{n+1}-x_n\| + |\lambda_n - \lambda_{n+1}|\|\nabla f x_n\| + \lambda_n|\alpha_n - \alpha_{n+1}|\|x_n\| \\
&\quad + \alpha_{n+1}|\lambda_n - \lambda_{n+1}|\|x_n\|.
\end{aligned}
$$

From conditions (ii), (iii) and (3.15), we obtain that

$$
\|y_{n+1} - y_n\| \to 0 \quad (\text{as } n \to \infty) \tag{3.17}
$$

and

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \left\| P_C(x_{n+1} - \lambda_{n+1}\nabla f_{\alpha_{n+1}}y_{n+1}) - P_C(x_n - \lambda_n\nabla f_{\alpha_n}y_n) \right\| \\
&\le \left\| (x_{n+1} - \lambda_{n+1}\nabla f_{\alpha_{n+1}}y_{n+1}) - (x_n - \lambda_n\nabla f_{\alpha_n}y_n) \right\| \\
&= \left\| (x_{n+1} - x_n) + (\lambda_n\nabla f_{\alpha_n}y_n - \lambda_{n+1}\nabla f_{\alpha_n+1}y_{n+1}) \right\| \\
&\le \|x_{n+1}-x_n\| + \|\lambda_n\nabla f_{\alpha_n}y_n - \lambda_{n+1}\nabla f_{\alpha_{n+1}}y_{n+1}\| \\
&= \|x_{n+1}-x_n\| + \left\| \lambda_n(\nabla f + \alpha_n)y_n - \lambda_{n+1}(\nabla f + \alpha_{n+1})y_{n+1} \right\| \\
&= \|x_{n+1}-x_n\| + \left\| \lambda_n\nabla f y_n + \lambda_n\alpha_n y_n - (\lambda_{n+1}\nabla f y_{n+1} + \lambda_{n+1}\alpha_{n+1}y_{n+1}) \right\| \\
&= \|x_{n+1}-x_n\| + \left\| (\lambda_n\nabla f y_n - \lambda_{n+1}\nabla f y_{n+1}) + \lambda_n\alpha_n y_n - \lambda_{n+1}\alpha_{n+1}y_{n+1} \right\| \\
&\le \|x_{n+1}-x_n\| + \|\lambda_n\nabla f y_n - \lambda_{n+1}\nabla f y_{n+1}\| + \|\lambda_n\alpha_n y_n - \lambda_{n+1}\alpha_{n+1}y_{n+1}\| \\
&= \|x_{n+1}-x_n\| + \left\| (\lambda_n\nabla f y_n - \lambda_n\nabla f y_{n+1}) + (\lambda_n\nabla f y_{n+1} - \lambda_{n+1}\nabla f y_{n+1}) \right\| \\
&\quad + \left\| (\lambda_n\alpha_n y_n - \lambda_n\alpha_n y_{n+1}) + (\lambda_n\alpha_n y_{n+1} - \lambda_{n+1}\alpha_{n+1}y_{n+1}) \right\| \\
&\le \|x_{n+1}-x_n\| + \lambda_n\|\nabla f y_n - \nabla f y_{n+1}\| + |\lambda_n - \lambda_{n+1}|\|\nabla f y_{n+1}\| \\
&\quad + \lambda_n\alpha_n\|y_n - y_{n+1}\| + |\lambda_n\alpha_n - \lambda_{n+1}\alpha_{n+1}|\|y_{n+1}\|.
\end{aligned}
$$

From conditions (ii), (iii), (3.15) and (3.17), we obtain that

$$
\|u_{n+1} - u_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.18}
$$

Since T is uniformly L-Lipschitzian continuous, then

$$\begin{aligned}\|u_n - Tu_n\| &\leq \|u_n - u_{n+1}\| + \left\|u_{n+1} - T^{n+1}u_{n+1}\right\| + \left\|T^{n+1}u_{n+1} - T^{n+1}u_n\right\| \\ &\quad + \left\|T^{n+1}u_n - Tu_n\right\| \\ &\leq \|u_n - u_{n+1}\| + \left\|u_{n+1} - T^{n+1}u_{n+1}\right\| + L\|u_n - u_{n+1}\| + L\left\|T^n u_n - u_n\right\|.\end{aligned}$$

Since $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$ and $\lim_{n\to\infty} \|u_n - T^n u_n\| = 0$, it follows that

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0. \tag{3.19}$$

Step 3. We will show that $\hat{x} \in \mathrm{Fix}(T) \cap \Gamma$.

We have from (3.11)

$$\|x_n - y_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.20}$$

Since $\nabla f = A^*(I - P_Q)A$ is Lipschitz continuous and from (3.11), we have

$$\lim_{n\to\infty} \left\|\nabla f(x_n) - \nabla f(y_n)\right\| = 0.$$

Since $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some $\hat{x}$.

First, we show that $\hat{x} \in \Gamma$. Since $\|x_n - y_n\| \to 0$, it is known that $y_{n_i} \rightharpoonup \hat{x}$.

Put

$$Aw = \begin{cases} \nabla f w + N_C w & \text{if } w \in C, \\ \emptyset & \text{if } w \notin C, \end{cases}$$

where $N_C w = \{z \in H_1 : \langle w - v, z\rangle \geq 0, \forall v \in C\}$. Then A is maximal monotone and $0 \in Aw$ if and only if $w \in VI(C, \nabla f)$; see [21] for more details. Let $(w, z) \in G(A)$, we have

$$z \in Aw = \nabla f w + N_C w,$$

and hence

$$z - \nabla f w \in N_C w.$$

So, we have

$$\langle w - v, z - \nabla f w\rangle \geq 0, \quad \forall v \in C.$$

On the other hand, from

$$u_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n) \quad \text{and} \quad w \in C,$$

we have

$$\langle x_n - \lambda_n \nabla f_{\alpha_n} y_n - u_n, u_n - w\rangle \geq 0,$$

and hence

$$\left\langle w - u_n, \frac{u_n - x_n}{\lambda_n} + \nabla f_{\alpha_n} y_n \right\rangle \geq 0.$$

Therefore from $z - \nabla f w \in N_C w$ and $\{u_{n_i}\} \in C$ it follows that

$$\begin{aligned}
\langle w - u_{n_i}, z\rangle &\geq \langle w - u_{n_i}, \nabla f w\rangle \\
&\geq \langle w - u_{n_i}, \nabla f w\rangle - \left\langle w - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} + \nabla f_{\alpha_{n_i}} y_{n_i} \right\rangle \\
&= \langle w - u_{n_i}, \nabla f w\rangle - \left\langle w - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} + \nabla f y_{n_i} \right\rangle - \alpha_{n_i} \langle w - u_{n_i}, y_{n_i}\rangle \\
&= \langle w - u_{n_i}, \nabla f w - \nabla f u_{n_i}\rangle + \langle w - u_{n_i}, \nabla f u_{n_i} - \nabla f y_{n_i}\rangle \\
&\quad - \left\langle w - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle - \alpha_{n_i} \langle w - u_{n_i}, y_{n_i}\rangle \\
&\leq \langle w - u_{n_i}, \nabla f u_{n_i} - \nabla f y_{n_i}\rangle - \left\langle w - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\quad - \alpha_{n_i} \langle w - u_{n_i}, y_{n_i}\rangle.
\end{aligned}$$

Hence, we obtain

$$\langle w - \hat{x}, z\rangle \geq 0 \quad \text{as } i \to \infty.$$

Since A is maximal monotone, we have $\hat{x} \in A_0^{-1}$, and hence $\hat{x} \in VI(C, \nabla f)$. Thus it is clear that $\hat{x} \in \Gamma$.

Next, we show that $\hat{x} \in \operatorname{Fix}(T)$. Indeed, since $y_{n_i} \rightharpoonup \hat{x}$ and $\|u_{n_i} - Tu_{n_i}\| \to 0$, by (3.16) and Lemma 2.7, we get $\hat{x} \in \operatorname{Fix}(T)$. Therefore, we have $\hat{x} \in \operatorname{Fix}(T) \cap \Gamma$.

Now we prove that $x_n \rightharpoonup \hat{x}$ and $y_n \rightharpoonup \hat{x}$.

Suppose the contrary and let $\{x_{n_k}\}$ be another subsequences of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup x^*$. Then $x^* \in \operatorname{Fix}(T) \cap \Gamma$. Let us show that $\hat{x} = x^*$. Assume that $\hat{x} \neq x^*$. From the Opial condition [22], we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \hat{x}\| &= \lim_{k\to\infty} \inf \|x_{n_k} - \hat{x}\| \\
&< \lim_{k\to\infty} \inf \|x_{n_k} - x^*\| \\
&= \lim_{n\to\infty} \|x_n - x^*\| \\
&= \lim_{k\to\infty} \inf \|x_{n_k} - x^*\| \\
&< \lim_{k\to\infty} \inf \|x_{n_k} - \hat{x}\| \\
&= \lim_{n\to\infty} \|x_n - \hat{x}\|.
\end{aligned}$$

This is a contradiction. Thus, we have $\hat{x} = x^*$. This implies

$$x_n \rightharpoonup \hat{x} \in \operatorname{Fix}(T) \cap \Gamma.$$

Further, from $\|x_n - y_n\| \to 0$ it follows that $y_n \rightharpoonup \hat{x}$. This shows that both sequences $\{y_n\}$ and $\{u_n\}$ converge weakly to $\hat{x} \in \mathrm{Fix}(T) \cap \Gamma$. This completes the proof. □

Utilising Theorem 3.1, we have the following new results in the setting of real Hilbert spaces.

Take $T^n \equiv T$ in Theorem 3.1. Therefore the conclusion follows.

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $T : C \to C$ be an uniformly L-Lipschitzian and quasi-nonexpansive mapping with* $\mathrm{Fix}(T) \cap \Gamma \neq \emptyset$. *Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences in C generated by the following algorithm*:

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} x_n), \\ u_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n), \\ x_{n+1} = \beta_n u_n + (1 - \beta_n) T^n u_n, \end{cases} \tag{3.21}$$

where $\nabla f_{\alpha_n} = \nabla f + \alpha_n I = A^*(I - P_Q)A + \alpha_n I$, and the sequences $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(ii) $\{\lambda_n\} \in (0, \frac{2}{\|A\|^2})$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$,
(iii) $\sum_{n=1}^{\infty} \alpha_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to an element $\hat{x} \in \mathrm{Fix}(T) \cap \Gamma$.

Take $T^n \equiv I$ (identity mappings) in Theorem 3.1. Therefore the conclusion follows.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $T : C \to C$ be an uniformly L-Lipschitzian with* $\mathrm{Fix}(T) \cap \Gamma \neq \emptyset$. *Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences in C generated by the following algorithm*:

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} x_n), \\ u_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n), \\ x_{n+1} = \beta_n u_n + (1 - \beta_n) T^n u_n, \end{cases} \tag{3.22}$$

where $\nabla f_{\alpha_n} = \nabla f + \alpha_n I = A^*(I - P_Q)A + \alpha_n I$, and the sequences $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(ii) $\{\lambda_n\} \in (0, \frac{2}{\|A\|^2})$,
(iii) $\sum_{n=1}^{\infty} \alpha_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to an element $\hat{x} \in \mathrm{Fix}(T) \cap \Gamma$.

Remark 3.4 Theorem 3.1 improves and extends [8, Theorem 5.7] in the following respects:

(a) The iterative algorithm [8, Theorem 5.7] is extended for developing our Mann's type extragradient algorithm in Theorem 3.1.

(b) The technique of proving weak convergence in Theorem 3.1 is different from that in [8, Theorem 5.7] because our technique uses asymptotically quasi-nonexpansive mappings and the property of maximal monotone mappings.
(c) The problem of finding a common element of $\mathrm{Fix}(T) \cap \Gamma$ for asymptotically quasi-nonexpansive mappings is more general than that for nonexpansive mappings and the problem of finding a solution of the (SFP) in [8, Theorem 5.7].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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