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Hybrid iterative algorithms for nonexpansive and nonspreading mappings in Hilbert spaces

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Abstract
Recently, Iemoto and Takahashi considered a weak convergence iterative scheme for a nonspreading mapping and a nonexpansive mapping in Hilbert spaces. In this paper, we suggest two hybrid iterative algorithms by modifying Iemoto and Takahashi's iterative scheme for a countable family of nonspreading mappings and a nonexpansive mapping in Hilbert spaces.
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1 Introduction and preliminaries

Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonlinear mapping of C into itself. We use $F(T)$ and P_C to denote the set of fixed points of T and the metric projection from H onto C, respectively.

Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in C$.

For approximating the fixed point of a nonexpansive mapping in a Hilbert space, Mann [1] in 1953 introduced the famous iterative scheme as follows:

$$\forall x_1 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1, \tag{1.2}$$

where T is a nonexpansive mapping of C into itself and $\{\alpha_n\}$ is a sequence in $(0,1)$. It is well known that $\{x_n\}$ defined in (1.2) converges weakly to a fixed point of T.

Attempts to modify the normal Mann iteration method (1.2) for nonexpansive mappings so that strong convergence is guaranteed have recently been made; see, *e.g.*, [2–9].

Let T be a mapping from C into itself. Then T is called nonspreading [3] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2$$

for all $x, y \in C$. A mapping $T : C \to C$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. If T is a nonspreading mapping from C into itself and $F(T)$ is nonempty, then T is quasi-nonexpansive. Further, we know that the set of fixed points of each quasi-nonexpansive mapping is closed and convex; see [10].

In [11], by using Moudafi's iterative scheme [12], Iemoto and Takahashi considered the following weak convergence theorem.

Theorem IT ([11]) *Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself, and let T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows*:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{\beta_n S x_n + (1-\beta_n)T x_n\} \end{cases} \tag{1.3}$$

for all $n \in N$, where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Then the following hold:

(i) *If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $\{x_n\}$ converges weakly to $v \in F(S)$*;

(ii) *If $\sum_{i=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\{x_n\}$ converges weakly to $v \in F(T)$*;

(iii) *If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges weakly to $v \in F(S) \cap F(T)$.*

In this paper, we modify (1.1) by a hybrid iterative scheme and obtain the strong convergence theorems for a family of nonspreading mappings and a nonexpansive mapping in a Hilbert space.

Let E be a Banach space and K be a nonempty closed convex subset of E. Let $\{T_n\} : K \to K$ be a family of mappings. Then $\{T_n\}$ is said to satisfy the *AKTT-condition* [13] if for each bounded subset B of K, one has

$$\sum_{n=1}^{\infty} \sup\bigl\{\|T_{n+1}z - T_n z\| : z \in B\bigr\} < \infty.$$

The following is an important result on a family of mappings $\{T_n\}_{n=1}^{\infty}$ satisfying the AKTT-condition.

Lemma 1.1 ([13]) *Let K be a nonempty and closed subset of a Banach space E, and let $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of K into itself which satisfies the AKTT-condition. Then, for each $x \in K$, $\{T_n x\}$ converges strongly to a point in K. Moreover, let the mapping $T : K \to K$ be defined by*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in K.$$

Then, for each bounded subset B of K,

$$\limsup_{n\to\infty} \bigl\{\|Tz - T_n z\| : z \in B\bigr\} = 0.$$

Obviously, if a family of mappings $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and $Tx = \lim_{n\to\infty} T_n x$ for each $x \in K$, then it is unnecessary that $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. To show this, see the following example.

Example 1.1 Let $E = \mathbb{R}$ and $K = [0,2]$. Define a family of mappings $\{T_n\}_{n=1}^{\infty} : K \to K$ by

$$T_1 x = 0, \qquad T_n = \frac{1}{n}(1+x), \quad n \geq 2.$$

Then $\{T_n\}_{n=1}^{\infty}$ satisfy the AKTT-condition. It is easy to see that for each $x \in K$, $\lim_{n\to\infty} T_n \times x = 0$. Define the mapping $T : K \to K$ by $Tx = \lim_{n\to\infty} T_n x$. That is, $Tx = 0$ for all $x \in K$. But $F(T) \neq \bigcap_{n=1}^{\infty} F(T_n)$.

In this paper, we call that $\{T_n, T\}$ satisfy the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfy the AKTT-condition with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 1.2 ([11]) *Let C be a nonempty closed subset of a Hilbert space H. Then a mapping $T : C \to C$ is nonspreading if and only if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$.

By using Lemma 1.2, we get the following simple but important result.

Lemma 1.3 *Let H be a Hilbert space and C be a nonempty subset of H. Let $\{T_n\}$ be a family of nonspreading mappings of C into itself, and assume that $\lim_{n\to\infty} T_n x$ exists for each $x \in C$. Define the mapping $T : C \to C$ by $Tx = \lim_{n\to\infty} T_n x$. Then the mapping T is a nonspreading mapping.*

Proof In fact, for all $x, y \in C$, we have

$$\begin{aligned} \|Tx - Ty\|^2 &= \Bigl\|\lim_{n\to\infty} T_n x - \lim_{n\to\infty} T_n y\Bigr\|^2 \\ &= \lim_{n\to\infty} \|T_n x - T_n y\|^2 \\ &\leq \lim_{n\to\infty} \bigl[\|x - y\|^2 + 2\langle x - T_n x, y - T_n y\rangle\bigr] \\ &= \|x - y\|^2 + 2\Bigl\langle x - \lim_{n\to\infty} T_n x, y - \lim_{n\to\infty} T_n y\Bigr\rangle \\ &= \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle. \end{aligned}$$

Lemma 1.2 shows that the mapping T is a nonspreading mapping. □

Lemma 1.4 *Let C be a closed convex subset of a real Hilbert space H, and let P_C be the metric projection from H onto C (i.e., for $x \in H$, $P_C x$ is the only point in C such that $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if the following relation holds*:

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

Lemma 1.5 ([14]) *Let H be a real Hilbert space. Then the following equation holds*:

$$\bigl\|tx + (1-t)y\bigr\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \quad \forall x \in C \text{ and } \forall t \in [0,1].$$

2 Main results

Theorem 2.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S: C \to C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty}: C \to C$ be a countable family of nonspreading mappings such that $F = F(S) \cap [\bigcap_{i=1}^{\infty} F(T_i)] \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n S x_n + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)T_i x_n], \\ C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n} x, \quad n \geq 1, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Assume that $\{\beta_n\}$ is strictly decreasing and $\beta_0 = 1$. Then the following hold:

(i) *If $\liminf_{n\to\infty} \alpha_n > 0$ and $\lim_{n\to\infty} \beta_n = 0$, then $\{x_n\}$ strongly converges to $q \in \bigcap_{i=1}^{\infty} F(T_i)$*;

(ii) *If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n > 0$, then $\{x_n\}$ converges strongly to $q \in F$.*

Proof Obviously, each C_n is closed and convex and hence D_n is closed and convex. Next, we show that $F \subset D_n$ for all $n \geq 1$. To end this, we need to prove that $F \subset C_n$ for all $n \geq 1$. Indeed, for each $p \in F$, we have

$$\begin{aligned} \|y_n - p\| &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\left[\beta_n\|Sx_n - p\| + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|T_i x_n - p\|\right] \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\left[\beta_n\|x_n - p\| + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|x_n - p\|\right] \\ &= \|x_n - p\|. \end{aligned} \tag{2.2}$$

This implies that

$$p \in C_n \quad \text{for all } n \geq 1.$$

Therefore, $F \subset C_n$ and hence C_n is nonempty for all $n \geq 1$. On the other hand, from the definition of D_n, we see that $F \subset D_n = \bigcap_{i=1}^{n} C_j$ for all $n \geq 1$.

From $x_{n+1} = P_{D_n} x$, we have

$$\|x_{n+1} - x\| \leq \|v - x\|, \quad \forall v \in D_n, n \geq 1.$$

Since $P_F x \in F \subset D_n$, one has

$$\|x_{n+1} - x\| \leq \|P_F x - x\|, \quad n \geq 1. \tag{2.3}$$

This implies that $\{x_n\}$ is bounded and hence $\{y_n\}$ is bounded.

On the other hand, since $D_{n+1} \subset D_n$ for all $n \geq 1$, we have

$$x_{n+2} = P_{D_{n+1}}x \in D_{n+1} \subset D_n$$

for all $n \geq 1$. From $x_{n+1} = P_{D_n}x$ one has

$$\|x_{n+1} - x\| \leq \|x_{n+2} - x\| \tag{2.4}$$

for all $n \geq 1$. It follows from (2.3) and (2.4) that the limit of $\{x_n - x\}$ exists.

Since $D_m \subset D_n$ and $x_{m+1} = P_{D_m}x \in D_m \subset D_n$ for all $m \geq n$ and $x_{n+1} = P_{D_n}x$, by Lemma 1.4 one has

$$\langle x_{n+1} - x, x_{m+1} - x_{n+1} \rangle \geq 0. \tag{2.5}$$

It follows from (2.5) that

$$\begin{aligned}
&\|x_{m+1} - x_{n+1}\|^2 \\
&\quad = \|x_{m+1} - x - (x_{n+1} - x)\|^2 \\
&\quad = \|x_{m+1} - x\|^2 + \|x_{n+1} - x\|^2 - 2\langle x_{n+1} - x, x_{m+1} - x \rangle \\
&\quad = \|x_{m+1} - x\|^2 + \|x_{n+1} - x\|^2 - 2\langle x_{n+1} - x, x_{m+1} - x_{n+1} + x_{n+1} - x \rangle \\
&\quad = \|x_{m+1} - x\|^2 - \|x_{n+1} - x\|^2 - 2\langle x_{n+1} - x, x_{m+1} - x_{n+1} \rangle \\
&\quad \leq \|x_{m+1} - x\|^2 - \|x_{n+1} - x\|^2.
\end{aligned} \tag{2.6}$$

Since the limit of $\|x_n - x\|$ exists, we get

$$\lim_{m,n \to \infty} \|x_m - x_n\| = 0.$$

It follows that $\{x_n\}$ is a Cauchy sequence. Since H is a Hilbert space and C is closed and convex, there exists $q \in C$ such that

$$x_n \to q, \quad \text{as } n \to \infty. \tag{2.7}$$

By taking $m = n + 1$ in (2.6), one arrives at

$$\lim_{n \to \infty} \|x_{n+2} - x_{n+1}\| = 0,$$

i.e.,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{2.8}$$

Noticing that $x_{n+1} = P_{D_n}x \in D_n \subset C_n$, we get

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0,$$

and hence

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{2.9}$$

From (2.7) and (2.9) it follows that

$$\lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|x_n - p\| = \|q - p\|, \quad \forall p \in F. \tag{2.10}$$

Now we prove (i). Note that

$$\begin{aligned} y_n &= (1-\alpha_n)x_n + \alpha_n\left[\beta_n S x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - x_n)\right] + \alpha_n(1-\beta_n)x_n \\ &= (1-\alpha_n\beta_n)x_n + \alpha_n\beta_n S x_n + \alpha_n \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - x_n). \end{aligned}$$

Hence,

$$\alpha_n \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x_n - x_n) = (1-\alpha_n\beta_n)(y_n - x_n) + \alpha_n\beta_n(y_n - S x_n). \tag{2.11}$$

On the other hand, for any $p \in F$, from Lemma 1.2 we have

$$\begin{aligned} \|x_n - p\|^2 &= 2\langle x_n - T_i x_n, p - T_i p\rangle + \|x_n - p\|^2 \\ &\ge \|T_i x_n - T_i p\|^2 = \|T_i x_n - p\|^2 = \big\|T_i x_n - x_n + (x_n - p)\big\|^2 \\ &= \|T_i x_n - x_n\|^2 + \|x_n - p\|^2 + 2\langle T_i x_n - x_n, x_n - p\rangle, \end{aligned}$$

and hence

$$\|T_i x_n - x_n\|^2 \le 2\langle x_n - T_i x_n, x_n - p\rangle, \quad \forall i \in \mathbb{N}. \tag{2.12}$$

Note that $\{\beta_n\}$ is strictly decreasing. Hence from (2.11) and (2.12) we get

$$\begin{aligned} \|T_i x_n - x_n\|^2 \le{} & \frac{1}{2\alpha_n(\beta_{i-1} - \beta_i)}\big[(1-\alpha_n\beta_n)\langle y_n - x_n, x_n - T_i p\rangle \\ & + \alpha_n\beta_n\langle y_n - S x_n, x_n - p\rangle\big], \quad i \ge 1. \end{aligned} \tag{2.13}$$

Since $\liminf_{n\to\infty} \alpha_n > 0$ and $\lim_{n\to\infty} \beta_n = 0$, from (2.9) and (2.13) it follows that

$$\lim_{n\to\infty} \|T_i x_n - x_n\| = 0, \quad \forall i \in \mathbb{N}. \tag{2.14}$$

Since each T_i is a nonspreading mapping, by Lemma 1.2, (2.7) and (2.10), we have

$$\|T_i q - T_i x_n\|^2 \le \|x_n - q\|^2 + 2\langle q - T_i q, x_n - T_i x_n\rangle \to 0, \quad \forall i \in \mathbb{N}. \tag{2.15}$$

Further, one has

$$\|q - T_i q\| \le \|q - x_n\| + \|x_n - T_i x_n\| + \|T_i x_n - T_i q\| \to 0, \quad \forall i \in \mathbb{N}. \tag{2.16}$$

So, we have $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

To prove (ii), first we show that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. For any $p \in F$, we have

$$\begin{aligned}\|y_n - p\|^2 &= \left\|\beta_n\big[(1-\alpha_n)x_n + \alpha_n Sx_n - p\big] + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\big[(1-\alpha_n)x_n + \alpha_n T_i x_n - p\big]\right\|^2 \\ &\le \beta_n\big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2 + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\big\|(1-\alpha_n)x_n + \alpha_n T_i x_n - p\big\|^2 \\ &\le \beta_n\big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2 \\ &\quad + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\big[(1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T_i x_n - p\|^2\big] \\ &\le \beta_n\big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2 + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|x_n - p\|^2 \\ &= \beta_n\big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2 + (1-\beta_n)\|x_n - p\|^2 \\ &\le \|x_n - p\|^2,\end{aligned}$$

and hence by (2.10) we get

$$\begin{aligned}0 &\le \|x_n - p\|^2 - \beta_n\big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2 - (1-\beta_n)\|x_n - p\|^2 \\ &= \beta_n\big[\|x_n - p\|^2 - \big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2\big] \\ &\le \|x_n - p\|^2 - \|y_n - p\|^2 \to 0.\end{aligned} \tag{2.17}$$

Since $\liminf_{n\to\infty}\beta_n > 0$, it follows from (2.17) that

$$\lim_{n\to\infty}\big(\|x_n - p\|^2 - \big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2\big) = 0. \tag{2.18}$$

From (2.18) and

$$\big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2 = (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|Sx_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sx_n\|^2,$$

we get

$$\begin{aligned}&\alpha_n(1-\alpha_n)\|x_n - Sx_n\|^2 \\ &\quad = \big(\|x_n - p\|^2 - \big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2\big) - \alpha_n\|x_n - p\|^2 + \alpha_n\|Sx_n - p\|^2 \\ &\quad \le \big(\|x_n - p\|^2 - \big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2\big) - \alpha_n\|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 \\ &\quad = \|x_n - p\|^2 - \big\|(1-\alpha_n)x_n + \alpha_n Sx_n - p\big\|^2 \to 0.\end{aligned}$$

Since $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we get

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{2.19}$$

Now, using (2.19), (2.7) and

$$\|q - Sq\| \le \|q - x_n\| + \|x_n - Sx_n\| + \|Sx_n - Sq\| \le 2\|q - x_n\| + \|x_n - Sx_n\| \to 0,$$

which implies that $q \in F(S)$.

Note that (2.9) and (2.19) imply that $\lim_{n\to\infty}\|y_n - Sx_n\| = 0$. Then, repeating (2.11) to (2.16), we get $q \in \bigcap_{i=1}^{\infty} F(T_i)$. So, $q \in F$. This completes the proof. □

Theorem 2.2 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of nonspreading mappings such that $F = F(S) \cap [\bigcap_{i=1}^{\infty} F(T_i)] \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1-\beta_n)T_n x_n], \\ C_n = \{v \in C : \|y_n - v\| \le \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n} x, \quad n \ge 1, \end{cases} \tag{2.20}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Assume that $\{T_n, T\}$ satisfies the AKTT-condition. Then the following hold:

(i) *If $\liminf_{n\to\infty}\alpha_n > 0$ and $\lim_{n\to\infty}\beta_n = 0$, then $\{x_n\}$ strongly converges to $v \in \bigcap_{i=1}^{\infty} F(T_i)$;*

(ii) *If $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}\beta_n > 0$, then $\{x_n\}$ converges strongly to $z \in F$.*

Proof By a process similar to the proof of Theorem 2.1, we can conclude that $\{x_n\}$ converges strongly to some $q \in C$ and

$$x_n - y_n \to 0.$$

We first prove (i). From (2.20) we have

$$T_n x_n - x_n = \frac{1}{\alpha_n(1-\beta_n)}(y_n - x_n) - \frac{\beta_n}{1-\beta_n}(Sx_n - x_n),$$

and hence

$$\|T_n x_n - x_n\| \le \frac{1}{\alpha_n(1-\beta_n)}\|y_n - x_n\| + \frac{\beta_n}{1-\beta_n}\|Sx_n - x_n\|.$$

Since $\liminf_{n\to\infty}\alpha_n > 0$ and $\lim_{n\to\infty}\beta_n = 0$, we get

$$\lim_{n\to\infty}\|T_n x_n - x_n\| = 0. \tag{2.21}$$

Further, by Lemma 1.1 and (2.21), we have

$$\begin{aligned} \|x_n - Tx_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\ &\le \|x_n - T_n x_n\| + \sup\bigl\{\|T_n z - Tz\| : z \in \{x_n\}\bigr\} \to 0. \end{aligned} \tag{2.22}$$

Since each T_n is a nonspreading mapping, Lemma 1.3 shows that T is a nonspreading mapping. Further, by using Lemma 1.2, we have

$$\|Tq - Tx_n\|^2 \le \|x_n - q\|^2 + 2\langle q - Tq, x_n - Tx_n\rangle \to 0, \quad \forall i \in \mathbb{N}. \tag{2.23}$$

From (2.21) and (2.23) it follows that

$$\|q - Tq\| \le \|q - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tq\| \to 0. \tag{2.24}$$

It follows that $q \in F(T)$. Since $(\{T_n\}, T)$ satisfies the AKTT-condition, one has $q \in \bigcap_{i=1}^{\infty} F(T_i) = F(T)$. This completes (i).

Next we show (ii). By a process similar to the proof of Theorem 2.1 and from (2.22) to (2.24), we can get that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0, \qquad \lim_{n\to\infty} \|x_n - T_n x_n\| = 0,$$
$$\lim_{n\to\infty} \|Tx_n - Tq\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Finally, by

$$\|q - Sq\| \le \|q - x_n\| + \|x_n - Sx_n\| + \|Sx_n - Sq\| \le 2\|x_n - q\| + \|x_n - Sx_n\| \to 0$$

and

$$\|q - Tq\| \le \|q - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tq\| \to 0,$$

we get $q \in F(S) \cap F(T)$. Since $(\{T_n\}, T)$ satisfies the AKTT-condition, we conclude that $q \in F$. This completes (ii). □

Letting $T_i = T$ for all $i \in \mathbb{N}$ in Theorem 2.1 and Theorem 2.2, we get the following corollary.

Corollary 2.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping and $T : C \to C$ be a nonspreading mapping such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1-\beta_n)Tx_n], \\ C_n = \{v \in C : \|y_n - v\| \le \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n}x, \quad n \ge 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Then the following hold:

(i) *If $\liminf_{n\to\infty} \alpha_n > 0$ and $\lim_{n\to\infty} \beta_n = 0$, then $\{x_n\}$ strongly converges to $x' \in F(T)$*;

(ii) *If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n > 0$, then $\{x_n\}$ converges strongly to $q \in F(S) \cap F(T)$ with $q = P_F x$.*

Letting $S = I$ in Theorems 2.1 and 2.2, we get the following corollary.

Corollary 2.2 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of nonspreading mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$*

be a sequence generated in the following manner:

$$\begin{cases} x_1 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = (1-\alpha_n(1-\beta_n))x_n + \alpha_n \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)T_i x_n, \\ C_n = \{v \in C : \|y_n - v\| \le \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n}x, \quad n \ge 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. *Assume that* $\{\beta_n\}$ *is strictly decreasing and* $\beta_0 = 1$. *Then if* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, *then* $\{x_n\}$ *strongly converges to* $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Corollary 2.3 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of nonspreading mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = (1-\alpha_n(1-\beta_n))x_n + \alpha_n(1-\beta_n)T_n x_n, \\ C_n = \{v \in C : \|y_n - v\| \le \|x_n - v\|\}, \\ D_n = \bigcap_{j=1}^{n} C_j, \\ x_{n+1} = P_{D_n}x, \quad n \ge 1, \end{cases}$$

where $\{\gamma_n\} \subset [0,1]$. *Assume that* $(\{T_n, T\})$ *satisfies the AKTT-condition*. *Then if* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, *then* $\{x_n\}$ *strongly converges to* $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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