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Convergence results for a common solution of a finite family of variational inequality problems for monotone mappings with Bregman distance function

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Abstract

In this paper, we introduce an iterative process which converges strongly to a common solution of a finite family of variational inequality problems for monotone mappings with Bregman distance function. Our convergence theorem is applied to the convex minimization problem. Our theorems extend and unify most of the results that have been proved for the class of monotone mappings.
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1 Introduction

Throughout this paper, E is a real reflexive Banach space with E^* as its dual and $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function. We denote by $\operatorname{dom} f$ the domain of f, defined by $\operatorname{dom} f := \{x \in E : f(x) < +\infty\}$. For any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, the *right-hand derivative* of f at x in the direction of y is defined by

$$f^0(x,y) := \lim_{t\to 0^+} \frac{f(x+ty) - f(x)}{t}. \tag{1.1}$$

The function f is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0^+}(f(x+ty) - f(x))/t$ exists for any $y \in E$. In this case, $f^0(x,y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. The function f is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. We say that f is *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, +\infty)$ defined by

$$D_f(x,y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle$$

is called the *Bregman distance with respect to* f [1].

A *Bregman projection* with respect to f [1] of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty, closed and convex set $C \subset \operatorname{int}(\operatorname{dom} f)$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f\big(P_C^f(x), x\big) = \inf\big\{D_f(y,x) : y \in C\big\}.$$

Remark 1.1 If E is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have that $\nabla f(x) = 2J(x)$ for all $x \in E$, where J the normalized duality mapping from E into E^*, and hence $D_f(x,y)$ becomes $D_f(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$, which is the Lyapunov function introduced by Alber [2] and studied by many authors (see, *e.g.*, [3–9] and the references therein). In addition, under the same condition, the Bregman projection $P_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$ (see, *e.g.*, [2]) which is defined by

$$\phi\big(\Pi_C(x), x\big) = \min_{y \in C} \phi(y,x). \tag{1.2}$$

If $E = H$, a Hilbert space, J is the identity mapping, and hence the Bregman projection $P_C^f(x)$ reduces to the metric projection of H on to C, $P_C(x)$.

A mapping $A : D(A) \subset E \to E^*$ is said to be γ*-inverse strongly monotone* if there exists a positive real number γ such that

$$\langle Ax - Ay, x - y\rangle \geq \gamma \|Ax - Ay\|^2 \quad \text{for all } x, y \in D(A). \tag{1.3}$$

A is said to be *monotone* if, for each $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y\rangle \geq 0. \tag{1.4}$$

Clearly, the class of monotone mappings includes the class of γ-inverse strongly monotone mappings.

Let C be a nonempty, closed and convex subset of E and $A : C \to E^*$ be a monotone mapping. The problem of finding

$$\text{a point } u \in C \text{ such that } \langle Au, v - u\rangle \geq 0 \text{ for all } v \in C \tag{1.5}$$

is called *the variational inequality problem*. The set of solutions of the variational inequality is denoted by $VI(C,A)$.

Variational inequality problems are related with the convex minimization problem, the zero of monotone mappings and the complementarity problem. Consequently, many researchers (see, *e.g.*, [3, 5, 10–15]) have made efforts to obtain iterative methods for approximating solutions of variational inequality problems.

If $E = H$, a Hilbert space, Iiduka *et al.* [16] introduced the following projection algorithm:

$$x_1 = x \in C, \quad x_{n+1} = P_C(x_n - \alpha_n A x_n) \quad \text{for any } n \geq 1, \tag{1.6}$$

where P_C is the metric projection from H onto C and $\{\alpha_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.6) *converges weakly* to some element of $VI(C,A)$ provided that A is a γ-inverse strongly monotone mapping.

If E is a 2-uniformly convex and uniformly smooth Banach space, and A is γ-inverse strongly monotone, Iiduka and Takahashi [17] introduced the following iteration scheme for finding a solution of the variational inequality problem:

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \alpha_n Ax_n) \quad \text{for any } n \geq 1, \tag{1.7}$$

where Π_C is the generalized projection from E onto C, J is the normalized duality mapping from E into E^* and $\{\alpha_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.7) *converges weakly* to some element of $VI(C,A)$.

It is worth mentioning that the convergence obtained above is *weak convergence*. Our concern now is to look for an iteration scheme which converges strongly to a solution of the variational inequality problem for a monotone mapping A.

In this regard, when E is a 2-uniformly convex and uniformly smooth Banach space and A is a γ-inverse strongly monotone mapping satisfying $\|Au\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(C,A)$ for $VI(C,A) \neq \emptyset$, Iiduka and Takahashi [10] studied the following iterative scheme for a solution of the variational inequality problem:

$$\begin{cases} x_0 \in K \quad \text{chosen arbitrarily,} \\ y_n = \Pi_C J^{-1}(Jx_n - \alpha_n Ax_n), \\ C_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \quad n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ is a positive real sequence satisfying certain mild conditions and $\Pi_{C_n \cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$, J is the duality mapping from E into E^*. Then they proved that the sequence $\{x_n\}$ converges strongly to an element of $VI(C,A)$.

Recently, Zegeye and Shahzad [18] studied the following iterative scheme for a common point of a solution of two variational inequality problems for continuous monotone mappings in a uniformly smooth and strictly convex real Banach space E which also enjoys the Kadec-Klee property:

$$\begin{cases} x_0 \in C_0 = C \quad \text{chosen arbitrarily,} \\ u_n = T_{1,\gamma_n} x_n; \qquad v_n = T_{2,\gamma_n} x_n, \\ w_n = J^{-1}(\beta J u_n + (1-\beta) J v_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad n \geq 0, \end{cases} \tag{1.9}$$

where $T_{i,\gamma} x := \{z \in C : \langle A_i z, y - z\rangle + \frac{1}{\gamma}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}$ for all $x \in E$, $i = 1, 2$, and $\beta, \gamma_n \in (0,1)$ satisfy certain mild conditions. Then they proved that the sequence $\{x_n\}$ converges strongly to $\Pi_F(x_0)$, where Π_F is the generalized projection from E onto $F := \bigcap_{i=1}^{2} VI(C, A_i) \neq \emptyset$.

In 1967, Bregman [1] discovered an elegant and effective technique for using the so-called *Bregman distance function* $D_f(\cdot,\cdot)$ in the process of designing and analyzing feasibility and optimization algorithms. Using Bregman's distance function and its properties, authors have opened a growing area of research not only for iterative algorithms of solving feasibility and optimization problems but also for algorithms of solving nonlinear, equilibrium, variational inequality, fixed point problems and others (see, *e.g.*, [19–25] and the references therein).

In 2010, Reich and Sabach [25] proposed an algorithm for finding a common zero point of a finite family of maximal monotone mappings $A_i : E \to 2^{E^*}$ $(i = 1, 2, \dots, N)$ in a general reflexive Banach space E as follows:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ y_n^i = \operatorname{Res}_{\lambda_n^i A_i}(x_n + e_n^i), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n^i = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = P^f_{C_n \cap Q_n}(x_0), \quad \forall n \geq 0, \end{cases} \tag{1.10}$$

where $\{\lambda_n^i\}_{i=1}^N \subset (0, \infty)$, $\{e_n^i\}_{i=1}^N$ are error sequences in E with $e_n^i \to 0$ and P_C^f is the Bregman projection with respect to f from E onto a closed and convex subset C of E. Those authors showed that the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common element in $\bigcap_{i=1}^N A^{-1}(0^*) = \bigcap_{i=1}^N VI(E, A_i)$ under some mild conditions. Similar results are also available in [26, 27].

Remark 1.2 But it is worth mentioning that the iteration processes (1.8)-(1.10) seem difficult in the sense that at each stage of iteration, the set(s) C_n and (or) Q_n is (are) computed and the next iterate is taken as the Bregman projection of x_0 onto the intersection of C_n and Q_n (or Q_n). This seems difficult to do in applications.

It is our purpose in this paper to introduce an iterative scheme $\{x_n\}$ which converges strongly to a common solution of a finite family of variational inequality problems for monotone mappings in real reflexive Banach spaces. Our scheme does not involve computations of C_n or Q_n for each $n \geq 1$. Furthermore, we apply our convergence theorem to a convex minimization problem. Our theorems extend and unify most of the results that have been proved for this important class of nonlinear operators.

2 Preliminaries

Let $x \in \operatorname{int}(\operatorname{dom} f)$. The subdifferential of f at x is the convex set defined by $\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \forall y \in E\}$, where the *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}$.

The function f is said to be:

(i) *Essentially smooth* if ∂f is both locally bounded and single-valued on its domain.

(ii) *Essentially strictly convex* if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} f$.

(iii) *Legendre* if it is both essentially smooth and essentially strictly convex.

We remark that we have the following:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [19], Theorem 5.4).

(ii) $(\partial f)^{-1} = \partial f^*$ (see [28]).

(iii) f is Legendre if and only if f^* is Legendre (see [19], Corollary 5.5).

(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}(\operatorname{dom} f)$ (see [19], Theorem 5.10).

When the subdifferential of f is single-valued, then $\partial f = \nabla f$ (see [29]).

A function f on E is *coercive* [30] if the sublevel set of f is bounded; equivalently, $\lim_{\|x\|\to\infty} f(x) = \infty$.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The *modulus of total convexity* of f at $x \in \operatorname{dom} f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in \operatorname{dom} f, \|y - x\| = t\}.$$

The function f is called *totally convex* at x if $\nu_f(x,t) > 0$, whenever $t > 0$. The function f is called totally convex if it is totally convex at any point $x \in \operatorname{int}(\operatorname{dom} f)$ and it is said to be totally convex on bounded sets if $\nu_f(B,t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B,t) := \inf\{V_f(x,t) : x \in B \cap \operatorname{dom} f\}.$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [22], Theorem 2.10). The following lemmas will be useful in the proof of our main result.

Lemma 2.1 [31] *The function $f : E \to (-\infty, +\infty]$ is totally convex on bounded subsets of E if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in* $\operatorname{int}(\operatorname{dom} f)$ *and* $\operatorname{dom} f$, *respectively, such that the first one is bounded*,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.2 [22] *Let C be a nonempty, closed and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function, and let $x \in E$. Then*:

(i) $z = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \le 0$, $\forall y \in C$.

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \le D_f(y,x)$, $\forall y \in C$.

Lemma 2.3 [29] *Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semicontinuous and convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \le \sum_{i=1}^N t_i D_f(z, x_i). \tag{2.1}$$

Lemma 2.4 [32] *Let $f : E \to \mathbb{R}$ be Gâteaux differentiable on* $\operatorname{int}(\operatorname{dom} f)$ *such that ∇f^* is bounded on bounded subsets of* $\operatorname{dom} f^*$. *Let $x^* \in X$ and $\{x_n\} \subset E$. If $\{D_f(x, x_n)\}$ is bounded, so is the sequence $\{x_n\}$.*

Let $f : E \to \mathbb{R}$ be a Legendre and Gâteaux differentiable function. Following [2] and [33], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*. \tag{2.2}$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \quad \text{for all } x \in E \text{ and } x^* \in E^*. \tag{2.3}$$

Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*), \tag{2.4}$$

$\forall x \in E$ and $x^*, y^* \in E^*$ (see [34]).

Lemma 2.5 [35] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, \quad n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\limsup_{n\to\infty} \delta_n \leq 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 [4] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all* (*sufficiently large*) *numbers $k \in \mathbb{N}$*:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

Following the agreement in [26], we have the following lemma.

Lemma 2.7 *Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function and C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be a continuous monotone mapping. For $r > 0$ and $x \in E$, define the mapping $F_r : E \to C$ as follows*:

$$F_r x := \left\{ z \in C : \langle Az, y - z \rangle + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in E$. Then the following hold:
(1) *F_r is single-valued*;
(2) $F(F_r) = VI(C, A)$;
(3) $\phi(p, F_r x) + \phi(F_r x, x) \leq \phi(p, x)$ *for* $p \in F(F_r)$;
(4) *$VI(C, A)$ is closed and convex.*

3 Main result

Let C be a nonempty, closed and convex subset of E. Let $A_i : C \to E^*$, for $i = 1, 2, \ldots, N$, be continuous monotone mappings. For $r > 0$, define $T_{i,r}x := \{z \in C : \langle A_i z, y - z \rangle + \frac{1}{r} \langle \nabla f(z) -$

$\nabla f(x), y - z\rangle \geq 0, \forall y \in C\}$ for all $x \in E$ and $i \in \{1, 2, \dots, N\}$, where f is a Legendre and convex function from E into $(-\infty, +\infty)$. Then, in what follows, we shall study the following iteration process:

$$\begin{cases} x_0 = u \in C \quad \text{chosen arbitrarily}, \\ w_n = T_{N,r_n} \circ T_{N-1,r_n} \circ \cdots \circ T_{1,r_n} x_n, \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(w_n)), \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} = \infty$, and $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$.

Theorem 3.1 *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} f)$ *and $A_i : C \to E^*$, for $i = 1, 2, \dots, N$, be a finite family of continuous monotone mappings with $\mathcal{F} := \bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$. Let $\{x_n\}_{n\geq 0}$ be a sequence defined by* (3.1). *Then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}}^f(u)$.*

Proof By Lemma 2.7 we have that each $VI(C, A_i)$ for each $i \in \{1, 2, \dots, N\}$ and hence $\mathcal{F}$ are closed and convex. Thus, we can take $x^* := P_{\mathcal{F}}^f u$. Let $u_{n,1} = T_{1,r_n} x_n, u_{n,2} = T_{2,r_n} u_{n,1}, \dots, u_{n,N-1} = T_{N-1,r_n} u_{n,N-2}$ and $u_{n,N} = T_{N,r_n} u_{n,N-1} = w_n$. Then, from (3.1), Lemmas 2.2, 2.3 and the property of ϕ, we get that

$$\begin{aligned} D_f(x^*, x_{n+1}) &= D_f\left(x^*, P_C^f \nabla f^*\left(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(w_n)\right)\right) \\ &\leq D_f\left(x^*, \nabla f^*\left(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(w_n)\right)\right) \\ &\leq \alpha_n D_f(x^*, u) + (1 - \alpha_n) D_f(x^*, w_n) \\ &= \alpha_n D_f(x^*, u) + (1 - \alpha_n) D_f(x^*, T_{N,r_n} \circ T_{N-1,r_n} \circ \cdots \circ T_{1,r_n} x_n) \\ &\leq \alpha_n D_f(x^*, u) + (1 - \alpha_n) D_f(x^*, x_n). \end{aligned} \tag{3.2}$$

Thus, by induction,

$$D_f(x^*, x_{n+1}) \leq \max\{D_f(x^*, x_n), D_f(x^*, u)\}, \quad \forall n \geq 0,$$

which implies by Lemma 2.4 that $\{x_n\}$ and hence $\{w_n\}$ are bounded. Now let $z_n = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(w_n))$. Then we have from (3.1) that $x_{n+1} = P_C^f z_n$. Using Lemmas 2.2, 2.3, 2.7(3), (2.3) and (2.4), we obtain that

$$\begin{aligned} D_f(x^*, x_{n+1}) &= D_f(x^*, P_C^f z_n) \leq D_f(x^*, z_n) = V(x^*, \nabla f(z_n)) \\ &\leq V\left(x^*, \nabla f(z_n) - \alpha_n\left(\nabla f(u) - \nabla f(x^*)\right)\right) + \left\langle \alpha_n\left(\nabla f(u) - \nabla f(x^*)\right), z_n - x^* \right\rangle \\ &= D_f\left(x^*, \nabla f^*\left(\alpha_n \nabla f(x^*) + (1 - \alpha_n)\nabla f(w_n)\right)\right) \\ &\quad + \alpha_n \left\langle \nabla f(u) - \nabla f(x^*), z_n - x^* \right\rangle \\ &\leq \alpha_n \phi(x^*, x^*) + (1 - \alpha_n) D_f(x^*, w_n) + \alpha_n \left\langle \nabla f(u) - \nabla f(x^*), z_n - x^* \right\rangle \\ &\leq (1 - \alpha_n) D_f(x^*, T_{N,r_n} u_{n,N-1}) + \alpha_n \left\langle \nabla f(u) - \nabla f(x^*), z_n - x^* \right\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} D_f(x^*, x_{n+1}) &\le (1-\alpha_n)\big[D_f(x^*, u_{n,N-1}) - D_f(w_n, u_{n,N-1})\big] \\ &\quad + \alpha_n\big\langle \nabla f(u) - \nabla f(x^*), z_n - x^*\big\rangle \\ &\le (1-\alpha_n)\big[D_f(x^*, u_{n_N-2}) - D_f(u_{n_N-1}, u_{n_N-2})\big] \\ &\quad - (1-\alpha_n)D_f(w_n, u_{n,N-1}) + \alpha_n\big\langle \nabla f(u) - \nabla f(x^*), z_n - x^*\big\rangle \\ &\quad \cdots \\ &\le (1-\alpha_n)D_f(x^*, x_n) - (1-\alpha_n)\big[D_f(u_{n,1}, x_n) + D_f(u_{n,2}, u_{n,1}) \\ &\quad + \cdots + D_f(u_{n,N-1}, u_{n,N-2}) + D_f(w_n, u_{n,N-1})\big] \\ &\quad + \alpha_n\big\langle \nabla f(u) - \nabla f(x^*), z_n - x^*\big\rangle \\ &\le (1-\alpha_n)D_f(x^*, x_n) + \alpha_n\big\langle \nabla f(u) - \nabla f(x^*), z_n - x^*\big\rangle. \end{aligned} \tag{3.3}$$

Now, we consider two possible cases.

Case 1. Suppose that there exists $n_0 \in N$ such that $\{D_f(x^*, x_n)\}$ is decreasing. Then we obtain that $\{D_f(x^*, x_n)\}$ is convergent. Thus, from (3.3) we have that $D_f(u_{n,1}, x_n), D_f(u_{n,2}, u_{n,1}), \ldots, D_f(w_n, u_{n,N-1}) \to 0$ as $n \to \infty$, and hence by Lemma 2.1 we get that

$$u_{n,1} - x_n \to 0, \qquad u_{n,2} - u_{n,1} \to 0, \qquad \ldots, \qquad w_n - u_{n,N-1} \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Furthermore, from the property of $D_f(\cdot,\cdot)$ and the fact that $\alpha_n \to 0$ as $n \to \infty$, we have that

$$\begin{aligned} D_f(w_n, z_n) &= D_f\big(w_n, \nabla f^*\big(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(w_n)\big)\big) \\ &\le \alpha_n D_f(w_n, u) + (1-\alpha_n)D_f(w_n, w_n) \\ &\le \alpha_n D_f(w_n, u) + (1-\alpha_n)D_f(w_n, w_n) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

and hence from Lemma 2.1 we have that $w_n - z_n \to 0$ and this with (3.4) implies that

$$z_n - u_{n,N-1} \to 0, \qquad z_n - u_{n,N-2} \to 0, \qquad \ldots, \qquad z_n - u_{n,1} \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Since $\{z_n\}$ is bounded and E is reflexive, we choose a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(x^*), z_n - x^*\rangle = \lim_{k\to\infty}\langle \nabla f(u) - \nabla f(x^*), z_{n_k} - x^*\rangle$. Then, from (3.5) and (3.4), we get that $u_{n_k,i} \rightharpoonup z$ for each $i \in \{1, 2, \ldots, N\}$.

Now, we show that $z \in VI(C, A_i)$ for each $i \in \{1, 2, \ldots, N\}$. But from the definition of $u_{n,i}$, we have that

$$\langle A_i u_{n,i}, y - u_{n,i}\rangle + \left\langle \frac{\nabla f(u_{n,i}) - \nabla f(x_n)}{r_n}, y - u_{n,i}\right\rangle \ge 0, \quad \forall y \in C,$$

and hence

$$\langle A_i u_{n_k,i}, y - u_{n_k,i}\rangle + \left\langle \frac{\nabla f(u_{n_k,i}) - \nabla f(x_{n_k})}{r_{n_k}}, y - u_{n_k,i}\right\rangle \ge 0, \quad \forall y \in C \tag{3.6}$$

for each $i \in \{1,2,\dots,N\}$. Set $v_t = ty + (1-t)z$ for all $t \in (0,1]$ and $y \in C$. Consequently, we get that $v_t \in C$. Now, from (3.6) it follows that

$$\begin{aligned}\langle A_i v_t, v_t - u_{n_k,i}\rangle &\geq \langle A_i v_t, v_t - u_{n_k,i}\rangle - \langle A_i u_{n_k,i}, v_t - u_{n_k,i}\rangle \\ &\quad - \left\langle \frac{\nabla f(u_{n_k,i}) - \nabla f(x_{n_k})}{r_{n_k}}, v_t - u_{n_k,i}\right\rangle \\ &= \langle A_i v_t - A_i u_{n_k,i}, v_t - u_{n_k,i}\rangle \\ &\quad - \left\langle \frac{\nabla f(u_{n_k,i}) - \nabla f(x_{n_k})}{r_{n_k}}, v_t - u_{n_k,i}\right\rangle.\end{aligned}$$

In addition, since f is uniformly Fréchet differentiable and bounded, we have that ∇f is uniformly continuous (see [36]). Thus, from (3.4) and the uniform continuity of ∇f, we obtain that

$$\frac{\nabla f(u_{n_k,i}) - \nabla f(x_{n_k})}{r_{n_k}} \to 0 \quad \text{as } k \to \infty,$$

and since A is monotone, we also have that $\langle A_i v_t - A_i u_{n_k,i}, v_t - u_{n_k,i}\rangle \geq 0$. Thus, it follows that

$$0 \leq \lim_{k\to\infty} \langle A_i v_t, v_t - u_{n_k,i}\rangle = \langle A_i v_t, v_t - z\rangle,$$

and hence

$$\langle A_i v_t, y - z\rangle \geq 0, \quad \forall y \in C, \text{for all } i \in \{1,2,\dots,N\}.$$

If $t \to 0$, the continuity of A_i implies that

$$\langle A_i z, y - z\rangle \geq 0, \quad \forall y \in C.$$

This implies that $z \in VI(C, A_i)$ for all $i \in \{1,2,\dots,N\}$.

Therefore, we obtain that $z \in \bigcap_{i=1}^N VI(C,A_i)$. Thus, by Lemma 2.2, we immediately obtain that $\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(x^*), z_n - x^*\rangle = \langle \nabla f(u) - \nabla f(x^*), z - x^*\rangle \leq 0$. It follows from Lemma 2.5 and (3.3) that $D_f(x^*, x_n) \to 0$ as $n \to \infty$. Consequently, $x_n \to x^*$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$D_f(x^*, x_{n_j}) < D_f(x^*, x_{n_j+1})$$

for all $j \in \mathbb{N}$. Then, by Lemma 2.6, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $D_f(x^*, x_{m_k}) \leq D_f(x^*, x_{m_k+1})$ and $D_f(x^*, x_k) \leq D_f(x^*, x_{m_k+1})$ for all $k \in \mathbb{N}$. From (3.3) and $\alpha_n \to 0$, we have

$$\begin{aligned}&(1-\alpha_{m_k})\big(D_f(u_{m_k,1}, x_{m_k}) + \cdots + D_f(w_{m_k}, u_{m_k,N-1})\big) \\ &\quad\leq \big(D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1})\big) + \alpha_{m_k} D_f(x^*, x_{m_k}) \\ &\qquad + \alpha_{m_k}\langle \nabla f(u) - \nabla f(x^*), z_{m_k} - x^*\rangle \to 0 \quad \text{as } k \to \infty,\end{aligned}$$

which implies that $D_f(u_{m_k,1}, x_{m_k}), \ldots, D_f(w_{m_k}, u_{m_k,N-1}) \to 0$, and hence $u_{m_k,1} - x_{m_k} \to 0, \ldots, w_{m_k} - u_{m_k,N-1} \to 0$ as $k \to \infty$. Thus, as in Case 1, we obtain that

$$\limsup_{k\to\infty} \langle \nabla f(u) - \nabla f(x^*), z_{m_k} - x^* \rangle \leq 0. \tag{3.7}$$

Furthermore, from (3.3) we have that

$$D_f(x^*, x_{m_k+1}) \leq (1-\alpha_{m_k}) D_f(x^*, x_{m_k}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(x^*), z_{m_k} - x^* \rangle. \tag{3.8}$$

Thus, since $D_f(x^*, x_{m_k}) \leq D_f(x^*, x_{m_k+1})$, we get that

$$\begin{aligned} \alpha_{m_k} D_f(x^*, x_{m_k}) &\leq D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1}) \\ &\quad + \alpha_{m_k} \langle \nabla f(u) - \nabla f(x^*), z_{m_k} - x^* \rangle \\ &\leq \alpha_{m_k} \langle \nabla f(u) - \nabla f(x^*), z_{m_k} - x^* \rangle. \end{aligned}$$

Moreover, since $\alpha_{m_k} > 0$, we obtain that

$$D_f(x^*, x_{m_k}) \leq \langle \nabla f(u) - \nabla f(x^*), z_{m_k} - x^* \rangle.$$

It follows from (3.7) that $D_f(x^*, x_{m_k}) \to 0$ as $k \to \infty$. This together with (3.8) implies that $D_f(x^*, x_{m_k+1}) \to 0$. Therefore, since $D_f(x^*, x_k) \leq D_f(x^*, x_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $x_k \to x^*$ as $k \to \infty$. Hence, both cases imply that $\{x_n\}$ converges strongly to $x^* = P_F^f u$ and the proof is complete. □

If in Theorem 3.1 $N = 1$, then we get the following corollary.

Corollary 3.2 *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} f)$, *and let $A : C \to E^*$ be a continuous monotone mapping with $VI(C, A) \neq \emptyset$. Let $\{x_n\}_{n\geq 0}$ be a sequence defined by* (3.1),

$$\begin{cases} x_0 = u \in C \quad \text{chosen arbitrarily}, \\ w_n = T_{r_n} x_n, \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(w_n)), \end{cases} \tag{3.9}$$

where $T_\gamma x := \{z \in C : \langle Az, y - z\rangle + \frac{1}{\gamma} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in C\}$ for all $x \in E$; $\alpha_n \in (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$ and $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$. Then the sequence $\{x_n\}_{n\geq 0}$ converges strongly to a point $x^* = P_{VI(C,A)}(u)$.

If $C = E$, then $VI(C, A) = A^{-1}(0)$ and hence the following corollary holds.

Corollary 3.3 *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $A_i : E \to E^*$, for $i = 1, 2, \ldots, N$, be a finite family of continuous monotone mappings. Let $\mathcal{F} := \bigcap_{i=1}^N VI(C, A_i) = \bigcap_{i=1}^N A^{-1}(0) \neq \emptyset$. Let $\{x_n\}_{n\geq 0}$ be a sequence defined by* (3.1). *Then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}}^f(u)$.*

If in Theorem 3.1 we assume $u = 0$, then the scheme converges strongly to the common minimum-norm zero of a finite family of continuous monotone mappings. In fact, we have the following corollary.

Corollary 3.4 *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of $\operatorname{int}(\operatorname{dom} f)$, and let $A_i : C \to E^*$, for $i = 1, 2, \dots, N$, be a finite family of continuous monotone mappings with $\mathcal{F} := \bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$. Let $\{x_n\}_{n\geq 0}$ be a sequence defined by (3.1) with $u = 0$. Then $\{x_n\}$ converges strongly to $x^* = P^f_{\mathcal{F}}(0)$, which is the common minimum-norm (with respect to the Bregman distance) solution of the variational inequalities.*

4 Application

In this section, we study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in Banach spaces.

Let g_i, for $i = 1, 2, \dots, N$, be continuously Fréchet differentiable convex functionals such that the gradients of g_i, $(\nabla g_i)|_C$ are continuous and monotone. For $r > 0$, let $K_{i,r}x := \{z \in C : \langle \nabla g_i(z), y - z\rangle + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \forall y \in C\}$ for all $x \in E$ and for each $i \in \{1, 2, \dots, N\}$. Then the following theorem holds.

Theorem 4.1 *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let g_i, $i = 1, 2, \dots, N$, be continuously Fréchet differentiable convex functionals such that the gradients of g_i, $(\nabla g_i)|_C$ are continuous, monotone and $\mathcal{F} := \bigcap_{i=1}^{N} \arg\min_{y\in C} g_i(y) \neq \emptyset$, where $\arg\min_{y\in C} g_i(y) := \{z \in C : g_i(z) = \min_{y\in C} g_i(y)\}$. Let $\{x_n\}_{n\geq 0}$ be a sequence defined by*

$$\begin{cases} x_0 = u \in C \quad \text{chosen arbitrarily}, \\ w_n = K_{N,r_n} \circ K_{N-1,r_n} \circ \cdots \circ K_{1,r_n} x_n, \\ x_{n+1} = P^f_C \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(w_n)), \quad \forall n \geq 0, \end{cases} \tag{4.1}$$

where $\alpha_n \in (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$. Then the sequence $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$.

Proof We note that from the convexity and Fréchet differentiability of f, we have $VI(C, (\nabla g_i)|_C) = \arg\min_{y\in C} g_i(y)$ for each $i \in \{1, 2, \dots, N\}$. Thus, by Theorem 3.1, $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$. □

Remark 4.2 Our results are new even if the convex function f is chosen to be $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ in uniformly smooth and uniformly convex spaces.

Remark 4.3 Our theorems extend and unify most of the results that have been proved for this important class of nonlinear operators. In particular, Theorem 3.1 extends Theorem 3.3 of [16], Theorem 3.1 of [17], Theorem 3.1 of [17] and Theorem 3.3 of [10] and Theorem 4.2 of [25] either to a more general class of continuous monotone operators or to a more general Banach space E. Moreover, in all our theorems and corollaries, the computation of C_n or Q_n for each $n \geq 1$ is not required.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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