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Some new results on fixed and best proximity points in preordered metric spaces

Alireza Amini-Harandi[1,2], Majid Fakhar[2,3], Hamid Reza Hajisharifi[3] and Nawab Hussain[4*]

Dedicated to Professor Wataru Takahashi on the occasion of his seventieth birthday

*Correspondence:
nhusain@kau.edu.sa
[4]Department of Mathematics, King
Abdulaziz University, P.O. Box 80203,
Jeddah, 21589, Saudi Arabia
Full list of author information is
available at the end of the article

Abstract

In this paper, we first introduce two new classes of (ω, δ)-contractions of the first and second kinds and establish some related new fixed point and best proximity point theorems in preordered metric spaces. Our theorems subsume the corresponding recent results of Samet (J. Optim. Theory Appl. (2013), doi:10.1007/s10957-013-0269-9) and extend and generalize many of the well-known results in the literature. An example is also provided to support our main results.

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1 Introduction and preliminaries

Given a metric space (X, d) and a self-mapping T on X, the theory on the existence of a solution to the equation of the form $Tx = x$ has gained impetus because of its applicability to solve many interesting problems that can be formulated as ordinary differential equations, matrix equations *etc.* For some recent fixed point results, see [1–6] and references therein. Let A and B be nonempty subsets of X, and let $T : A \rightarrow B$ be a non-self mapping. The equation $Tx = x$ is unlikely to have a solution, because of the fact that a solution of the preceding equation demands the nonemptiness of $A \cap B$. Eventually, it is quite natural to seek an approximate solution x that is optimal in the sense that the distance $d(x, Tx)$ is minimum. The well-known best approximation theorem, due to Fan [7], states that if A is a nonempty, compact, and convex subset of a normed linear space X and T is a continuous function from A to X, then there exists a point x in A such that $\|x - Tx\| = d(Tx, A) = \inf\{\|Tx - u\| : u \in A\}$. Such a point x is called a best approximant point of T in A. Many generalizations and extensions of this theorem appeared in the literature (see [8–11] and references therein).

Best proximity problem for the pairs (A, B) is to find an element $x \in A$ such that $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. Since $d(A, B)$ is a lower bound for the function $x \rightarrow d(x, Tx)$ on A, then the solutions of the best proximity problem are the minimum points of the function $x \rightarrow d(x, Tx)$ on A. Every solution of the best proximity problem is said to be a best proximity point of T in A. Moreover, if $A = B$ then every best proximity point of T is a fixed point. According to this fact, many authors by motivation of well-known fixed point results obtained sufficient conditions to solving best proximity problems; for more details, see [12–27] and the references therein.

Existence of best proximity and fixed points in partially ordered metric spaces has been considered recently by many authors (see [6, 13, 20, 28]). Recently Samet [29] studied the existence of best proximity points for a class of non-self almost (φ, θ)-contractive mappings. In this work we define two new classes of contractions called (ω, δ)-contractions of the first and second kind and establish some related new fixed point results in the setting of preordered metric spaces, and then we derive some new best proximity point theorems for these new classes of non-self contractive mappings. The presented theorems extend and generalize many of the well-known fixed point and best proximity point results.

2 Fixed point theory

Definition 2.1 Let (X, d) be a metric space, and let $R_+ = [0, \infty)$.

(a) Denote by Ω the family of functions $\omega : R_+ \to R_+$ such that $\omega(0) = 0$, $\omega(t) < t$ for each $t > 0$ and for each sequence $\{x_n\}$ in X with,

$$d(x_n, x_{n+1}) \leq \omega\big(d(x_{n-1}, x_n)\big) \quad \forall n \in \mathbb{N} \Rightarrow \{x_n\} \text{ is a Cauchy sequence;}$$

(b) Denote by Δ the family of functions $\delta : R_+^4 \to R$ such that δ is continuous and if $t_i = 0$ for some $i \in \{1, 2, 3, 4\}$, then $\delta(t_1, t_2, t_3, t_4) = 0$;

(c) Denote by Φ the family of non-decreasing functions $\phi : R_+ \to R_+$ such that $\Sigma_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$;

(d) Denote by Σ the family of functions $\sigma : R_+ \to R_+$ such that $\sigma(t) = \alpha(t)t$ for each $t > 0$ and $\alpha : R_+ \to [0, 1)$ satisfies

$$\limsup_{s \to t} \alpha(s) < 1 \quad \text{for each } t > 0; \tag{1}$$

(e) Denote by Ψ the family of non-decreasing functions $\psi : R_+ \to R_+$ such that $\lim_{n \to \infty} \psi^n(t) = 0$ for each $t > 0$;

(f) Denote by Λ the family of non-decreasing and upper semicontinuous from the right functions $\lambda : R_+ \to R_+$ such that $\lambda(t) < t$ for each $t > 0$;

(g) Let Θ be a collection of the following functions:
$\theta(t_1, t_2, t_3, t_4) = \tau \min\{t_1, t_2, t_3, t_4\}, \tau > 0;$
$\theta(t_1, t_2, t_3, t_4) = \tau \ln(1 + t_1 t_2 t_3 t_4), \tau > 0;$
$\theta(t_1, t_2, t_3, t_4) = \tau t_1 t_2 t_3 t_4, \tau > 0.$

Lemma 2.2 *Let (X, d) be a metric space. Then the following statements hold*:

(i) $\Phi \subseteq \Omega$,

(ii) $\Sigma \subseteq \Omega$,

(iii) $\Psi \subseteq \Omega$,

(iv) $\Lambda \subseteq \Psi \subseteq \Omega$,

(v) $\Theta \subseteq \Delta$.

Proof Let $\{x_n\}$ be a sequence in X. To prove (i), assume that $d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n))$ for each $n \in \mathbb{N}$, where $\phi \in \Phi$. Since ϕ is non-decreasing, then by induction we get

$$d(x_n, x_{n+1}) \leq \phi^n\big(d(x_0, x_1)\big) \quad \text{for each } n \in \mathbb{N}.$$

Then, for each sufficiently large $m < n$, we have

$$d(x_m, x_n) \leq \sum_{j=m}^{n-1} d(x_j, x_{j+1}) \leq \sum_{j=m}^{n-1} \phi^n \big(d(x_0, x_1)\big) < \epsilon,$$

and so $\{x_n\}$ is a Cauchy sequence.

(ii) Let us suppose that $d(x_n, x_{n+1}) \leq \sigma(d(x_{n-1}, x_n))$ for each $n \in \mathbb{N}$, where $\sigma(t) = \alpha(t)t$ for each $t > 0$ and α satisfying (1). Then

$$d(x_n, x_{n+1}) \leq \alpha\big(d(x_{n-1}, x_n)\big)d(x_{n-1}, x_n) \tag{2}$$

for each $n \in \mathbb{N}$. Since $\alpha(t) < 1$ for each $t \in R_+$, then $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of non-negative numbers and so is convergent to a real number, say r_0. We will show that $r_0 = 0$. On the contrary, assume that $r_0 > 0$. Then from (2) we get

$$1 = \lim_{n \to \infty} \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \limsup_{n \to \infty} \alpha\big(d(x_{n-1}, x_n)\big) \leq \limsup_{s \to r_0} \alpha(s),$$

a contradiction and so $r_0 = 0$. To show that $\{x_n\}$ is a Cauchy sequence, on the contrary assume that $\limsup_{m,n \to \infty} d(x_m, x_n) = \infty$. Thus there exist subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that $\lim_{k \to \infty} d(x_{m_k}, x_{n_k}) > 0$. Then, by the triangle inequality, we get

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k})$$

$$\leq d(x_{m_k}, x_{m_k+1}) + \alpha\big(d(x_{m_k}, x_{n_k})\big)d(x_{m_k}, x_{n_k}) + d(x_{n_k+1}, x_{n_k}).$$

Then

$$\big(1 - \alpha\big(d(x_{m_k}, x_{n_k})\big)\big)d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k})$$

for each $k \in \mathbb{N}$. From the above, we obtain $\lim_{k \to \infty} \alpha(d(x_{m_k}, x_{n_k})) = 1$. Then from (1) we get $\lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = 0$, a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence.

(iii) Notice first that $\psi(t) < t$ for each $t > 0$. To see this, suppose that there exists $t_0 > 0$ with $\psi(t_0) > t_0$, then since ψ is non-decreasing, we see that $t_0 \leq \psi^n(t_0)$ for all $n \in \mathbb{N}$ and it is a contradiction with $\lim_{n \to \infty} \psi^n(t) = 0$ for each $t > 0$. Note also that $\psi(0) = 0$.

Now assume that $d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n))$ for each $n \in \mathbb{N}$, where $\psi \in \Psi$. Since ψ is non-decreasing, then by induction we get

$$d(x_n, x_{n+1}) \leq \psi^n\big(d(x_0, x_1)\big) \quad \text{for each } n \in \mathbb{N}.$$

Let $\epsilon > 0$ be fixed. Choose $n \in \mathbb{N}$ such that

$$d(x_{n+1}, x_n) < \epsilon - \psi(\epsilon).$$

Now we have

$$d(x_{n+2}, x_n) \leq d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)$$

$$\leq \psi\big(d(x_{n+1}, x_n)\big) + \big(\epsilon - \psi(\epsilon)\big)$$

$$\leq \psi\big(\epsilon - \psi(\epsilon)\big) + \big(\epsilon - \psi(\epsilon)\big)$$

$$\leq \psi(\epsilon) + \big(\epsilon - \psi(\epsilon)\big) = \epsilon.$$

Also we have

$$d(x_{n+3}, x_n) \leq d(x_{n+3}, x_{n+1}) + d(x_{n+1}, x_n)$$

$$\leq \psi\big(d(x_{n+2}, x_n)\big) + \big(\epsilon - \psi(\epsilon)\big)$$

$$\leq \psi(\epsilon) + \big(\epsilon - \psi(\epsilon)\big) = \epsilon.$$

So, by induction, for each $k \in \mathbb{N}$, we have

$$d(x_{n+k}, x_n) \leq \epsilon.$$

This implies that (x_n) is Cauchy and the proof of (iii) is complete.

(iv) For each $\lambda \in \Lambda$, we have $\lim_{n \to \infty} \lambda^n(t) = 0$ for each $t > 0$ (see Remark 2.2 in [30]). Then the conclusion follows from (iii).

(v) obviously holds.   □

Let X be a nonempty set. A preorder $\preceq$ on X is a binary relation which is reflexive and transitive. Let $(X, \preceq)$ be a preordered set, and let $T : X \to X$ be a mapping. We say that T is non-decreasing if for each $x, y \in X$, $x \preceq y \Rightarrow Tx \preceq Ty$.

Definition 2.3 Let $(X, \preceq)$ be a preordered set and d be a metric on X. We say that $(X, \preceq, d)$ is regular if and only if the following condition holds:

$\{x_n\}$ is non-decreasing and $x_n \to x$ for some $x \in X \Rightarrow x_n \preceq x$ for each $n \in \mathbb{N}$.

Definition 2.4 Let $(X, \preceq, d)$ be a preordered metric space, and let $\omega : R_+ \to R_+$ and $\delta : R_+^4 \to R_+$ be arbitrary mappings.

(a) A mapping $T : X \to X$ is said to be $(\omega, \delta, \preceq)$-contraction of the first kind if for all $x, y \in X$ with $x \preceq y$,

$$d(Tx, Ty) \leq \omega\big(d(x, y)\big) + \delta\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big);$$

(b) A mapping $T : X \to X$ is said to be $(\omega, \delta, \preceq)$-contraction of the second kind if for all $x, y \in X$ with $x \preceq y$,

$$d(Tx, Ty) \leq \omega\big(M(x, y)\big) + \delta\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$.

Remark 2.5 If $\preceq = X \times X$, that is, $x \preceq y$ for each $x, y \in X$, then $(\omega, \delta, \preceq)$-contractions of the first and second kind are called (ω, δ)-contractions of the first and second kind in brief. The class of (ω, δ)-contraction maps of the first and second kind include the mappings with condition (B) [3] and almost generalized contractions [6], respectively.

Theorem 2.6 *Let $(X, \preceq, d)$ be a complete preordered metric space, and let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(i) *T is continuous or $(X, \preceq, d)$ is regular,*

(ii) *T is non-decreasing,*

(iii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*

(iv) *T is an $(\omega, \delta, \preceq)$-contraction mapping of the first kind, where $\omega \in \Omega$ and $\delta \in \Delta$.*

Then T has a fixed point. Moreover, the sequence $\{T^n x_0\}$ converges to the fixed point of T.

Proof Let $x_n = Tx_{n-1}$ for any $n \in \mathbb{N}$. Since $x_0 \preceq Tx_0$ and T is non-decreasing, then we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

Now since T is an $(\omega, \delta, \preceq)$-contraction mapping of the first kind, we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \le \omega\big(d(x_n, x_{n-1})\big)$$
$$+ \delta\big(d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)\big) \tag{3}$$

for all $n \in \mathbb{N}$. Since $d(x_n, Tx_{n-1}) = d(x_n, x_n) = 0$ and $\delta \in \Delta$, so for all $n \in \mathbb{N}$,

$$\delta\big(d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)\big) = 0. \tag{4}$$

Now, from (3) and (4), we have

$$d(x_{n+1}, x_n) \le \omega\big(d(x_n, x_{n-1})\big)$$

for all $n \in \mathbb{N}$. Since $\omega \in \Omega$, so $\{x_n\}$ is a Cauchy sequence, hence there exists $x^* \in X$ such that $\{x_n\}$ converges to x^*. Now we show that x^* is a fixed point of T. If T is continuous, then from the equality $x_n = Tx_{n-1}$, we get $x^* = Tx^*$. Now assume that $(X, \preceq, d)$ is regular. Then, for each $n \in \mathbb{N}$, we have $x_n \preceq x^*$. On the contrary, assume that $d(x^*, Tx^*) > 0$. For any $n \in \mathbb{N}$,

$$d\big(Tx^*, x_{n+1}\big) = d\big(Tx^*, Tx_n\big)$$
$$\le \omega\big(d\big(x_n, x^*\big)\big) + \delta\big(d(x_n, Tx_n), d\big(x^*, Tx^*\big), d\big(x_n, Tx^*\big), d\big(x^*, Tx_n\big)\big)$$
$$\le d\big(x_n, x^*\big) + \delta\big(d(x_n, x_{n+1}), d\big(x^*, Tx^*\big), d\big(x_n, Tx^*\big), d\big(x^*, x_{n+1}\big)\big). \tag{5}$$

Since $\lim_{n \to \infty} d(x^*, x_{n+1}) = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0$ and $\delta \in \Delta$, then

$$\lim_{n \to \infty} \delta\big(d(x_n, x_{n+1}), d\big(x^*, Tx^*\big), d\big(x_n, Tx^*\big), d\big(x^*, x_{n+1}\big)\big) = 0, \tag{6}$$

and so from (5) we get $x^* = Tx^*$. $\qquad\square$

Corollary 2.7 *Let (X, d) be a complete metric space, and let $T : X \to X$ be an (ω, δ)-contraction mapping of the first kind, where $\omega \in \Omega$ and $\delta \in \Delta$. Then*

(i) *T has a unique fixed point. Moreover, for all $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the fixed point of T, that is, T is the Picard operator.*

(ii) *T is continuous at $\mathrm{Fix}(T) = \{x^*\}$.*

Proof (i) Let $\preceq = X \times X$. Then from Theorem 2.6 we deduce that T has a fixed point. To prove the uniqueness, on the contrary, assume that $x, y \in X$ are distinct fixed points of T. So,

$$
\begin{aligned}
d(x, y) = d(Tx, Ty) &\leq \omega\big(d(x, y)\big) + \delta\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) \\
&= \omega\big(d(x, y)\big) < d(x, y),
\end{aligned}
\tag{7}
$$

a contradiction. By the uniqueness of a fixed point and from Theorem 2.6, we get that the sequence $\{T^n x_0\}$ converges to the fixed point of T for all $x_0 \in X$.

(ii) Let $\mathrm{Fix}(T) = \{x^*\}$ and $\{y_n\}$ be a sequence in X such that $y_n \to x^*$. Since T is an (ω, δ)-contraction mapping of the first kind, so for all $n \in \mathbb{N}$ we have

$$
d\big(Tx^*, Ty_n\big) \leq \omega\big(d\big(x^*, y_n\big)\big) + \delta\big(d\big(x^*, Tx^*\big), d(y_n, Ty_n), d\big(x^*, Ty_n\big), d\big(y_n, Tx^*\big)\big).
$$

Since $Tx^* = x^*$ and $\delta \in \Delta$, we have

$$
\delta\big(d\big(x^*, Tx^*\big), d(y_n, Ty_n), d\big(x^*, Ty_n\big), d\big(y_n, Tx^*\big)\big) = 0.
$$

Thus, for any $n \in \mathbb{N}$,

$$
d\big(Tx^*, Ty_n\big) \leq \omega\big(d\big(x^*, y_n\big)\big) < d\big(x^*, y_n\big).
$$

Thus $Ty_n \to Tx^*$, and so T is continuous at x^*. □

Remark 2.8 Theorem 2.6 extends the main result of Babu *et al.* [3], Corollary 1 of Berinde *et al.* [4], Corollary 3.1 of Samet [29] and Theorem 2.1 of Agarwal *et al.* [30].

Theorem 2.9 *Let $(X, \preceq, d)$ be a complete preordered metric space, and let $T : X \to X$ be a mapping. Suppose that the following conditions hold*:

(i) *T is continuous or $(X, \preceq, d)$ is regular,*

(ii) *T is non-decreasing,*

(iii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*

(iv) *T is an $(\omega, \delta, \preceq)$-contraction mapping of the second kind, where $\omega \in \Omega$ and $\delta \in \Delta$.*

Then T has a fixed point. Moreover, the sequence $\{T^n x_0\}$ converges to the fixed point of T.

Proof Let $x_n = Tx_{n-1}$ for any $n \in \mathbb{N}$. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then $x_{n-1} = x_n = Tx_{n-1}$, and so x_{n-1} is a fixed point of T, and we are finished. So, we may assume that $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. Now, since $x_0 \preceq Tx_0$ and T is non-decreasing, so

$$
x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.
$$

Since T is an $(\omega, \delta, \preceq)$-contraction of the second kind, so for all $n \in \mathbb{N}$ we have

$$
\begin{aligned}
d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\leq \omega\big(M(x_n, x_{n-1})\big) \\
&\quad + \delta\big(d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)\big).
\end{aligned}
\tag{8}
$$

Since $d(x_n, Tx_{n-1}) = d(x_n, x_n) = 0$ and $\delta \in \Delta$, for all $n \in \mathbb{N}$,

$$\delta\big(d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)\big) = 0. \tag{9}$$

For all $n \in \mathbb{N}$, we have

$$M(x_n, x_{n-1}) = \max\left\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2} \right\}. \tag{10}$$

By the triangle inequality, we have

$$\frac{d(x_{n-1}, x_{n+1})}{2} \leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}$$
$$\leq \max\big\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}) \big\}.$$

Hence, by (8), (9) and (10),

$$d(x_{n+1}, x_n) \leq \omega\big(\max\big\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}) \big\}\big). \tag{11}$$

Now, if $\max\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}) \} = d(x_{n+1}, x_n)$, then by (11) we have

$$d(x_{n+1}, x_n) \leq \omega\big(d(x_n, x_{n+1})\big) < d(x_n, x_{n+1}),$$

a contradiction. So, for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_n) \leq \omega\big(d(x_{n-1}, x_n)\big).$$

As $\omega \in \Omega$, so $\{x_n\}$ is a Cauchy sequence and so, by the completeness of (X, d), there exists $x^* \in X$ such that $\{x_n\}$ converges to x^*. Now we show that x^* is a fixed point of T. If T is continuous, then from the equality $x_n = Tx_{n-1}$, we get $x^* = Tx^*$. Now, assume that $(X, \preceq, d)$ is regular. Then, for each $n \in \mathbb{N}$, we have $x_n \preceq x^*$. Now, on the contrary, assume that $d(x^*, Tx^*) > 0$. So, for any $n \in \mathbb{N}$,

$$d\big(Tx^*, x_{n+1}\big) = d\big(Tx^*, Tx_n\big)$$
$$\leq \omega\big(M\big(x_n, x^*\big)\big) + \delta\big(d(x_n, x_{n+1}), d\big(x^*, Tx^*\big), d\big(x_n, Tx^*\big), d\big(x^*, x_{n+1}\big)\big). \tag{12}$$

Since $\lim_{n\to\infty} d(x^*, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and $\delta \in \Delta$, we have

$$\lim_{n\to\infty} \delta\big(d(x_n, x_{n+1}), d\big(x^*, Tx^*\big), d\big(x_n, Tx^*\big), d\big(x^*, x_{n+1}\big)\big) = 0. \tag{13}$$

Now let $a = d(x^*, Tx^*)$ and choose $n \in \mathbb{N}$ such that for $n \geq N$, we have $d(x^*, x_n) < \frac{a}{2}$, then

$$d(x_{n+1}, x_n) \leq d\big(x^*, x_{n+1}\big) + d\big(x^*, x_n\big) < \frac{a}{2} + \frac{a}{2} = a,$$

and

$$\frac{d(x_n, Tx^*) + d(x_{n+1}, x^*)}{2} < \frac{1}{2}\left(\frac{a}{2} + d\big(x^*, Tx^*\big) + d\big(x^*, x_n\big) \right) < \frac{1}{2}\left(\frac{a}{2} + a + \frac{a}{2} \right) = a.$$

So, for $n \geq N$, we have

$$\omega\left(\max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, x_{n+1})}{2}\right\}\right) \leq \omega(d(x^*, Tx^*)).$$

Then, from (12) and (13), we get

$$d(Tx^*, x^*) = \lim_{n \to \infty} d(Tx^*, x_{n+1}) \leq \limsup_{n \to \infty} \omega(M(x_n, x^*))$$

$$= \limsup_{n \to \infty} \omega\left(\max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*),\right.\right.$$

$$\left.\left.\frac{d(x_n, Tx^*) + d(x^*, x_{n+1})}{2}\right\}\right)$$

$$\leq \omega(d(x^*, Tx^*)) < d(x^*, Tx^*), \tag{14}$$

a contradiction. $\qquad\square$

Corollary 2.10 *Let (X, d) be a complete metric space, and let $T : X \to X$ be an (ω, δ)-contraction mapping of the second kind, where $\omega \in \Omega$ and $\delta \in \Delta$. Then T has a unique fixed point. Moreover, for all $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the fixed point of T, that is, T is the Picard operator.*

Proof By Theorem 2.9 it is sufficient to prove the uniqueness of the fixed point. On the contrary assume that $x, y \in X$ are distinct fixed points of T. Then

$$d(x, y) = d(Tx, Ty) \leq \omega(M(x, y)) + \delta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$$

$$= \omega(d(x, y)) < d(x, y), \tag{15}$$

a contradiction. $\qquad\square$

Remark 2.11 Theorem 2.9 is a generalization of Theorem 2.2 and Theorem 2.3 of Agarwal *et al.* [30].

Remark 2.12 When for all $t \in [0, \infty)$ we set $\omega(t) = \alpha t$ where $\alpha \in (0, 1)$ and $\delta(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\}$ where $L \geq 0$, in Corollary 2.10, we obtain Theorem 2.4 of Berinde [5].

3 Best proximity point theory

Let A and B be two nonempty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}, \tag{16}$$

where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

Definition 3.1 Let (A, B) be a pair of nonempty subsets of the metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property [31] if and only if

$$
\begin{cases}
d(x_1, y_1) = d(A, B), \\
d(x_2, y_2) = d(A, B),
\end{cases}
\implies \quad d(x_1, x_2) = d(y_1, y_2),
$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

The following lemma is crucial in proving our best proximity point results.

Lemma 3.2 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $B_0 \neq \emptyset$ and that (A, B) satisfies the P-property. Then there exists a mapping $Q : B_0 \to A_0$ satisfying*

$$
d(x, Qx) = d(A, B) \quad and \quad d(Qx, Qy) = d(x, y) \quad \forall x, y \in B_0. \tag{17}
$$

Furthermore, B_0 is closed.

Proof Let $x \in B_0$, then we show that there exists a unique $y \in A_0$ such that $d(x, y) = d(A, B)$. To prove the uniqueness, let us assume that there exists $z \in A_0$ such that $d(x, y) = d(A, B) = d(x, z)$. Since (A, B) has the P-property, we have $d(y, z) = d(x, x) = 0$ and so $y = z$. Let $y = Qx$, then $d(x, Qx) = d(x, y) = d(A, B)$. Now, assume that $d(x, Qx) = d(A, B) = d(y, Qy)$, where $x, y \in B_0$. Then, by the P-property of (A, B), we get $d(x, y) = d(Qx, Qy)$. Therefore, there exists a mapping $Q : B_0 \to A_0$ such that

$$
d(x, Qx) = d(A, B) \quad and \quad d(Qx, Qy) = d(x, y) \quad \text{for each } x, y \in B_0.
$$

Now, we show that B_0 is closed. To prove the claim, let $\{x_n\}$ be a sequence in B_0 with $x_n \to x \in B$ (note that B is closed). Since A is a closed subset of a complete metric space, $d(Qx_m, Qx_n) = d(x_m, x_n)$ for each $m, n \in \mathbb{N}$ and $\{x_n\}$ is a Cauchy sequence, we deduce that $Qx_n \to y \in A$. Since $d(x_n, Qx_n) = d(A, B)$ for each $n \in \mathbb{N}$, we have

$$
d(x, y) = \lim_{n \to \infty} d(x_n, Qx_n) = d(A, B)
$$

and so $x \in B_0$. Hence, B_0 is closed. $\qquad \square$

Remark 3.3 It is clear that the mapping Q in Lemma 3.2 is a bijection and for any $x \in A_0$, we have $d(x, Q^{-1}x) = d(Q(Q^{-1}x), Q^{-1}x) = d(A, B)$.

Definition 3.4 Let $(X, \preceq)$ be a preordered set. A non-self mapping $T : M \subseteq A \to B$ is said to be proximally non-decreasing if and only if

$$
\begin{cases}
d(x_1, Ty_1) = d(A, B), \\
d(x_2, Ty_2) = d(A, B), \\
y_1 \preceq y_2,
\end{cases}
\implies \quad x_1 \preceq x_2,
$$

where $x_1, x_2 \in A$, $y_1, y_2 \in M$.

The following lemma follows from Lemma 14 in [32].

Lemma 3.5 *Let* $(X, \preceq, d)$ *be a preordered metric space, and let* $T : A \to B$ *be a non-self mapping such that* $TA_0 \subseteq B_0$. *Let* (A, B) *and* Q *be as in the statement of Lemma* 3.2. *Suppose that* $T : A_0 \to B_0$ *is proximally non-decreasing. Then the mapping* $S : A_0 \to A_0$ *defined by* $Sx = QTx$ *for each* $x \in A_0$ *is non-decreasing.*

The following lemma follows from Lemma 15 in [32].

Lemma 3.6 *Let* $(X, d, \preceq)$ *be a preordered metric space* (A, B), *and* Q *be as in Lemma* 3.2 *and* $T : A \to B$ *be a non-self mapping such that* $TA_0 \subseteq B_0$. *Suppose that there exist* $x_0, x_1 \in A_0$ *such that* $d(x_1, Tx_0) = d(A, B)$ *and* $x_0 \preceq x_1$. *Let the mapping* $S : A_0 \to A_0$ *be defined by* $Sx = QTx$ *for each* $x \in A_0$. *Then* $x_0 \preceq Sx_0$.

Now, we are ready to establish our best proximity point theorems.

Theorem 3.7 *Let* (A, B) *be a pair of nonempty closed subsets of a complete preordered metric space* $(X, \preceq, d)$ *such that* $A_0 \neq \emptyset$. *Let* $T : A \to B$ *be a non-self mapping. Suppose that the following conditions hold*:

(i) $TA_0 \subseteq B_0$ *and* (A, B) *satisfy the P-property,*

(ii) T *is continuous or* $(A_0, \preceq, d)$ *is regular,*

(iii) T *is proximally non-decreasing,*

(iv) *there exist* $x_0, x_1 \in A_0$ *such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \preceq x_1,$$

(v) *For all* $x, y \in A$ *such that* $x \preceq y$, *we have*

$$d(Tx, Ty) \leq \omega\big(d(x, y)\big) + \delta\big(d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty)$$
$$- d(A, B), d(y, Tx) - d(A, B)\big), \tag{18}$$

where $\omega \in \Omega$, $\delta \in \Delta$ *and* δ *is non-decreasing in each of its variables.*
Then T *has a best proximity point in* A.

Proof Since $A_0 \neq \emptyset$, so $B_0 \neq \emptyset$. By Lemma 3.2, B_0 is closed and there exists an isometry $Q : B_0 \to A_0$ which satisfies (17). Let $S : A_0 \to A_0$ be defined by $Sx = QTx$ for each $x \in A_0$. Let $x, y \in A_0$ and $x \preceq y$, then from (18) we have

$$d(Sx, Sy) = d(QTx, QTy) = d(Tx, Ty) \leq \omega\big(d(x, y)\big) + \delta\big(d(x, Tx) - d(A, B), d(y, Ty)$$
$$- d(A, B), d(x, Ty) - d(A, B), d(y, Tx) - d(A, B)\big) \tag{19}$$

but

$$d(x, Tx) - d(A, B) \leq d(x, Sx) + d(Sx, Tx) - d(A, B) = d(x, Sx) + d(QTx, Tx) - d(A, B)$$
$$= d(x, Sx) + d(A, B) - d(A, B) = d(x, Sx)$$

and

$$d(x, Ty) - d(A, B) \leq d(x, Sy) + d(Sy, Ty) - d(A, B) = d(x, Sy) + d(QTy, Ty) - d(A, B)$$

$$= d(x, Sy) + d(A, B) - d(A, B) = d(x, Sy).$$

So, from (19) we have

$$d(Sx, Sy) \leq \omega\big(d(x, y)\big) + \delta\big(d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)\big).$$

Thus S is an ordered $(\omega, \delta, \preceq)$-contraction mapping of the first kind. Now conditions (ii), (iii) and (iv) with Lemma 3.5 and Lemma 3.6 imply that S satisfies conditions (i), (ii) and (iii) of Theorem 2.6. Consequently, S has a fixed point $x^* \in A_0$ such that $x^* = Sx^* = QTx^*$ and $Q^{-1}x^* = Tx^*$. That is, $d(x^*, Tx^*) = d(x^*, Q^{-1}x^*) = d(Q(Q^{-1}x^*), Q^{-1}x^*) = d(A, B)$. Thus $x^* \in A_0$ is the required best proximity point for T. □

Corollary 3.8 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the P-property. Let $T : A \to B$ such that for all $x, y \in A$,*

$$d(Tx, Ty) \leq \omega\big(d(x, y)\big) + \delta\big(d(x, Tx) - d(A, B), d(y, Ty)$$

$$- d(A, B), d(x, Ty) - d(A, B), d(y, Tx) - d(A, B)\big), \tag{20}$$

where $\omega \in \Omega$, $\delta \in \Delta$ and δ is non-decreasing in each of its variables. Moreover, assume that $TA_0 \subseteq B_0$. Then T has a best proximity point in A.

Theorem 3.9 *Let (A, B) be a pair of nonempty closed subsets of a complete preordered metric space $(X, \preceq, d)$ such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a non-self mapping. Suppose that the following conditions hold:*

  (i) *$TA_0 \subseteq B_0$ and (A, B) satisfy the P-property,*

 (ii) *T is continuous or $(A_0, \preceq, d)$ is regular,*

(iii) *T is proximally non-decreasing,*

(iv) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \preceq x_1,$$

 (v) *For all $x, y \in A$ such that $y \preceq x$, we have*

$$d(Tx, Ty) \leq \omega\bigg(\max\bigg\{ d(x, y), d(x, Tx) - d(A, B), d(y, Ty) - d(A, B),$$

$$\frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)\bigg\}\bigg)$$

$$+ \delta\big(d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty)$$

$$- d(A, B), d(y, Tx) - d(A, B)\big), \tag{21}$$

where $\omega \in \Omega$ is non-decreasing, $\delta \in \Delta$ and δ is non-decreasing in each of its variables.

Then T has a best proximity point in A.

Proof Since $A_0 \neq \emptyset$, so $B_0 \neq \emptyset$. By Lemma 3.2, B_0 is closed and there exists an isometry $Q : B_0 \to A_0$ which satisfies (17). Let $S : A_0 \to A_0$ be defined by $Sx = QTx$ for each $x \in A_0$. Let $x, y \in A_0$ and $y \preceq x$, then from (21) we have

$$
\begin{aligned}
d(Sx, Sy) = d(QTx, QTy) &= d(Tx, Ty) \\
&\leq \omega\bigg(\max\bigg\{ d(x,y), d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), \\
&\qquad \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)\bigg\}\bigg) \\
&\quad + \delta\big(d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty) \\
&\quad - d(A, B), d(y, Tx) - d(A, B)\big).
\end{aligned}
\tag{22}
$$

Since ω is non-decreasing and δ is non-decreasing in each of its variables, in view of the proof of Theorem 3.7, we get

$$
d(Sx, Sy) \leq \omega\big(M(x, y)\big) + \delta\big(d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)\big),
$$

for each $x, y \in A_0$, where $M(x, y) = \max\{d(x, y), d(x, Sx), d(y, Sy), \frac{d(x,Sy)+d(y,Sx)}{2}\}$. Thus S is an ordered $(\omega, \delta, \preceq)$-contraction mapping of the second kind. Now conditions (ii), (iii) and (iv) with Lemma 3.5 and Lemma 3.6 imply that S satisfies conditions (i), (ii) and (iii) of Theorem 2.9, so by Theorem 2.9 S has a fixed point $x^* \in A_0$ such that $x^* = Sx^* = QTx^*$ and $Q^{-1}x^* = Tx^*$. Thus $d(x^*, Tx^*) = d(x^*, Q^{-1}x^*) = d(Q(Q^{-1}x^*), Q^{-1}x^*) = d(A, B)$, as required. □

Corollary 3.10 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the P-property. Let $T : A \to B$ be such that for all $x, y \in A$,*

$$
\begin{aligned}
d(Tx, Ty) &\leq \omega\bigg(\max\bigg\{ d(x,y), d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), \\
&\qquad \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)\bigg\}\bigg) \\
&\quad + \delta\big(d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty) \\
&\quad - d(A, B), d(y, Tx) - d(A, B)\big),
\end{aligned}
\tag{23}
$$

where $\omega \in \Omega$ is non-decreasing, $\delta \in \Delta$ and δ is non-decreasing in each of its variables. Moreover, assume that $TA_0 \subseteq B_0$. Then T has a best proximity point in A.

Remark 3.11

(a) Theorem 3.9 is a generalization of Theorem 20 of Jleli *et al.* [32].
(b) From Lemma 2.2 and Theorem 3.7, we deduce the ordered version of Theorem 3.1 in [29].

From Lemma 2.2 and Corollary 3.8, we deduce the following result due to Samet [29].

Theorem 3.12 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$, (A, B) satisfies the P-property. Let $T : A \to B$ such that for all $x, y \in A$,*

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big) + \theta\big(d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty)$$
$$- d(A, B), d(y, Tx) - d(A, B)\big),$$

where $\varphi \in \Phi$, $\theta \in \Theta$. Moreover, assume that $TA_0 \subseteq B_0$. Then T has a best proximity point in A.

Now we provide the following example to show that Corollary 3.8 is an essential extension of the above mentioned theorem of Samet.

Example 3.13 Consider the complete metric space $X = [0, 2] \times [0, \infty)$ with the Euclidean metric. Let $A = \{(0, x) : 0 \leq x\}$ and $B = \{(2, y) : 0 \leq y\}$. Then $d(A, B) = 2$, $A_0 = A$, $B_0 = B$ and (A, B) has the P-property.

Let $T : A \to B$ be defined by

$$T(0, x) = \big(2, \ln(1 + x)\big) \quad \text{for each } x \geq 0.$$

Let $\alpha(t) = \frac{\ln(1+t)}{t}$ for each $t > 0$, let $\alpha(0) = 0$ and let $\omega(t) = \alpha(t)t$ for each $t \in [0, \infty)$. Then, for all $x, y \in [0, \infty)$, we have

$$d\big(T(0, x), T(0, y)\big) = \big|\ln(1 + x) - \ln(1 + y)\big|$$
$$\leq \ln\big(1 + |x - y|\big) = \omega\big(|x - y|\big) = \omega\big(d((0, x), (0, y))\big),$$

and so the conditions of Corollary 3.8 are satisfied. Thus T has a best proximity point (indeed, $P = (0, 0)$ is a best proximity point of T). But we cannot invoke the above mentioned theorem of Samet to show that the mapping T has a best proximity point in A because T is not an almost (φ, θ) contraction. On the contrary, assume that there exist $\varphi \in \Phi$ and $\theta \in \Theta$ such that for all $x, y \in A$,

$$d\big(T(0, x), T(0, y)\big) \leq \varphi\big(d((0, x), (0, y))\big) + \theta\big(d((0, y), T(0, x))$$
$$- d(A, B), d((0, x), T(0, y)) - d(A, B), d((0, x), T(0, x))$$
$$- d(A, B), d((0, y), T(0, y)) - d(A, B)\big).$$

Letting $y = 0$, we get

$$\ln(1 + x) \leq \varphi(x) \quad \text{for each } x \geq 0.$$

Let $f(x) = \ln(1 + x)$ for each $x \in [0, \infty)$. Then it is easy to see that f on $(0, \infty)$ is an increasing positive function. So, we have

$$f(x) \leq \varphi(x) \quad \text{for each } x \geq 0.$$

Let $x_0 > 0$, and let $x_{n+1} = f(x_n)$ for each $n \in \mathbb{N}$, then $\{x_n\}$ is a decreasing sequence of positive numbers. From the above, we have $x_1 = f(x_0) \leq \varphi(x_0)$. Since φ is non-decreasing, then from the above, we get

$$x_2 = f(x_1) \leq \varphi(x_1) \leq \varphi^2(x_0).$$

Proceeding in this manner, we get $x_n \leq \varphi^n(x_0)$ for each $n \in \mathbb{N}$, and so $\sum x_n$ is convergent (note that $\sum \varphi^n(x_0)$ is convergent). Let $0 < c < \min\{x_1, 1\}$ be a constant. Now we show that

$$x_n > \frac{c}{n} \quad \text{for each } n \in \mathbb{N}. \tag{24}$$

Obviously, the inequality holds for $n = 1$. Now we proceed by induction. Assume that (24) holds for $n \in \mathbb{N}$. Then we have

$$x_{n+1} = f(x_n) \geq f\left(\frac{c}{n}\right) = \ln\left(1 + \frac{c}{n}\right).$$

Then, from the above, we obtain (note that $\ln(1 + x) \geq x - \frac{x^2}{2}$)

$$x_{n+1} \geq \ln\left(1 + \frac{c}{n}\right) \geq \frac{c}{n} - \frac{c^2}{2n^2}$$
$$= \frac{c}{n+1} + \frac{c}{n(n+1)} - \frac{c^2}{2n^2} \geq \frac{c}{n+1},$$

and so (24) holds for each $n \in \mathbb{N}$. Since $x_n > \frac{c}{n}$ for each $n \in \mathbb{N}$ and $\sum \frac{c}{n} = \infty$, then we get $\sum x_n = \infty$, a contradiction.

Author details
[1] Department of Pure Mathematics, University of Shahrekord, Shahrekord, 88186-34141, Iran. [2] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran. [3] Department of Mathematics, University of Isfahan, Isfahan, 81745-163, Iran. [4] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

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