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Graph convergence for the $H(\cdot,\cdot)$-mixed mapping with an application for solving the system of generalized variational inclusions

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Abstract

In this paper, we investigate a class of accretive mappings called the $H(\cdot,\cdot)$-mixed mappings in Banach spaces. We prove that the proximal-point mapping associated with the $H(\cdot,\cdot)$-mixed mapping is single-valued and Lipschitz continuous. Some examples are given to justify the definition of $H(\cdot,\cdot)$-mixed mapping. Further, a concept of graph convergence concerned with the $H(\cdot,\cdot)$-mixed mapping is introduced in Banach spaces and some equivalence theorems between graph-convergence and proximal-point mapping convergence for the $H(\cdot,\cdot)$-mixed mappings sequence are proved. As an application, we consider a system of generalized variational inclusions involving $H(\cdot,\cdot)$-mixed mappings in real q-uniformly smooth Banach spaces. Using the proximal-point mapping method, we prove the existence and uniqueness of solution and suggest an iterative algorithm for the system of generalized variational inclusions. Furthermore, we discuss the convergence criteria for the iterative algorithm under some suitable conditions.

MSC: 47J19; 49J40; 49J53

Keywords: $H(\cdot,\cdot)$-mixed mapping; graph convergence; proximal-point mapping method; system of generalized variational inclusions; iterative algorithm

1 Introduction

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years. Some of the most interesting and important problems in the theory of variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. For applications of variational inclusions, we refer to [1]. Various kinds of iterative methods have been studied to solve the variational inclusions. Among these methods, the proximal-point mapping technique for the study of variational inclusions has been widely used by many authors. For details, we refer to [2–20].

In 2001, Huang and Fang [5] were the first to introduce the generalized m-accretive mapping and give the definition of the proximal-point mapping for the generalized m-accretive mapping in Banach spaces. Since then a number of researchers have investigated several classes of generalized m-accretive mappings such as H-accretive, H,η-accretive, (P,η)-proximal-point, (P,η)-accretive, A-maximal relaxed accretive, (A,η)-accretive mappings. For details, we refer to [2, 3, 6, 7, 11, 14, 16, 18].

Recently, Zou and Huang [19, 20] introduced and studied $H(\cdot,\cdot)$-accretive mappings; Kazmi *et al.* [8–10] introduced and studied generalized $H(\cdot,\cdot)$-accretive mappings, $H(\cdot,\cdot)$-

η-proximal-point mappings. Very recently, Li and Huang [12] studied the graph convergence for the $H(\cdot,\cdot)$-accretive mapping and showed the equivalence between graph convergence and proximal-point mapping convergence for the $H(\cdot,\cdot)$-accretive mapping sequence in a Banach space, and Verma [17] studied the graph convergence for an A-maximal relaxed monotone mapping and gave the equivalence between the graph convergence and the proximal-point mapping convergence for the A-maximal relaxed monotone mapping sequence in a Hilbert space. They extended the concept of graph convergence introduced and considered by Attouch [21].

Motivated by the research work going on in this direction, we consider a class of accretive mappings called $H(\cdot,\cdot)$-mixed mappings, a natural generalization of accretive (monotone) mappings in Banach spaces. For related work, we refer to [2–4, 11, 14, 16, 18–20]. We prove that the proximal-point mapping of the $H(\cdot,\cdot)$-mixed mapping is single-valued and Lipschitz continuous and extends the concept of proximal-point mappings associated with the $H(\cdot,\cdot)$-accretive mappings to the $H(\cdot,\cdot)$-mixed mappings. Further, we study the graph convergence for the $H(\cdot,\cdot)$-mixed mappings. We present an equivalence theorem between graph convergence and proximal-point mapping convergence for the $H(\cdot,\cdot)$-mixed mapping sequence in Banach spaces. As an application, we consider a system of generalized variational inclusions involving the $H(\cdot,\cdot)$-mixed mappings in real q-uniformly smooth Banach spaces. Using the proximal-point mapping method, we prove the existence and uniqueness of solution and suggest an iterative algorithm for the system of generalized variational inclusions. Furthermore, we discuss the convergence criteria of the iterative algorithm under some suitable conditions. Our results can be viewed as a generalization of some known results given in [12, 17, 19–21].

2 Preliminaries

Let X be a real Banach space equipped with the norm $\|\cdot\|$, and let X^* be the topological dual space of X. Let $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, and let 2^X be the power set of X.

Definition 2.1 [22] For $q > 1$, a mapping $J_q : X \to 2^{X^*}$ is said to be a *generalized duality mapping* if it is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X.$$

In particular, J_2 is the *usual normalized duality mapping* on X. It is known that, in general,

$$J_q(x) = \|x\|^{q-1}J_2(x) \quad \forall x(\neq 0) \in X.$$

If $X \equiv H$ a real Hilbert space, then J_2 becomes an *identity mapping* on H.

Definition 2.2 [22] A Banach space X is called *smooth* if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^*$ such that $\|f\| = f(x) = 1$.

The *modulus of smoothness* of X is a function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

Definition 2.3 [22] A Banach space X is called

(i) *uniformly smooth* if

$$\lim_{t\to 0}\frac{\rho_X(t)}{t}=0;$$

(ii) *q-uniformly smooth*, for $q>1$, if there exists a constant $c>0$ such that

$$\rho_X(t)\le ct^q,\quad t\in[0,\infty).$$

Note that J_q is single-valued if X is uniformly smooth. Concerned with the characteristic inequalities in q-uniformly smooth Banach spaces, Xu [22] proved the following result.

Lemma 2.4 *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q>0$ such that, for all $x,y\in X$,*

$$\|x+y\|^q\le\|x\|^q+q\langle y,J_q(x)\rangle+c_q\|y\|^q.$$

From Lemma 2 of Liu [13], it is easy to have the following lemma.

Lemma 2.5 *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying*

$$a_{n+1}\le ka_n+b_n$$

with $0<k<1$ and $b_n\to 0$. Then $\lim_{n\to\infty}a_n=0$.

Definition 2.6 Let $G:X\to X$ be a single-valued mapping. Then

(i) G is said to be *accretive* if

$$\langle G(x)-G(y),J_q(x-y)\rangle\ge 0,\quad\forall x,y\in X;$$

(ii) G is said to be *ξ-strongly accretive* if there exists a constant $\xi>0$ such that

$$\langle G(x)-G(y),J_q(x-y)\rangle\ge\xi\|x-y\|^q,\quad\forall x,y\in X;$$

(iii) G is said to be *μ-cocoercive* if there exists a constant $\mu>0$ such that

$$\langle G(x)-G(y),J_q(x-y)\rangle\ge\mu\|G(x)-G(y)\|^q,\quad\forall x,y\in X;$$

(iv) G is said to be *λ_G-Lipschitz continuous* if there exists a constant $\lambda_G>0$ such that

$$\|G(x)-G(y)\|\le\lambda_G\|x-y\|,\quad\forall x,y\in X;$$

(v) G is said to be *α-expansive* if there exists a constant $\alpha>0$ such that

$$\|G(x)-G(y)\|\ge\alpha\|x-y\|,\quad\forall x,y\in X;$$

if $\alpha=1$, then it is *expansive*.

Definition 2.7 Let $H : X \times X \to X$ and $A, B : X \to X$ be three single mappings. Then

(i) $H(A,\cdot)$ is said to be μ-cocoercive with respect to A if there exists a constant $\mu > 0$ such that

$$\langle H(Ax,u) - H(Ay,u), J_q(x-y)\rangle \geq \mu\|Ax - Ay\|^q, \quad \forall x,y,u \in X;$$

(ii) $H(\cdot,B)$ is said to be *γ-relaxed accretive with respect to B* if there exists a constant $\gamma > 0$ such that

$$\langle H(u,Bx) - H(u,By), J_q(x-y)\rangle \geq (-\gamma)\|x - y\|^q, \quad \forall x,y,u \in X;$$

(iii) $H(A,\cdot)$ is said to be *r_1-Lipschitz continuous with respect to A* if there exists a constant $r_1 > 0$ such that

$$\|H(Ax,\cdot) - H(Ay,\cdot)\| \leq r_1\|x - y\|, \quad \forall x,y \in X;$$

(iv) $H(\cdot,B)$ is said to be *r_2-Lipschitz continuous with respect to B* if there exists a constant $r_2 > 0$ such that

$$\|H(\cdot,Bx) - H(\cdot,By)\| \leq r_2\|x - y\|, \quad \forall x,y \in X.$$

Example 2.8 Let us consider the 2-uniformly smooth Banach space $X = \mathbb{R}^2$ with the usual inner product. Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$Ax = \begin{pmatrix} mx_1 - mx_2 \\ -mx_1 + 2mx_2 \end{pmatrix}, \qquad By = \begin{pmatrix} -my_1 + my_2 \\ -my_1 - my_2 \end{pmatrix}$$

for all scalers $m \in \mathbb{R}$ and for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

Suppose that $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $H(Ax, By) = Ax + By$, then $H(A,B)$ is $\frac{1}{3m}$-cocoercive with respect to A and m-relaxed accretive with respect to B, and $\sqrt{5}m$-Lipschitz continuous with respect to A and $\sqrt{2}m$-Lipschitz continuous with respect to B.

Indeed, let for any $u \in X$,

$$\begin{aligned}
&\langle H(Ax,u) - H(Ay,u), x - y\rangle \\
&\quad = \langle Ax - Ay, x - y\rangle \\
&\quad = \langle (mx_1 - mx_2, -mx_1 + 2mx_2) - (my_1 - my_2, -my_1 + 2my_2), \\
&\qquad (x_1 - y_1, x_2 - y_2)\rangle \\
&\quad = \langle \big(m(x_1 - y_1) - m(x_2 - y_2), -m(x_1 - y_1) + 2m(x_2 - y_2)\big), \\
&\qquad (x_1 - y_1, x_2 - y_2)\rangle \\
&\quad = m(x_1 - y_1)^2 - 2m(x_1 - y_1)(x_2 - y_2) + 2m(x_2 - y_2)^2, \\
&\|Ax - Ay\|^2 = \langle Ax - Ay, Ax - Ay\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \big\langle \big((mx_1 - mx_2, -mx_1 + 2mx_2) - (my_1 - my_2, -my_1 + 2my_2)\big), \\
&\quad \big((mx_1 - mx_2, -mx_1 + 2mx_2) - (my_1 - my_2, -my_1 + 2my_2)\big)\big\rangle \\
&= 2m^2(x_1 - y_1)^2 - 6m^2(x_1 - y_1)(x_2 - y_2) + 5m^2(x_2 - y_2)^2 \\
&\leq 3m^2(x_1 - y_1)^2 - 6m^2(x_1 - y_1)(x_2 - y_2) + 6m^2(x_2 - y_2)^2 \\
&= 3m\big\{m(x_1 - y_1)^2 - 2m(x_1 - y_1)(x_2 - y_2) + 2m(x_2 - y_2)^2\big\} \\
&= 3m\big\{\big\langle H(Ax,u) - H(Ay,u), x - y\big\rangle\big\},
\end{aligned}
$$

which implies that

$$
\big\langle H(Ax,u) - H(Ay,u), x - y\big\rangle \geq \frac{1}{3m}\|Ax - Ay\|^2,
$$

that is, $H(A,B)$ is $\frac{1}{3m}$-cocoercive with respect to A.

$$
\begin{aligned}
&\big\langle H(u,Bx) - H(u,By), x - y\big\rangle \\
&\quad = \langle Bx - By, x - y\rangle \\
&\quad = \langle Bx - By, x - y\rangle \\
&\quad = \big\langle (-mx_1 + mx_2, -mx_1 - mx_2) - (-my_1 + my_2, -my_1 - my_2), \\
&\qquad (x_1 - y_1, x_2 - y_2)\big\rangle \\
&\quad = \big\langle \big(-m(x_1 - y_1) + m(x_2 - y_2), -m(x_1 - y_1) - m(x_2 - y_2)\big), \\
&\qquad (x_1 - y_1, x_2 - y_2)\big\rangle \\
&\quad = -m(x_1 - y_1)^2 - m(x_2 - y_2)^2 \\
&\quad = -m\big\{(x_1 - y_1)^2 + (x_2 - y_2)^2\big\} \\
&\quad \geq -m\|x - y\|^2,
\end{aligned}
$$

which implies that

$$
\big\langle H(u,Bx) - H(u,By), x - y\big\rangle \geq (-m)\|x - y\|^2,
$$

that is, $H(A,B)$ is m-relaxed accretive with respect to B.

$$
\begin{aligned}
\big\|H(Ax,u) - H(Ay,u)\big\|^2 &= \|Ax - Ay\|^2 = \langle Ax - Ay, Ax - Ay\rangle \\
&= \big\langle \big((mx_1 - mx_2, -mx_1 + 2mx_2) - (my_1 - my_2, -my_1 + 2my_2)\big), \\
&\quad \big((mx_1 - mx_2, -mx_1 + 2mx_2) - (my_1 - my_2, -my_1 + 2my_2)\big)\big\rangle \\
&= 2m^2(x_1 - y_1)^2 - 6m^2(x_1 - y_1)(x_2 - y_2) + 5m^2(x_2 - y_2)^2 \\
&\leq 5m^2(x_1 - y_1)^2 + 5m^2(x_2 - y_2)^2,
\end{aligned}
$$

which implies that

$$
\big\|H(Ax,u) - H(Ay,u)\big\| \leq \sqrt{5}m\|x - y\|,
$$

that is, $H(A,B)$ is $\sqrt{5}m$-Lipschitz continuous with respect to A.

$$\begin{aligned}\|H(u,Bx)-H(u,By)\|^2 &= \|Bx-By\|^2 = \langle Bx-By, Bx-By\rangle \\ &= \big\langle\big((-mx_1+mx_2,-mx_1-mx_2)-(-my_1+my_2,-my_1-my_2)\big), \\ &\quad \big((-mx_1+mx_2,-mx_1-mx_2)-(-my_1+my_2,-my_1-my_2)\big)\big\rangle \\ &= 2m^2(x_1-y_1)^2+2m^2(x_2-y_2)^2\end{aligned}$$

which implies that

$$\|H(u,Bx)-H(u,By)\| \leq \sqrt{2}m\|x-y\|,$$

that is, $H(A,B)$ is $\sqrt{2}m$-Lipschitz continuous with respect to B.

Definition 2.9 Let $\eta : X \times X \to X$ and $H, A, B : X \to X$ be mappings. Let $M : X \to 2^X$ be a set-valued mapping. Then

(i) η is said to be *τ-Lipschitz continuous* if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau\|x-y\|, \quad \forall x,y \in X;$$

(ii) M is said to be *accretive* if

$$\langle u-v, J_q(x-y)\rangle \geq 0, \quad \forall x,y \in X, u \in Mx, v \in My;$$

(iii) M is said to be *μ'-strongly accretive* if there exists a constant $\mu' > 0$ such that

$$\langle u-v, J_q(x-y)\rangle \geq \mu'\|x-y\|^q, \quad \forall x,y \in X, u \in Mx, v \in My;$$

(iv) M is said to be *m-relaxed accretive* if there exists a constant $m > 0$ such that

$$\langle u-v, J_q(x-y)\rangle \geq -m\|x-y\|^q, \quad \forall x,y \in X, u \in Mx, v \in My;$$

(v) M is said to be *η-accretive* if

$$\langle u-v, J_q(\eta(x,y))\rangle \geq 0, \quad \forall x,y \in X, u \in Mx, v \in My;$$

(vi) M is said to be *strictly η-accretive* if M is η-accretive and equality holds if and only if $x = y$;

(vii) M is said to be *γ-strongly η-accretive* if there exists a constant $\gamma > 0$ such that

$$\langle u-v, J_q(\eta(x,y))\rangle \geq \gamma\|x-y\|^q, \quad \forall x,y \in X, u \in Mx, v \in My;$$

(viii) M is said to be *α-relaxed η-accretive* if there exists a constant $\alpha > 0$ such that

$$\langle u-v, J_q(\eta(x,y))\rangle \geq (-\alpha)\|x-y\|^q, \quad \forall x,y \in X, u \in Mx, v \in My;$$

(ix) M is said to be *m-accretive* if M is accretive and $(I + \rho M)(X) = X$ for all $\rho > 0$, where I denotes the identity operator on X;

(x) M is said to be *generalized m-accretive* if M is η-accretive and $(I + \rho M)(X) = X$ for all $\rho > 0$;

(xi) M is said to be *H-accretive* if M is accretive and $(H + \rho M)(X) = X$ for all $\rho > 0$;

(xii) M is said to be (H,η)*-accretive* if M is η-accretive and $(H + \rho M)(X) = X$ for all $\rho > 0$;

(xiii) M is said to be (A,η)*-accretive* if M is m-relaxed η-accretive and $(A + \rho M)(X) = X$ for all $\rho > 0$.

Definition 2.10 [19] Let $A, B : X \to X$, $H : X \times X \to X$ be three single-valued mappings. Let $M : X \to 2^X$ be a set-valued mapping. Then M is said to be $H(\cdot,\cdot)$*-accretive* with respect to A and B if M is accretive and $(H(\cdot,\cdot) + \rho M)(X) = X$ for all $\rho > 0$.

3 $H(\cdot,\cdot)$-mixed mappings

In this section, we introduce the $H(\cdot,\cdot)$-mixed mapping and show some of its properties.

Definition 3.1 Let $H : X \times X \to X$, $A, B : X \to X$ be three single-valued mappings. Let $H(A,B)$ be μ-cocoercive with respect to A, γ-relaxed accretive with respect to B. Then the set-valued mapping $M : X \to 2^X$ is said be $H(\cdot,\cdot)$*-mixed* with respect to mappings A and B if

(i) M is m-relaxed accretive;

(ii) $(H(A,B) + \rho M)(X) = X$ for all $\rho > 0$.

Example 3.2 Let X, H, A, B be the same as in Example 2.8, and $M : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $M(x) = (-3\pi, -3x_2)$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$.

We claim that M is a 3-relaxed accretive mapping. Indeed, for any $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$

$$
\begin{aligned}
\langle Mx - My, x - y\rangle &= \bigl\langle(-3\pi, -3x_2) - (-3\pi, -3y_2), \bigl((x_1 - y_1), (x_2 - y_2)\bigr)\bigr\rangle \\
&= \bigl\langle\bigl(0, -3(x_2 - y_2)\bigr), \bigl((x_1 - y_1), (x_2 - y_2)\bigr)\bigr\rangle \\
&= -3(x_2 - y_2)^2 \\
&\geq -3\bigl\{(x_1 - y_1)^2 + (x_2 - y_2)^2\bigr\} \\
&\geq -3\|x - y\|^2, \\
\langle Mx - My, x - y\rangle &\geq (-3)\|x - y\|^2.
\end{aligned}
$$

Furthermore, M is also an $H(\cdot,\cdot)$-mixed mapping since $(H(A,B) + \rho M)(\mathbb{R}^2) = \mathbb{R}^2$ for any $\rho > 0$.

Proposition 3.3 *Let the set-valued mapping $M : X \to 2^X$ be an $H(\cdot,\cdot)$-mixed mapping with respect to mappings A and B. If A is α-expansive and $\mu > \gamma$ with $r = \mu\alpha^q - \gamma > m$, then the following inequality holds*:

$$
\bigl\langle x - y, J_q(u - v)\bigr\rangle \geq 0, \quad \forall (v,y) \in \operatorname{graph}(M), \quad \textit{implies} \quad x \in Mu.
$$

Proof Suppose on contrary that there exists $(u_0, x_0) \notin \operatorname{graph} M$ such that

$$\langle x_0 - y, J_q(u_0 - v)\rangle \geq 0, \quad \forall (v, y) \in \operatorname{graph}(M). \tag{3.1}$$

Since M is an $H(\cdot,\cdot)$-mixed mapping, we know that $(H(A,B) + \rho M)(X) = X$ holds for every $\rho > 0$, and so there exists $(u_1, x_1) \in \operatorname{graph}(M)$ such that

$$H(Au_0, Bu_0) + \rho x_0 = H(Au_1, Bu_1) + \rho x_1 \in X. \tag{3.2}$$

Now

$$\begin{aligned}
&\rho x - \rho x_1 = H(Au_1, Bu_1) - H(Au, Bu) \in X,\\
&\langle \rho x - \rho x_1, J_q(u - u_1)\rangle = \langle H(Au_1, Bu_1) - H(Au, Bu_0), J_q(u - u_1)\rangle.
\end{aligned}$$

Setting $(v, y) = (u_1, x_1)$ in (3.1) and then from the resultant (3.2) and m-relaxed accretivity of M, we obtain

$$\begin{aligned}
-m\|u_0 - u_1\|^q &\leq \rho\langle x_0 - x_1, J_q(u_0 - u_1)\rangle\\
&= -\langle H(Au_0, Bu_0) - H(Au_1, Bu_1), J_q(u_0 - u_1)\rangle\\
&= -\langle H(Au_0, Bu_0) - H(Au_1, Bu_0), J_q(u_0 - u_1)\rangle\\
&\quad - \langle H(Au_1, Bu_0) - H(Au_1, Bu_1), J_q(u_0 - u_1)\rangle.
\end{aligned} \tag{3.3}$$

Since $H(A,B)$ is μ-cocoercive with respect to A and γ-relaxed accretive with respect to B, and A is α-expansive, thus (3.3) becomes

$$\begin{aligned}
-m\|u_0 - u_1\|^q &\leq -\mu\|Au - Au_1\|^q + \gamma\|u - u_1\|^q\\
&\leq -\mu\alpha^2\|u_0 - u_1\|^q + \gamma\|u - u_1\|^q\\
&\leq -\big(\mu\alpha^q - \gamma\big)\|u_0 - u_1\|^q\\
&= -r\|u_0 - u_1\|^q \leq 0, \quad \text{where } r = \mu\alpha^q - \gamma\\
&\leq -(r - m)\|u_0 - u_1\|^q \leq 0.
\end{aligned}$$

It implies that $u_0 = u_1$ since $r > m$. By (3.1), we have $x_0 = x_1$, a contradiction. This completes the proof. □

Theorem 3.4 *Let the set-valued mapping $M : X \to 2^X$ be an $H(\cdot,\cdot)$-mixed mapping with respect to mappings A and B. If A is α-expansive and $\mu > \gamma$ with $r = \mu\alpha^q - \gamma > \rho m$, then $(H(A,B) + \rho M)^{-1}$ is single-valued.*

Proof For any given $u \in X$, let $x, y \in (H(A,B) + \rho M)^{-1}(u)$. It follows that

$$\begin{aligned}
&-H(Ax, Bx) + u \in \rho Mx,\\
&-H(Ay, By) + u \in \rho My.
\end{aligned}$$

Since M is m-relaxed accretive, we have

$$
\begin{aligned}
&-m\|x-y\|^q \leq \frac{1}{\rho}\big\langle -H(Ax,Bx)+u-\big(-H(Ay,By)+u\big),J_q(x-y)\big\rangle,\\
&-m\rho\|x-y\|^q = -\big\langle H(Ax,Bx)-H(Ay,Bx),J_q(x-y)\big\rangle\\
&\qquad\qquad - \big\langle H(Ay,Bx)-H(Ay,By),J_q(x-y)\big\rangle,
\end{aligned}
$$

which is like (3.3). Hence it follows that $\|x-y\| \leq 0$. This implies that $x=y$ and so $(H(A,B)+\rho M)^{-1}$ is single-valued. □

Definition 3.5 Let the set-valued mapping $M : X \to 2^X$ be an $H(\cdot,\cdot)$-mixed mapping with respect to mappings A and B. If A is α-expansive and $\mu > \gamma$ with $r = \mu\alpha^q - \gamma > \rho m$, then the *proximal-point mapping* $R^{H(\cdot,\cdot)}_{\rho,M} : X \to X$ is defined by

$$
R^{H(\cdot,\cdot)}_{\rho,M}(u) = \big(H(A,B)+\rho M\big)^{-1}(u), \quad \forall u \in X. \tag{3.4}
$$

Now we prove that the proximal-point mapping defined by (3.4) is Lipschitz continuous.

Theorem 3.6 *Let the set-valued mapping $M : X \to 2^X$ be an $H(\cdot,\cdot)$-mixed mapping with respect to mappings A and B. If A is α-expansive and $\mu > \gamma$ with $r = \mu\alpha^q - \gamma > \rho m$, then the proximal-point mapping $R^{H(\cdot,\cdot)}_{\rho,M} : X \to X$ is $\frac{1}{r-\rho m}$-Lipschitz continuous, that is,*

$$
\big\|R^{H(\cdot,\cdot)}_{\rho,M}(u) - R^{H(\cdot,\cdot)}_{\rho,M}(v)\big\| \leq \frac{1}{r-\rho m}\|u-v\|, \quad \forall u,v \in X.
$$

Proof Let u and $v \in X$ be any given points in X. It follows from (3.2) that

$$
\begin{aligned}
&\begin{cases}
R^{H(\cdot,\cdot)}_{\rho,M}(u) = H((A,B)+\rho M))^{-1}(u),\\
R^{H(\cdot,\cdot)}_{\rho,M}(v) = H((A,B)+\rho M))^{-1}(v),
\end{cases}\\
&\frac{1}{\rho}\big(u - H\big(A\big(R^{H(\cdot,\cdot)}_{\rho,M}(u)\big),B\big(R^{H(\cdot,\cdot)}_{\rho,M}(u)\big)\big)\big) \in M\big(R^{H(\cdot,\cdot)}_{\rho,M}(u)\big),\\
&\frac{1}{\rho}\big(v - H\big(A\big(R^{H(\cdot,\cdot)}_{\rho,M}(v)\big),B\big(R^{H(\cdot,\cdot)}_{\rho,M}(v)\big)\big)\big) \in M\big(R^{H(\cdot,\cdot)}_{\rho,M}(v)\big).
\end{aligned}
$$

Let $z_1 = R^{H(\cdot,\cdot)}_{\rho,M}(u)$ and $z_2 = R^{H(\cdot,\cdot)}_{\rho,M}(v)$.

Since M is m-relaxed accretive, we have

$$
\begin{aligned}
&\frac{1}{\rho}\big\langle\big(u - H\big(A(z_1),B(z_1)\big)\big) - \big(v - H\big(A(z_2),B(z_2)\big)\big),J_q(z_1-z_2)\big\rangle \geq -m\|z_1-z_2\|^q,\\
&\big\langle u - v - \big(H\big(A(z_1),B(z_1)\big) - H\big(A(z_2),B(z_2)\big)\big),J_q(z_1-z_2)\big\rangle \geq -\rho m\|z_1-z_2\|^q,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|u-v\|\|z_1-z_2\|^{q-1} &\geq \big\langle u-v,J_q(z_1-z_2)\big\rangle\\
&\geq \big\langle H\big(A(z_1),B(z_1)\big) - H\big(A(z_2),B(z_2)\big),J_q(z_1-z_2)\big\rangle - \rho m\|z_1-z_2\|^q
\end{aligned}
$$

$$\begin{aligned}
&\geq \langle H(A(z_1), B(z_1)) - H(A(z_2), B(z_1)), J_q(z_1 - z_2)\rangle \\
&\quad - \langle H(A(z_2), B(z_1)) - H(A(z_2), B(z_2)), J_q(z_1 - z_2)\rangle - \rho m\|z_1 - z_2\|^q \\
&\geq \mu \|A(z_1) - A(z_2)\|^q - \gamma \|z_1 - z_2\|^q - \rho m\|z_1 - z_2\|^q \\
&\geq \mu\alpha^q \|z_1 - z_2\|^q - \gamma \|z_1 - z_2\|^q - \rho m\|z_1 - z_2\|^q \\
&= (\mu\alpha^q - \gamma - \rho m)\|z_1 - z_2\|^q, \\
&= (r - \rho m)\|z_1 - z_2\|^q, \quad \text{where } r = \mu\alpha^q - \gamma,
\end{aligned}$$

and hence

$$\|u - v\|\|z_1 - z_2\|^{q-1} \geq (r - \rho m)\|z_1 - z_2\|^q,$$

that is,

$$\|R^{H(\cdot,\cdot)}_{\rho,M}(u) - R^{H(\cdot,\cdot)}_{\rho,M}(v)\| \leq \frac{1}{r - \rho m}\|u - v\|, \quad \forall u, v \in X.$$

This completes the proof. □

4 Graph convergence for an $H(\cdot,\cdot)$-mixed mapping

Let $M : X \to 2^X$ be a set-valued mapping. The graph of the map M is defined by

$$\operatorname{graph}(M) = \{(x, y) \in X \times X : y \in M(X)\}.$$

In this section we shall introduce the graph convergence for the $H(\cdot,\cdot)$-mixed mapping.

Definition 4.1 Let $M_n, M : X \to 2^X$ be the set-valued mappings such that M, M_n are $H(\cdot,\cdot)$-mixed mappings with respect to the mappings A and B for $n = 0, 1, 2, \ldots$. The sequence $\{M_n\}$ is said to be *graph convergent* to M, denoted by $M_n \xrightarrow{G} M$, if for every $(x, y) \in \operatorname{graph}(M)$, there exists a sequence $(x_n, y_n) \in \operatorname{graph}(M_n)$ such that

$$x_n \to x, \qquad y_n \to y \quad \text{as } n \to \infty.$$

Theorem 4.2 *Let $M_n, M : X \to 2^X$ be the set-valued mappings such that M, M_n are $H(\cdot,\cdot)$-mixed mappings with respect to the mappings A and B for $n = 0, 1, 2, \ldots$. Let $H(A, B)$ be s-Lipschitz continuous with respect to A and t-Lipschitz continuous with respect to B. If A is α-expansive and $\mu > \gamma$ with $r = \mu\alpha^q - \gamma > \rho m$, then $M_n \xrightarrow{G} M$ if and only if*

$$R^{H(\cdot,\cdot)}_{\rho,M_n}(u) \to R^{H(\cdot,\cdot)}_{\rho,M}(u), \quad \forall u \in X, \rho > 0,$$

where

$$R^{H(\cdot,\cdot)}_{\rho,M_n}(u) = (H(A, B) + \rho M_n)^{-1}(u), \qquad R^{H(\cdot,\cdot)}_{\rho,M}(u) = (H(A, B) + \rho M)^{-1}(u).$$

Proof It follows from Theorem 3.6 that $R^{H(\cdot,\cdot)}_{\rho,M_n}$ and $R^{H(\cdot,\cdot)}_{\rho,M}$ are both $\frac{1}{r-\rho m}$-Lipschitz continuous.

If part: Suppose that $M_n \xrightarrow{G} M$. For any given $x \in X$, let

$$z_n = R^{H(\cdot,\cdot)}_{\rho,M_n}(x), \qquad z = R^{H(\cdot,\cdot)}_{\rho,M}(x).$$

Then

$$\frac{1}{\rho}\big[x - H(Az, Bz)\big] \in M(z), \quad \text{that is} \quad \left(z, \frac{1}{\rho}\big[x - H(Az, Bz)\big]\right) \in \operatorname{graph}(M).$$

In the light of Definition 4.1, we know that there exists a sequence $(z'_n, y'_n) \in \operatorname{graph}(M_n)$ such that

$$z'_n \to z, \qquad y'_n \to \frac{1}{\rho}\big[x - H(Az, Bz)\big] \quad \text{as } n \to \infty. \tag{4.1}$$

Since $y'_n \in M_n(z'_n)$, we have

$$H\big(Az'_n, Bz'_n\big) + \rho y'_n \in \big[H(A,B) + \rho M_n\big]\big(z'_n\big)$$

and so

$$z'_n = \big[H\big(Az'_n, Bz'_n\big) + \rho y'_n\big].$$

From the Lipschitz continuity of M_n, we get

$$\begin{aligned}
\|z_n - z\| &\le \big\|z_n - z'_n\big\| + \big\|z'_n - z\big\| \\
&= \big\|R^{H(\cdot,\cdot)}_{\rho,M_n}(x) - R^{H(\cdot,\cdot)}_{\rho,M_n}\big[H\big(Az'_n, Bz'_n\big) + \rho y'_n\big]\big\| + \big\|z'_n - z\big\| \\
&\le \frac{1}{r - \rho m}\big\|x - H\big(Az'_n, Bz'_n\big) - \rho y'_n\big\| + \big\|z'_n - z\big\| \\
&\le \frac{1}{r - \rho m}\big[\big\|x - H(Az, Bz) - \rho y'_n\big\| \\
&\quad + \big\|H(Az, Bz) - H\big(Az'_n, Bz'_n\big)\big\|\big] + \big\|z'_n - z\big\|.
\end{aligned} \tag{4.2}$$

From the Lipschitz continuity of $H(A,B)$, we have

$$\begin{aligned}
&\big\|H(Az, Bz) - H\big(Az'_n, Bz'_n\big)\big\| \\
&\quad \le \big\|H(Az, Bz) - H\big(Az, Bz'_n\big)\big\| + \big\|H\big(Az, Bz'_n\big) - H\big(Az'_n, Bz'_n\big)\big\| \\
&\quad \le (s + t)\big\|z'_n - z\big\|.
\end{aligned} \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\|z_n - z\| \le \frac{1}{r - \rho m}\big\|x - H(Az, Bz) - \rho y'_n\big\| + \left[1 + \frac{1}{r - \rho m}(s + t)\right]\big\|z'_n - z\big\|.$$

By (4.1), we have

$$\big\|z'_n - z\big\| \to 0, \qquad \left\|\frac{1}{\rho}\big[x - H(Az, Bz) - y'_n\big]\right\| \to 0$$

and so

$$\|z_n - z\| \to 0 \quad \text{as } n \to \infty.$$

Only if part: Suppose that

$$R^{H(\cdot,\cdot)}_{\rho,M_n} \to R^{H(\cdot,\cdot)}_{\rho,M}, \quad \forall u \in X, \rho > 0.$$

For any given $(x, y) \in \text{graph}(M)$, we have

$$H(Ax, Bx) + \rho y \in \big[H(A, B) + \rho M\big](x)$$

and so

$$x = R^{H(\cdot,\cdot)}_{\rho,M}\big[H(Ax, Bx) + \rho y\big].$$

Let

$$x_n = R^{H(\cdot,\cdot)}_{\rho,M_n}\big[H(Ax, Bx) + \rho y\big].$$

Then

$$\frac{1}{\rho}\big[H(Ax, Bx) - H(Ax_n, Bx_n) + \rho y\big] \in M_n(x_n).$$

Let

$$y_n = \frac{1}{\rho}\big[H(Ax, Bx) - H(Ax_n, Bx_n) + \rho y\big].$$

It follows from (4.3) that

$$\begin{aligned}\|y_n - y\| &\leq \left\|\frac{1}{\rho}\big[H(Ax, Bx) - H(Ax_n, Bx_n) + \rho y\big] - y\right\| = \frac{1}{\rho}\big\|H(Ax, Bx) - H(Ax_n, Bx_n)\big\| \\ &\leq \frac{1}{\rho}(s + t)\|x_n - x\|. \end{aligned} \tag{4.4}$$

Since $R^{H(\cdot,\cdot)}_{\rho,M_n} \to R^{H(\cdot,\cdot)}_{\rho,M}$ for any $u \in X$, we know that $\|x_n - x\| \to 0$. Now (4.4) implies that

$$y_n \to y \quad \text{as } n \to \infty,$$

and so $M_n \xrightarrow{G} M$. This completes the proof. □

5 An application of the $H(\cdot, \cdot)$-mixed mapping for solving the system of generalized variational inclusions

Throughout the rest of the paper, unless otherwise stated, we assume that for each $i = 1, 2$, E_i is a q_i-uniformly smooth Banach space with the norm $\|\cdot\|_i$.

Let $A_1, B_1 : X_1 \to X_1$, $A_2, B_2 : X_2 \to X_2$, $N_1, H_1 : X_1 \times X_2 \to X_1$ and $N_2, H_2 : X_1 \times X_2 \to X_2$ be nonlinear mappings. Let $M_1 : X_1 \to 2^{X_1}$ be $H_1(\cdot,\cdot)$-mixed and $M_2 : X_2 \to 2^{X_2}$ be $H_2(\cdot,\cdot)$-mixed mappings, respectively. We consider the following system of generalized variational inclusions (SGVI): Find $(x, y) \in X_1 \times X_2$ such that

$$\begin{cases} \theta_1 \in N_1(x,y) + M_1(x); \\ \theta_2 \in N_2(x,y) + M_2(y), \end{cases} \tag{5.1}$$

where θ_1, θ_2 are zero vectors of X_1 and X_2, respectively. The problem of type (5.1) was studied by Zou and Huang [20].

Definition 5.1 Let $A : X_1 \to X_1$. A mapping $N : X_1 \times X_2 \to X_1$ is said to be:

(i) κ-strongly accretive in the first argument with respect to A if there exists a constant $\kappa > 0$ such that

$$\big\langle N(x_1,y) - N(x_2,y), J_q\big(A(x_1) - A(x_2)\big)\big\rangle_1 \geq \kappa \|x - y\|_1^{q_1}, \quad \forall x_1, x_2 \in X_1, y \in X_2;$$

(ii) L_N-*Lipschitz continuous in the first argument* if there exists a constant $L_N > 0$ such that

$$\big\|N(x_1,y) - N(x_2,y)\big\|_1 \leq L_N \|x_1 - x_2\|_1^{q_1}, \quad \forall x_1, x_2 \in X_1, y \in X_2;$$

(iii) l_N-*Lipschitz continuous in the second argument* if there exists a constant $l_N > 0$ such that

$$\big\|N(x,y_1) - N(x,y_2)\big\|_1 \leq l_N \|y_1 - y_2\|_2^{q_2}, \quad \forall x \in X_1, y_1, y_2 \in X_2.$$

The following lemma, which will be used in the sequel, is an immediate consequence of the definitions of $R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}$, $R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}$.

Lemma 5.2 *For any given* $(x, y) \in X_1 \times X_2$, (x, y) *is a solution of* (*SGVI*) (5.1) *if and only if* (x, y) *satisfies*

$$x = R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\big[H_1(A_1,B_1)(x) - \rho_1 N_1(x,y)\big], \tag{5.2}$$

$$y = R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}\big[H_2(A_2,B_2)(y) - \rho_2 N_2(x,y)\big], \tag{5.3}$$

where $R^{H_1(\cdot,\cdot)}_{\rho_1,M_1} = (H_1(A_1,B_1) + \rho_1 M_1)^{-1}$ *and* $R^{H_2(\cdot,\cdot)}_{\rho_2,M_2} = (H_2(A_2,B_2) + \rho_2 M_2)^{-1}$, *and* $\rho_1, \rho_2 > 0$ *are constants.*

Proof Consider first that an element $(x, y) \in X_1 \times X_2$ is a solution to (5.1). Then it follows that

$$\begin{aligned} &\theta_1 \in N_1(x,y) + M_1(x) \\ &\quad \Rightarrow \quad H_1(A_1x, B_1x) \in H_1(A_1x, B_1x) + \rho_1 N_1(x,y) + \rho_1 M_1(x) \\ &\quad \Rightarrow \quad H_1(A_1x, B_1x) - \rho_1 N_1(x,y) \in H_1(Ax, Bx) + \rho_1 M_1(x) \\ &\quad \Rightarrow \quad x = R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\big[H_1(A_1x, B_1x) - \rho_1 N_1(x,y)\big]. \end{aligned}$$

In a similar way, we can show that

$$y = R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}\big[H_2(A_2y,B_2y) - \rho_2 N_2(x,y)\big]. \qquad \square$$

A similar proof follows for the converse part:

$$\begin{aligned} x &= R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\big[H_1(A_1,B_1)(x) - \rho_1 N_1(x,y)\big] \\ &\Rightarrow \quad x = \big(H_1(A_1,B_1) + \rho_1 M_1\big)^{-1}\big[H_1(A_1,B_1)(x) - \rho_1 N_1(x,y)\big] \\ &\Rightarrow \quad \big(H_1(A_1,B_1) + \rho_1 M_1\big)(x) \ni \big(H_1(A_1x,B_1x)\big) - \rho_1 N_1(x,y) \\ &\Rightarrow \quad \theta_1 \in N_1(x,y) + M_1(x). \end{aligned}$$

In a similar way, we can show that

$$\theta_2 \in N_2(x,y) + M_2(y).$$

Theorem 5.3 *For each $i = 1,2$, let X_i be q_i-uniformly smooth Banach spaces, let $A_i, B_i : X_i \to X_i$ be single-valued mappings. Let the set-valued mappings $M_i : X_i \to 2^{X_i}$ be such that M_i are $H_i(\cdot,\cdot)$-mixed mappings with respect to mappings A_i and B_i, and A_i are α_i-expansive and $\mu_i > \gamma_i$ with $r_i = \mu_i\alpha_i^{q_i} - \gamma_i > \rho_i m_i$. Let $H_i : X_1 \times X_2 \to X_i$ be s_i-Lipschitz continuous with respect to A_i and t_i-Lipschitz continuous with respect to B_i, and let $N_i : X_1 \times X_2 \to X_i$ be a κ_i-strongly accretive mapping in the ith argument, L_{N_i}-Lipschitz continuous in the first argument and l_{N_i}-Lipschitz continuous in the second argument. Suppose that there are two constants $\rho_1, \rho_2 > 0$ satisfying the following conditions*:

$$\begin{cases} \tau_1 = a_1 + \rho_2 L_2 L_{N_2} < 1; \\ \tau_2 = a_2 + \rho_2 L_1 l_{N_1} < 1, \end{cases} \tag{5.4}$$

where

$$\begin{aligned} a_1 &= L_1\Big[\big(1 - 2q_1r_1 + c_{q_1}(s_1+t_1)^{q_1}\big)^{\frac{1}{q_1}} + \big(1 - 2\rho_1q_1\kappa_1 + c_{q_1}\rho_1^{q_1}L_1^{q_1}\big)^{\frac{1}{q_1}}\Big]; \\ a_2 &= L_2\Big[\big(1 - 2q_2r_2 + c_{q_2}(s_2+t_2)^{q_2}\big)^{\frac{1}{q_2}} + \big(1 - 2\rho_2q_2\kappa_2 + c_{q_2}\rho_2^{q_2}L_2^{q_2}\big)^{\frac{1}{q_2}}\Big]; \\ L_1 &= \frac{1}{r_1 - \rho_1 m_1}; \\ L_2 &= \frac{1}{r_2 - \rho_2 m_2}. \end{aligned}$$

Then SGVI (5.1) *has a unique solution $(x,y) \in X_1 \times X_2$.*

Proof For $i = 1,2$, it follows that for $(x,y) \in X_1 \times X_2$, the proximal-point mappings $R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}$ and $R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}$ are $\frac{1}{r_1-\rho_1 m_1}$-Lipschitz continuous and $\frac{1}{r_2-\rho_2 m_2}$-Lipschitz continuous, respectively.

Let $R : X_1 \times X_2 \to X_1 \times X_2$ be defined as follows:

$$R(x,y) = \big(P(x,y), Q(x,y)\big), \quad \forall (x,y) \in X_1 \times X_2, \tag{5.5}$$

where $P : X_1 \times X_2 \to X_1$ and $Q : X_1 \times X_2 \to X_2$ are defined by

$$P(x,y) = R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\bigl[H_1(A_1,B_1)(x) - \rho_1 N_1(x,y)\bigr] \tag{5.6}$$

and

$$Q(x,y) = R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}\bigl[H_2(A_2,B_2)(x) - \rho_2 N_2(x,y)\bigr] \tag{5.7}$$

for $\rho_1, \rho_2 > 0$, respectively.

For any $(x_1,y_1),(x_2,y_2) \in X_1 \times X_2$, it follows from (5.6) and (5.7) and the Lipschitz continuity of $R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}$ and $R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}$ that

$$\begin{aligned}
&\bigl\|P(x_1,y_1) - P(x_2,y_2)\bigr\|_1 \\
&\quad = \bigl\|R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\bigl[H_1(A_1,B_1)(x_1) - \rho N_1(x_1,y_1)\bigr] \\
&\qquad - R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\bigl[H_1(A_1,B_1)(x_2) - \rho N_1(x_2,y_2)\bigr]\bigr\|_1 \\
&\quad \le Ł_1\bigl[\bigl\|H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2) - \rho_1\bigl(N_1(x_1,y_1) - N_1(x_2,y_1)\bigr)\bigr\|_1 \\
&\qquad + \rho_1\bigl\|N_1(x_2,y_1) - N_1(x_2,y_2)\bigr\|_1\bigr]
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
&\bigl\|Q(x_1,y_1) - Q(x_2,y_2)\bigr\|_2 \\
&\quad \le Ł_2\bigl[\bigl\|H_2(A_2,B_2)(y_1) - H_2(A_2,B_2)(y_2) - \rho_2\bigl(N_2(x_1,y_1) - N_2(x_1,y_2)\bigr)\bigr\|_2 \\
&\qquad + \rho_2\bigl\|N_2(x_1,y_2) - N_2(x_2,y_2)\bigr\|_2\bigr].
\end{aligned} \tag{5.9}$$

Now

$$\begin{aligned}
&\bigl\|H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2) - \rho_1\bigl(N_1(x_1,y_1) - N_1(x_2,y_1)\bigr)\bigr\|_1 \\
&\quad = \bigl\|H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2) - (x_1 - x_2)\bigr\|_1 \\
&\qquad + \bigl\|x_1 - x_2 - \rho_1\bigl(N_1(x_1,y_1) - N_1(x_2,y_1)\bigr)\bigr\|_1.
\end{aligned} \tag{5.10}$$

Also,

$$\begin{aligned}
&\bigl\|H_2(A_2,B_2)(y_1) - H_2(A_2,B_2)(y_2) - (y_1 - y_2)\bigr\|_2 \\
&\quad = \bigl\|H_2(A_2,B_2)(y_1) - H_2(A_2,B_2)(y_2) - (y_1 - y_2)\bigr\|_2 \\
&\qquad + \bigl\|y_1 - y_2 - \rho_2\bigl(N_2(x_1,y_1) - N_1(x_1,y_2)\bigr)\bigr\|_2.
\end{aligned} \tag{5.11}$$

Now

$$\begin{aligned}
&\bigl\|H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2) - (x_1 - x_2)\bigr\|_1^{q_1} \\
&\quad \le \|x_1 - x_2\|_1^{q_1} - 2q_1\bigl\langle H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2) - J_{q_1}(x_1 - x_2)\bigr\rangle_1 \\
&\qquad + c_{q_1}\bigl\|H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2)\bigr\|_1^{q_1}.
\end{aligned} \tag{5.12}$$

Since M_1 is an $H(\cdot,\cdot)$-mixed mapping, then $H_1(A_1,B_1)$ is μ_1-cocoercive with respect to A_1 and γ_1-relaxed accretive with respect to B_1, and from the fact that A_1 is α_1-expansive, we can obtain

$$\begin{aligned}
&\langle H_1(A_1,B_1)(x_1) - H_1(A_2,B_2)(x_2) - J_{q_1}(x_1-x_2)\rangle_1 \\
&\quad = \langle H_1(A_1,B_1)(x_1) - H_1(A_2,B_1)(x_2) - J_{q_1}(x_1-x_2)\rangle_1 \\
&\qquad + \langle H_1(A_2,B_1)(x_1) - H_1(A_2,B_2)(x_2) - J_{q_1}(x_1-x_2)\rangle_1 \\
&\quad \le \mu_1 \| A_1(x_1) - A_1(x_2)\|^{q_1} - \gamma_1\|x_1-x_2\|^{q_1} \\
&\quad \le \mu_1\alpha_1^{q_1}\|x_1-x_2\|^{q_1} - \gamma_1\|x_1-x_2\|^{q_1} \\
&\quad = (\mu_1\alpha_1^{q_1} - \gamma_1)\|x_1-x_2\|^{q_1}.
\end{aligned} \tag{5.13}$$

Since $H_1(A_1,B_1)$ is s_1-Lipschitz continuous with respect to A_1 and t_1-Lipschitz continuous with respect to B_1, we have

$$\begin{aligned}
&\| H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2) - (x_1-x_2)\|_1 \\
&\quad \le \| H_1(A_1,B_1)(x_1) - H_1(A_2,B_1)(x_2)\|_1 + \| H_1(A_2,B_1)(x_1) - H_1(A_2,B_2)(x_2)\|_1 \\
&\quad \le (s_1+t_1)\|x_1-x_2\|_1.
\end{aligned} \tag{5.14}$$

Using (5.12), (5.13) and (5.14), we have

$$\begin{aligned}
&\| H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2) - (x_1-x_2)\|_1^{q_1} \\
&\quad \le \|x_1-x_2\|_1^{q_1} - 2q_1\langle H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2) - J_{q_1}(x_1-x_2)\rangle_1 \\
&\quad \le [1 - 2q_1r_1 + c_{q_1}(s_1+t_1)^{q_1}]\|x_1-x_2\|_1^{q_1},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\| H_1(A_1,B_1)(x_1) - H_1(A_1,B_1)(x_2) - (x_1-x_2)\|_1 \\
&\quad \le [1 - 2q_1r_1 + c_{q_1}(s_1+t_1)^{q_1}]^{\frac{1}{q_1}}\|x_1-x_2\|_1, \quad \text{where } r_1 = \mu_1\alpha_1^{q_1} - \gamma_1.
\end{aligned} \tag{5.15}$$

In the light of (5.15), we can obtain

$$\begin{aligned}
&\| H_2(A_2,B_2)(y_1) - H_2(A_2,B_2)(y_2) - (y_1-y_2)\|_2 \\
&\quad \le [1 - 2q_2r_2 + c_{q_2}(s_2+t_2)^{q_2}]^{\frac{1}{q_2}}\|y_1-y_2\|_2, \quad \text{where } r_2 = \mu_2\alpha_2^{q_2} - \gamma_2.
\end{aligned} \tag{5.16}$$

Again, since N_i is κ_i-strongly accretive in the first argument and L_{N_i}-Lipschitz continuous in the first argument and l_{N_i}-Lipschitz continuous in the second argument, then using Lemma 2.4, we have

$$\begin{aligned}
&\| x_1 - x_2 - \rho_1(N_1(x_1,y_1) - N_1(x_2,y_1))\|_1^{q_1} \\
&\quad \le \|x_1-x_2\|_1^{q_1} - 2\rho_1q_1\langle N_1(x_1,y_1) - N_1(x_2,y_1), J_{q_1}(x_1-x_2)\rangle_1
\end{aligned}$$

$$\begin{aligned}
&+ \rho_1^{q_1} c_{q_1} \big\| N_1(x_1, y_1) - N_1(x_2, y_1) \big\|_1^{q_1} \\
&\le \big(1 - 2\rho_1 q_1 \kappa_1 + c_{q_1} \rho_1^{q_1} L_1^{q_1}\big) \|x_1 - x_2\|_1^{q_1},
\end{aligned}$$

which implies that

$$\big\| x_1 - x_2 - \rho_1\big(N_1(x_1, y_1) - N_1(x_2, y_1)\big)\big\|_1 \le \big(1 - 2\rho_1 q_1 \kappa_1 + c_{q_1} \rho_1^{q_1} L_1^{q_1}\big)^{\frac{1}{q_1}} \|x_1 - x_2\|_1. \tag{5.17}$$

In the light of (5.17), we have

$$\big\| y_1 - y_2 - \rho_2\big(N_2(x_1, y_1) - N_2(x_1, y_2)\big)\big\|_2 \le \big(1 - 2\rho_2 q_2 \kappa_2 + c_{q_2} \rho_2^{q_2} L_2^{q_2}\big)^{\frac{1}{q_2}} \|y_1 - y_2\|_2. \tag{5.18}$$

Using (5.8), (5.15) and (5.17), we have

$$\begin{aligned}
&\big\| P(x_1, y_1) - P(x_2, y_2) \big\|_1 \\
&\quad \le \big(L_1\big[\big(1 - 2q_1 r_1 + c_{q_1}(s_1 + t_1)^{q_1}\big)^{\frac{1}{q_1}} + \big(1 - 2\rho_1 q_1 \kappa_1 + c_{q_1} \rho_1^{q_1} L_1^{q_1}\big)^{\frac{1}{q_1}}\big]\big) \|x_1 - x_2\|_1 \\
&\qquad + L_1 \rho_1 l_{N_1} \|y_1 - y_2\|_2.
\end{aligned} \tag{5.19}$$

Using (5.9), (5.16) and (5.18), we have

$$\begin{aligned}
&\big\| Q(x_1, y_1) - Q(x_2, y_2) \big\|_2 \\
&\quad \le \big(L_2\big[\big(1 - 2q_2 r_2 + c_{q_2}(s_2 + t_2)^{q_2}\big)^{\frac{1}{q_2}} + \big(1 - 2\rho_2 q_2 \kappa_2 + c_{q_2} \rho_2^{q_2} L_2^{q_2}\big)^{\frac{1}{q_2}}\big]\big) \|y_1 - y_2\|_2 \\
&\qquad + L_2 \rho_2 l_{N_2} \|x_1 - x_2\|_1.
\end{aligned} \tag{5.20}$$

From (5.19) and (5.20), we have

$$\begin{aligned}
&\big\| P(x_1, y_1) - P(x_2, y_2) \big\|_1 + \big\| Q(x_1, y_1) - Q(x_2, y_2) \big\|_2 \\
&\quad \le \tau_1 \|x_1 - x_2\|_1 + \tau_2 \|y_1 - y_2\|_2 \\
&\quad \le \max\{\tau_1, \tau_2\}\big(\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2\big),
\end{aligned} \tag{5.21}$$

where

$$\begin{cases}
\tau_1 = a_1 + \rho_2 L_2 L_{N_2}; \\
\tau_2 = a_2 + \rho_2 L_1 l_{N_1},
\end{cases} \tag{5.22}$$

and

$$\begin{aligned}
&a_1 = L_1\big[\big(1 - 2q_1 r_1 + c_{q_1}(s_1 + t_1)^{q_1}\big)^{\frac{1}{q_1}} + \big(1 - 2\rho_1 q_1 \kappa_1 + c_{q_1} \rho_1^{q_1} L_1^{q_1}\big)^{\frac{1}{q_1}}\big]; \\
&a_2 = L_2\big[\big(1 - 2q_2 r_2 + c_{q_2}(s_2 + t_2)^{q_2}\big)^{\frac{1}{q_2}} + \big(1 - 2\rho_2 q_2 \kappa_2 + c_{q_2} \rho_2^{q_2} L_2^{q_2}\big)^{\frac{1}{q_2}}\big]; \\
&L_1 = \frac{1}{r_1 - \rho_1 m_1}; \\
&L_2 = \frac{1}{r_2 - \rho_2 m_2}.
\end{aligned}$$

Now define the norm $\|\cdot\|_\star$ on $X_1 \times X_2$ by

$$\|(x,y)\|_\star = \|x\|_1 + \|y\|_2, \quad \forall (x,y) \in X_1 \times X_2. \tag{5.23}$$

We observe that $(X_1 \times X_2, \|\cdot\|_\star)$ is a Banach space. Hence it follows from (5.5), (5.21) and (5.23) that

$$\|R(x_1,y_1) - R(x_2,y_2)\|_\star = \max\{\tau_1,\tau_2\}\|(x_1,y_1) - (x_2,y_2)\|_\star. \tag{5.24}$$

Since $\max\{\tau_1,\tau_2\} < 1$ by (5.2), it follows from (5.24) that R is a contraction mapping. Hence, by the Banach contraction principle, there exists a unique point $(x,y) \in X_1 \times X_2$ such that

$$R(x,y) = (x,y),$$

which implies that

$$x = R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\big[H_1(A_1,B_1)(x) - \rho_1 N_1(x,y)\big],$$
$$y = R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}\big[H_2(A_2,B_2)(y) - \rho_2 N_2(x,y)\big].$$

It follows from Lemma 5.2 that (x,y) is a unique solution of SGVI (5.1). This completes the proof. □

6 Convergence of an iterative algorithm for SGVI (5.1)

Based on Lemma 5.2, we suggest and analyze the following iterative algorithm for finding an approximate solution for SGVI (5.1).

Algorithm 6.1 For any given $(x_0,y_0) \in X_1 \times X_2$, $(x_n,y_n) \in X_1 \times X_2$ by an iterative scheme

$$x_{n+1} = R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\big[H_1(A_1,B_1)(x_n) - \rho_1 N_1(x_n,y_n)\big], \tag{6.1}$$
$$y_{n+1} = R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}\big[H_2(A_2,B_2)(y_n) - \rho_2 N_2(x_n,y_n)\big], \tag{6.2}$$

where $n = 0,1,2,\ldots$ and $\rho_1, \rho_2 > 0$ are constants.

Theorem 6.2 *For each $i = 1,2$, let X_i be q_i-uniformly smooth Banach spaces, let $A_i, B_i : X_i \to X_i$ be single-valued mappings. Let the set-valued mappings $M_{n_i}, M_i : X_i \to 2^{X_i}$ be such that M_{n_i}, M_i are $H_i(\cdot,\cdot)$-mixed mappings with respect to mappings A_i and B_i such that $M_{n_i} \xrightarrow{G} M_i$ for $n = 0,1,2,\ldots$, and A_i is α_i-expansive and $\mu_i > \gamma_i$ with $r_i = \mu_i\alpha_i^{q_i} - \gamma_i > \rho_i m_i$. Let $H_i : X_1 \times X_2 \to X_i$ be s_i-Lipschitz continuous with respect to A_i and t_i-Lipschitz continuous with respect to B_i, and let $N_i : X_1 \times X_2 \to X_i$ be a κ_i-strongly accretive mapping in the ith argument, L_{N_i}-Lipschitz continuous in the first argument and l_{N_i}-Lipschitz continuous in the second argument. Suppose that there are two constants $\rho_1, \rho_2 > 0$ satisfying the following conditions*:

$$\begin{cases} \tau_1 = a_1 + \rho_2 L_2 L_{N_2} < 1; \\ \tau_2 = a_2 + \rho_2 L_1 l_{N_1} < 1, \end{cases} \tag{6.3}$$

where

$$a_1 = L_1\big[\big(1 - 2q_1r_1 + c_{q_1}(s_1 + t_1)^{q_1}\big)^{\frac{1}{q_1}} + \big(1 - 2\rho_1q_1\kappa_1 + c_{q_1}\rho_1^{q_1}L_1^{q_1}\big)^{\frac{1}{q_1}}\big];$$
$$a_2 = L_2\big[\big(1 - 2q_2r_2 + c_{q_2}(s_2 + t_2)^{q_2}\big)^{\frac{1}{q_2}} + \big(1 - 2\rho_2q_2\kappa_2 + c_{q_2}\rho_2^{q_2}L_2^{q_2}\big)^{\frac{1}{q_2}}\big];$$
$$L_1 = \frac{1}{r_1 - \rho_1m_1};$$
$$L_2 = \frac{1}{r_2 - \rho_2m_2}.$$

Then the approximate solution (x_n, y_n) *generated by Algorithm* 6.1 *converges strongly to the unique solution* (x, y) *of SGVI* (5.1).

Proof By Theorem 5.3, there exists a unique solution $(x, y) \in X_1 \times X_2$ of SGVI (5.1). It follows from Algorithm 6.1 and Theorem 3.6 that

$$\begin{aligned}\|x_{n+1} - x\| &= \big\|R^{H_1(\cdot,\cdot)}_{\rho_1,M_{n_1}}\big[H_1(A_1,B_1)(x_n) - \rho_1N_1(x_n,y_n)\big] \\ &\quad - R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\big[H_1(A_1,B_1)(x) - \rho_1N_1(x,y)\big]\big\|_1 \\ &\leq \big\|R^{H_1(\cdot,\cdot)}_{\rho_1,M_{n_1}}\big[H_1(A_1,B_1)(x_n) - \rho_1N_1(x_n,y_n)\big] \\ &\quad - R^{H_1(\cdot,\cdot)}_{\rho_1,M_{n_1}}\big[H_1(A_1,B_1)(x) - \rho_1N_1(x,y)\big]\big\|_1 \\ &\quad + \big\|R^{H_1(\cdot,\cdot)}_{\rho_1,M_{n_1}}\big[H_1(A_1,B_1)(x) - \rho_1N_1(x,y)\big] \\ &\quad - R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\big[H_1(A_1,B_1)(x) - \rho_1N_1(x,y)\big]\big\|_1\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\|y_{n+1} - y\| &\leq \big\|R^{H_2(\cdot,\cdot)}_{\rho_2,M_{n_2}}\big[H_2(A_2,B_2)(y_n) - \rho_2N_2(x_n,y_n)\big] \\ &\quad - R^{H_2(\cdot,\cdot)}_{\rho_2,M_{n_2}}\big[H_2(A_2,B_2)(y) - \rho_2N_2(x,y)\big]\big\|_2 \\ &\quad + \big\|R^{H_2(\cdot,\cdot)}_{\rho_2,M_{n_2}}\big[H_2(A_2,B_2)(y) - \rho_2N_2(x,y)\big] \\ &\quad - R^{H_2(\cdot,\cdot)}_{\rho_2,M_{n_2}}\big[H_2(A_2,B_2)(y) - \rho_2N_2(x,y)\big]\big\|_2.\end{aligned} \tag{6.5}$$

By (5.8) and (5.19), we have

$$\begin{aligned}&\big\|R^{H_1(\cdot,\cdot)}_{\rho_1,M_{n_1}}\big[H_1(A_1,B_1)(x_n) - \rho_1N_1(x_n,y_n)\big] \\ &\qquad - R^{H_1(\cdot,\cdot)}_{\rho_1,M_{n_1}}\big[H_1(A_1,B_1)(x) - \rho_1N_1(x,y)\big]\big\|_1 \\ &\quad \leq Ł_1\big[\big\|H_1(A_1,B_1)(x_n) - H_1(A_1,B_1)(x) - \rho_1\big(N_1(x_n,y_n) - N_1(x,y_n)\big)\big\|_1 \\ &\qquad + \rho_1\big\|N_1(x,y_n) - N_1(x,y)\big\|_1\big] \\ &\quad \leq \big(L_1\big[\big(1 - 2q_1r_1 + c_{q_1}(s_1 + t_1)^{q_1}\big)^{\frac{1}{q_1}} + \big(1 - 2\rho_1q_1\kappa_1 + c_{q_1}\rho_1^{q_1}L_1^{q_1}\big)^{\frac{1}{q_1}}\big]\big) \\ &\qquad \times \|x_n - x\|_1 + L_1\rho_1l_{N_1}\|y_n - y\|_2\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}
&\big\| R^{H_2(\cdot,\cdot)}_{\rho_2,M_{n_2}}\big[H_2(A_2,B_2)(y_n)-\rho_2N_2(x_n,y_n)\big]-R^{H_2(\cdot,\cdot)}_{\rho_2,M_{n_2}}\big[H_2(A_2,B_2)(y)-\rho_2N_2(x,y)\big]\big\|_2\\
&\quad\le Ł_2\big[\big\|H_2(A_2,B_2)(y_n)-H_2(A_2,B_2)(y)-\rho_2\big(N_2(x_n,y_n)-N_1(x_n,y)\big)\big\|_2\\
&\qquad+\rho_2\big\|N_2(x_n,y)-N_2(x,y)\big\|_2\big]\\
&\quad\le \big(L_2\big[\big(1-2q_2r_2+c_{q_2}(s_2+t_2)^{q_2}\big)^{\frac{1}{q_2}}+\big(1-2\rho_2q_2\kappa_2+c_{q_2}\rho_2^{q_2}L_2^{q_2}\big)^{\frac{1}{q_2}}\big]\big)\\
&\qquad\times\|y_n-y\|_2+L_2\rho_2l_{N_2}\|x_n-x\|_1. \qquad (6.7)
\end{aligned}$$

By Theorem 4.2, we have

$$R^{H_1(\cdot,\cdot)}_{\rho_1,M_{n_1}}\big[H_1(A_1,B_1)(x)-\rho_1N_1(x,y)\big]\to R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\big[H_1(A_1,B_1)(x)-\rho_1N_1(x,y)\big], \qquad (6.8)$$

$$R^{H_2(\cdot,\cdot)}_{\rho_2,M_{n_2}}\big[H_2(A_2,B_2)(y)-\rho_2N_2(x,y)\big]\to R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}\big[H_1(A_2,B_2)(y)-\rho_2N_2(x,y)\big]. \qquad (6.9)$$

Let

$$\begin{aligned}
b_{n_1}&=\big\|R^{H_1(\cdot,\cdot)}_{\rho_1,M_{n_1}}\big[H_1(A_1,B_1)(x)-\rho_1N_1(x,y)\big]\\
&\qquad-R^{H_1(\cdot,\cdot)}_{\rho_1,M_1}\big[H_1(A_1,B_1)(x)-\rho_1N_1(x,y)\big]\big\|_1, \qquad (6.10)\\
b_{n_2}&=\big\|R^{H_2(\cdot,\cdot)}_{\rho_2,M_{n_2}}\big[H_2(A_2,B_2)(y)-\rho_2N_2(x,y)\big]\\
&\qquad-R^{H_2(\cdot,\cdot)}_{\rho_2,M_2}\big[H_2(A_2,B_2)(y)-\rho_2N_2(x,y)\big]\big\|_2. \qquad (6.11)
\end{aligned}$$

From (6.4)-(6.11), we have

$$\|x_{n+1}-x\|_1\le\tau_1\|x_n-x\|_1+b_{n_1}, \qquad (6.12)$$

$$\|y_{n+1}-y\|_1\le\tau_2\|y_n-y\|_1+b_{n_2}. \qquad (6.13)$$

From (6.12) and (6.13), we have

$$\|x_{n+1}-x\|_1+\|y_{n+1}-y\|_2\le\max\{\tau_1,\tau_2\}\big\{\|x_n-x\|_1+\|y_n-y\|_2\big\}+\{b_{n_1}+b_{n_2}\}. \qquad (6.14)$$

Since $(X_1\times X_2,\|\cdot\|_\star)$ is a Banach space with the norm $\|\cdot\|_\star$ defined by (5.23), it follows from (5.5), (5.23) and (6.14) that

$$\begin{aligned}
\big\|(x_{n+1},y_{n+1})-(x,y)\big\|_\star&=\|x_{n+1}-x\|_1+\|y_{n+1}-y\|_2\\
&\le\max\{\tau_1,\tau_2\}\big\|(x_n,y_n)-(x,y)\big\|+\{b_{n_1}+b_{n_2}\}. \qquad (6.15)
\end{aligned}$$

By condition (6.3), it follows that $\max\{\tau_1,\tau_2\}<1$ and Lemma 2.5 implies that $\|(x_{n+1},y_{n+1})-(x,y)\|_\star\to 0$ as $n\to\infty$.

Thus $\{(x_n,y_n)\}$ converges strongly to the unique solution (x,y) of SGVI (5.1). This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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References

1. Aubin, JP, Cellina, A: Differential Inclusions. Springer, Berlin (1984)
2. Fang, Y-P, Huang, N-J: *H*-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces. Appl. Math. Lett. **17**(6), 647-653 (2004)
3. Fang, YP, Cho, Y-J, Kim, JK: (H,η)-accretive operator and approximating solutions for systems of variational inclusions in Banach spaces. Appl. Math. Lett. (to appear)
4. Huang, N-J, Fang, Y-P: Generalized *m*-accretive mappings in Banach spaces. J. Sichuan Univ. **38**(4), 591-592 (2001)
5. Huang, N-J: A new class of generalized set-valued implicit variational inclusions in Banach spaces with an application. Comput. Math. Appl. **41**(7-8), 937-943 (2001)
6. Kazmi, KR, Khan, FA: Iterative approximation of a solution of multi-valued variational-like inclusion in Banach spaces: a P-η-proximal-point mapping approach. J. Math. Anal. Appl. **325**(1), 665-674 (2007)
7. Kazmi, KR, Khan, FA: Sensitivity analysis for parametric generalized implicit quasi-variational-like inclusions involving P-η-accretive mappings. J. Math. Anal. Appl. **337**(2), 1198-1210 (2008)
8. Kazmi, KR, Ahmad, N, Shahzad, M: Convergence and stability of an iterative algorithm for a system of generalized implicit variational-like inclusions in Banach spaces. Appl. Math. Comput. **218**, 9208-9219 (2012)
9. Kazmi, KR, Bhat, MI, Ahmad, N: An iterative algorithm based on *M*-proximal mappings for a system of generalized implicit variational inclusions in Banach spaces. J. Comput. Appl. Math. **233**, 361-371 (2009)
10. Kazmi, KR, Khan, FA, Shahzad, M: A system of generalized variational inclusions involving generalized $H(\cdot,\cdot)$-accretive mapping in real *q*-uniformly smooth Banach spaces. Appl. Math. Comput. **217**(2), 9679-9688 (2011)
11. Lan, H-Y, Cho, YJ, Verma, RU: Nonlinear relaxed cocoercive variational inclusions involving (A,η)-accretive mappings in Banach spaces. Comput. Math. Appl. **51**(9-10), 1529-1538 (2006)
12. Li, X, Huang, N-J: Graph convergence for the $H(\cdot,\cdot)$-accretive operator in Banach spaces with an application. Appl. Math. Comput. **217**, 9053-9061 (2011)
13. Liu, LS: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**, 114-127 (1995)
14. Peng, JW: Set-valued variational inclusions with *T*-accretive operator in Banach spaces. Appl. Math. Lett. **16**(6), 273-282 (2006)
15. Sun, JH, Zhang, LW, Xiao, XT: An algorithm based on resolvent operators for solving variational inequalities in Hilbert spaces. Nonlinear Anal. TMA **69**, 3344-3357 (2008)
16. Verma, RU: The generalized relaxed proximal point algorithm involving *A*-maximal-relaxed accretive mappings with applications to Banach spaces. Math. Comput. Model. **50**(7-8), 1026-1032 (2009)
17. Verma, RU: General class of implicit variational inclusions and graph convergence on *A*-maximal relaxed monotonicity. J. Optim. Theory Appl. **129**(3), 457-467 (2012)
18. Xia, FQ, Huang, N-J: Variational inclusions with a general *H*-monotone operator in Banach spaces. Comput. Math. Appl. **54**, 24-30 (2007)
19. Zou, Y-Z, Huang, N-J: $H(\cdot,\cdot)$-accretive operator with an application for solving variational inclusions in Banach spaces. Appl. Math. Comput. **204**(2), 809-816 (2008)
20. Zou, Y-Z, Huang, N-J: A new system of variational inclusions involving $H(\cdot,\cdot)$-accretive operator in Banach spaces. Appl. Math. Comput. **212**, 135-144 (2009)
21. Attouch, H: Variational Convergence for Functions and Operators. Applied Mathematics Series. Pitman, London (1984)
22. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**(12), 1127-1138 (1991)

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