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A multiplicity result for the non-homogeneous Klein-Gordon-Maxwell system in rotationally symmetric bounded domains

Yuhu Wu[1*] and Bin Ge[2]

*Correspondence:
wuyuhu51@gmail.com
[1] Department of Mathematics,
Harbin University of Science and
Technology, Harbin, 150080, PR
China
Full list of author information is
available at the end of the article

Abstract

This paper is concerned with the Klein-Gordon equation coupled with the Maxwell equation in the rotationally symmetric bounded domains when a non-homogeneous term breaks the symmetry of the associated functional. Under some suitable assumption on nonlinear perturbation, we obtain infinitely many radially symmetric solutions to the non-homogeneous Klein-Gordon-Maxwell system.

1 Introduction

This paper is concerned with the existence and multiplicity of solutions for a class of Klein-Gordon-Maxwell (KGM for short) systems in rotationally symmetric bounded domains. We fix $0 \leq \rho < \infty$ and define

$$\Omega = \Omega_\rho := \text{int}\{x \in R^N \ (N = 3, 4 \text{ or } 5) : \rho \leq |x| < \rho + 1\},$$

when $\rho = 0$, $\Omega = \Omega_0$ is a ball and when $\rho > 0$, $\Omega = \Omega_\rho$ is an *annulus* in R^N ($N = 3, 4, 5$).

We consider the KGM system in this type symmetric bounded domains (the model in a bounded domain was introduced by D'Avenia *et al.* [1])

$$\begin{cases} -\Delta u - (q\phi - \omega)^2 u + m^2 u = f(|x|, u) + g(x), \\ \Delta\phi = 4\pi q(q\phi - \omega)u^2 \end{cases} \quad \text{in } \Omega \tag{1}$$

with Dirichlet boundary conditions

$$\begin{cases} u = 0, \\ \phi = \zeta \end{cases} \quad \text{on } \partial\Omega, \tag{2}$$

where m, q and ω are real constants and $\zeta \in C(\partial\Omega)$.

This system appears as a model which describes the nonlinear Klein-Gordon field in a three-dimensional space interacting with the electromagnetic field. Specifically, in 2002 Benci and Fortunato [2] proposed this couplement, and in their subsequent article [3]

they proved the existence of infinitely many radially symmetric solutions for the Klein-Gordon-Maxwell system when the nonlinearity exhibits subcritical behavior. D'Aprile and Mugnai [1] established the existence of infinitely many radially symmetric solutions for the subcritical KGM system in R^3. They extended the interval of definitions of power in the nonlinearity exhibited in [3]. Non-existence results of the KGM system in the whole R^3 can be found in [4] and [5]. Positive and ground state solutions for the critical KGM system with potentials in R^3 were obtained in [6]. For related works, see [7–9], and [10].

The KGM system in the case of bounded domains was firstly considered by D'Avenia *et al.* [11, 12]. They established the existence and multiplicity of solutions to the KGM system in bounded domains under suitable Dirichlet or mixed boundary conditions.

All the above mentioned results are mainly focused on the homogeneous case $g \equiv 0$. However, on the non-homogeneous case $g \neq 0$, only a few results are known for KGM systems. In [13], Chen and Tang proved two different solutions for the non-homogeneous KGM equations in the whole R^3. Candela and Salvatore [14, 15] dealt with multiplicity of solutions to non-homogeneous Schrödinger-Maxwell and Klein-Gordon-Maxwell systems with homogenous boundary conditions in a bounded ball, respectively. Inspired by Candela and Salvatore's results, in this paper we deal with the non-homogeneous KGM system in the special bounded symmetric domains - balls or annuli - and obtain infinitely many radially symmetric solutions by using the variational method.

In the uncoupled case $q = 0$, problem (1)-(2) can be split into

$$\begin{cases} -\Delta u + (m^2 - \omega^2)u = f(|x|, u) + g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

and

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega, \\ \phi = \zeta & \text{on } \partial\Omega. \end{cases} \tag{4}$$

The existence and uniqueness of solutions of problem (3) and (4) are independent of each other. And it is clear that problem (4) has a unique solution. In this paper, we mainly consider the coupled case $q \neq 0$. In this case, the change of variables

$$\begin{cases} u_q = \sqrt{4\pi}qu, \\ \varphi_q = q\varphi - \omega, \end{cases}$$

transforms system (1)-(2) into

$$\begin{cases} -\Delta u_q + \phi_q^2 u + m^2 u_q = f_q(|x|, u_q) + g(x) & \text{in } \Omega, \\ \Delta\phi_q = \phi_q u_q^2 & \text{in } \Omega, \\ u_q = 0 & \text{on } \partial\Omega, \\ \phi_q = q\zeta - \omega & \text{on } \partial\Omega \end{cases} \tag{5}$$

with

$$f_q(|x|, u_q) = \sqrt{4\pi}qf\left(|x|, \frac{u}{\sqrt{4\pi}q}\right).$$

Based on the observation above, we will focus on studying the existence and multiplicity of solutions for problem (5), instead of problem (1)-(2). For the sake of simplicity, from now on, we shall omit the subscript q in problem (5). We assume that the nonlinearity term f in (5) satisfies the following conditions.

(f_1) There exist $a_1, a_2 \geq 0$ and $p \in (2, \frac{2N}{N-2})$ such that $|f(r, u)| \leq a_1 + a_2|u|^{p-1}$.

(f_2) f is odd with respect to the second variable, that is, for all $r \geq 0$, $u \in R$, $f(r, -u) = -f(r, u)$.

(f_3) There exists a constant $s \in (2, p]$ such that for all $r \geq 0$, $u > 0$, $0 < sF(r, u) \leq uf(r, u)$, with $F(r, u) = \int_0^t f(r, \tau)\, d\tau$.

And we also assume that the non-homogeneous term g satisfies

(g_1) $g \in L^2(\Omega)$ and $g(|x|) = g(x)$, $\forall x \in \Omega$.

Condition (f_3) is known in the literature as the Ambrosetti-Rabinowitz type condition. The main result we provide in this paper is the following theorem.

Theorem 1.1 *Assume that conditions* (f_1)-(f_3) *and* (g_1) *hold and*

$$-\frac{s-2}{s}|m| < q\zeta - \omega \leq 0 \quad a.e.\ \partial\Omega. \tag{6}$$

If ζ is a radial function on $\partial\Omega$, then system (5) has infinitely many radial solutions $(u_i, \phi_i) \in H^1_{0,r}(\Omega) \times H^1_r(\Omega)$ with $\|\nabla u_i\|_2 \to +\infty$ and $\{\phi_i\}$ bounded in $L^\infty(\Omega)$.

Remark 1.1 Theorem 1.1 can be viewed as a reasonable extension of Theorem 1.1 of [14], where the multiplicity of solutions to the non-homogeneous KGM system with homogenous boundary conditions in a bounded ball of R^3 can be found.

This paper is organized as follows. In Section 2 we introduce the variational tools: a variational principle, stated as in [3], which allows us to reduce the previous systems to a semilinear elliptic equation in the only variable u, and a perturbation method, introduced by Bolle [16, 17], useful for stating our multiplicity results. Finally, in Section 3 we prove our main theorem.

2 Preliminary results

In this section, we present the variational framework to deal with problem (5) and also give some preliminaries which are useful later.

To get homogeneous boundary conditions, we change variables as follows:

$$\varphi = \phi - \phi_0,$$

where ϕ_0 is the unique solution of

$$\begin{cases} \Delta\phi_0 = 0 & \text{in } \Omega, \\ \phi_0 = q\zeta - \omega & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Then problem (5) can be rewritten as

$$\begin{cases} -\Delta u - (\varphi + \phi_0)^2 u + m^2 u = f(|x|, u) + g(x) & \text{in } \Omega, \\ \Delta \varphi = (\varphi + \phi_0) u^2 & \text{in } \Omega, \\ u = 0, \quad \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Remark 2.1 By the maximum principle, (6) implies

$$-\frac{s-2}{s}|m| < \phi_0 \le 0 \quad \text{a.e. } \partial\Omega. \tag{9}$$

System (8) contains the Euler-Lagrange equations related to the functional

$$I : H_0^1(\Omega) \times H_0^1(\Omega) \to R$$

defined as

$$I(u, \varphi) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 - |\nabla \varphi|^2 + \left[m^2 - (\varphi + \phi_0)^2 \right] u^2 \right) dx \\ - \int_\Omega F(|x|, u) \, dx - \int_\Omega gu \, dx. \tag{10}$$

By the standard argument, $I(u, \varphi)$ is C^1 on $H_0^1(\Omega) \times H_0^1(\Omega)$.

The functional I is strongly indefinite, that is, unbounded from below and from above on an infinite dimensional subspace. So we apply a well-known reduction argument (see, *e.g.*, [1]) to avoid this indefiniteness.

Similar to Lemma 2.1 of [4], which deals with the case of the entire domain R^3, one easily obtains the following auxiliary result. To avoid repetition, the details of the proof, which is mainly based on the Lax-Milgram lemma, are omitted.

Lemma 2.1 *For any $u \in H_0^1(\Omega)$ and for any $h \in H^{-1}(\Omega)$, there exists a unique solution $\varphi := (\Delta - u^2)^{-1}[h] \in H_0^1(\Omega)$ of the equation*

$$\begin{cases} \Delta \varphi = u^2 \varphi + h & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, for every $u \in H_0^1(\Omega)$ and for every $h, k \in H^{-1}(\Omega)$,

$$\langle h, (\Delta - u^2)^{-1}[k] \rangle = \langle k, (\Delta - u^2)^{-1}[h] \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H_1^0(\Omega)$ and $H^{-1}(\Omega)$.

By taking $h = \phi_0 u^2$, we can deduce the following result.

Proposition 2.1 *For any $u \in H_0^1(\Omega)$, there exists unique $\varphi = \varphi_u \in H_0^1(\Omega)$ which satisfies*

$$\begin{cases} \Delta \varphi = (\varphi + \phi_0) u^2 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Furthermore, the map $\Phi : u \in H_0^1(\Omega) \to \varphi_u \in H_0^1(\Omega)$ *is of class* C^1 *and for every* $u, v \in H_0^1(\Omega)$,

$$\big(\Phi'[u]\big)[v] = 2\big(\Delta - u^2\big)^{-1}\big[(\varphi_u + \phi_0)uv\big]. \tag{12}$$

From the above proposition, one directly gets the following assertion.

Corollary 2.1 *Let* $u \in H_0^1(\Omega)$ *and set* $\Psi_u := (\Phi'[u])[u] \in H_0^1(\Omega)$. *Then* Ψ_u *is a solution to the integral equation*

$$\int_\Omega \phi_0 \Psi_u u^2 \, dx = \int_\Omega (\phi_0 + \varphi_u)\varphi_u u^2 \, dx$$

and

$$\Psi_u \le 0.$$

Since the map $\Phi : u \in H_0^1(\Omega) \to \varphi_u \in H_0^1(\Omega)$ is continuously differentiable, we define the reduced C^1 functional $K(u) := I(u, \varphi_u)$. It is easy to see that the pair $(u, \varphi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of a critical point for I if and only if $u \in H_0^1(\Omega)$ is a critical point for K and $\varphi = \Phi[u]$ (see [2]). Hence, we look for critical points of K.

Since, by Proposition 2.1, the functional φ_u satisfies

$$\|\nabla \varphi_u\|_2^2 = -\int_\Omega \varphi_u(\varphi_u + \phi_0)u^2 \, dx,$$

we obtain, for any $u \in H_0^1(\Omega)$,

$$K(u) = I(u, \varphi_u) = \frac{1}{2} \int_\Omega \big(|\nabla u|^2 + \big[m^2 - \phi_0(\varphi_u + \phi_0)\big]u^2\big) \, dx$$
$$- \int_\Omega F\big(|x|, u\big) \, dx - \int_\Omega gu \, dx. \tag{13}$$

According to Lemma 2.1 and Proposition 2.1, for every $u, v \in H_0^1(\Omega)$, we have

$$K'(v)[u] = \int_\Omega \nabla v \nabla u \, dx + \big(m^2 - \phi_0^2\big) \int_\Omega vu \, dx - \int_\Omega \varphi_v \phi_0 vu \, dx$$
$$- \int_\Omega \phi_0 v^2 \big(\Delta - v^2\big)^{-1}\big[(\phi_0 + \varphi_v)uv\big] \, dx - \int_\Omega f\big(|x|, v\big)u \, dx - \int_\Omega gu \, dx$$
$$= \int_\Omega \nabla v \nabla u \, dx + \big(m^2 - \phi_0^2\big) \int_\Omega vu \, dx - \int_\Omega \varphi_v \phi_0 vu \, dx$$
$$- \int_\Omega (\phi_0 + \varphi_v)uv\big(\Delta - v^2\big)^{-1}\big[\phi_0 v^2\big] \, dx - \int_\Omega f\big(|x|, v\big)u \, dx - \int_\Omega gu \, dx$$
$$= \int_\Omega \nabla v \nabla u \, dx + \int_\Omega \big[m^2 - (\phi_0 + \varphi_v)^2\big]vu \, dx - \int_\Omega f\big(|x|, v\big)u \, dx - \int_\Omega gu \, dx. \tag{14}$$

Hence we get, as an operator in $H^{-1}(\Omega)$,

$$K'(v) = \frac{\partial I}{\partial v}(v, \varphi_v) = -\Delta v + \big[m^2 - (\varphi_v + \phi_0)^2\big]v - f\big(|x|, v\big) - g. \tag{15}$$

Lemma 2.2 (Lemma 9 of [11]) *For every $u \in H_0^1(\Omega)$, the function φ_u satisfies the following inequalities:*

$$-\max\{0, \phi_0\} \leq \varphi_u \leq \max\{0, -\phi_0\} \quad a.e.\ in\ \Omega.$$

Remark 2.2 In fact, according to (9), we observe that

$$0 \leq \varphi_u \leq -\phi_0 \quad \text{a.e. in } \Omega. \tag{16}$$

Now we recall Bolle's method for dealing with problems with broken symmetry.

Let E be a Hilbert space equipped with the norm $\|\cdot\|$. Assume that $E = E_- \oplus E_+$, where $\dim(E_-) < +\infty$, and let $(e_k)_{k \geq 1}$ be an orthonormal base of E_+. Consider

$$E_0 = E_-, \qquad E_{k+1} = E_k \oplus \mathbf{R}e_{k+1}, \quad k \geq 0.$$

So $\{E_k\}_k$ is an increasing sequence of a finite dimensional subspace of E.

Let $I : [0,1] \times E \to R$ be a C^1-functional, and set

$$c_k = \inf_{\gamma \in \Gamma} \sup_{v \in E_k} I\big(0, \gamma(v)\big), \tag{17}$$

with $\Gamma = \{\gamma \in C(E, E) : \gamma \text{ is odd and } \exists R > 0 \text{ s.t. } \gamma(v) = v \text{ for } \|v\| \geq R\}$.

We make the following hypotheses:

(H_1) I satisfies the following weaker form of the Palais-Smale condition: for every sequence $\{(\theta_n, v_n)\}_n \subset [0,1] \times H$ such that

$$\big\{I(\theta_n, v_n)\big\}_n \text{ are bounded} \quad \text{and} \quad \lim_{n \to +\infty} \frac{\partial I}{\partial v}(\theta_n, v_n) = 0,$$

there is a subsequence converging in $[0,1] \times E$.

(H_2) For all $b > 0$, there exists a constant $C_b > 0$ such that

$$\left|I(\theta, v)\right| < b \quad \text{implies} \quad \left|\frac{\partial I}{\partial \theta}(\theta, v)\right| < C_b\left(\left\|\frac{\partial I}{\partial v}(\theta, v)\right\| + 1\right)\big(\|v\| + 1\big).$$

(H_3) There exist two continuous maps $\eta_1, \eta_2 : [0,1] \times R \to R$, which are Lipschitz continuous with respect to the second variable, such that $\eta_1(\theta, \cdot) \leq \eta_2(\theta, \cdot)$ and such that, for all critical points v of $I(\theta, \cdot)$,

$$\eta_1\big(\theta, I(\theta, v)\big) \leq \frac{\partial}{\partial \theta}I(\theta, v) \leq \eta_2\big(\theta, I(\theta, v)\big).$$

(H_4) $I(0, v)$ is even and for any finite dimensional space W of E, we have

$$\lim_{\substack{|v| \to +\infty \\ v \in A}} \sup_{\theta \in [0,1]} I(\theta, v) = -\infty.$$

Let $\chi_i : [0,1] \times R \to R$ $(i = 1, 2)$ be the flow associated to η_i that is the solution of problem

$$
\begin{cases}
\frac{\partial}{\partial \theta} \chi_i(\theta, s) = \eta_i(\theta, \chi_i(\theta, s)), \\
\eta_i(0, s) = s.
\end{cases}
\tag{18}
$$

Note that χ_1 and χ_2 are continuous and that for all $\theta \in [0,1]$, $\chi_1(\theta, \cdot)$ and $\chi_2(\theta, \cdot)$ they are non-decreasing on R. Moreover, since $\eta_1 \le \eta_2$, we have $\chi_1 \le \chi_2$. Set

$$
\overline{\eta}_1(s) = \sup_{\theta \in [0,1]} \left| \eta_1(\theta, s) \right|, \qquad \overline{\eta}_2(s) = \sup_{\theta \in [0,1]} \left| \eta_2(\theta, s) \right|.
$$

In this framework, the following result of Bolle can be proved (for more details and the proof, see [17, Theorem 2.2]).

Theorem 2.1 *Assume that* $I : [0,1] \times E \to R$ *is* C^1 *and satisfies* (H_1)-(H_4). *Then there is* $C > 0$ *such that for every* k,
  (1) *either* $I(1, x)$ *has a critical level* $\overline{c}_k$ *such that* $\chi_2(1, c_k) \le \chi_1(1, c_{k+1}) \le \overline{c}_k$,
  (2) *or* $c_{k+1} - c_k \le C(\overline{\eta}_1(c_{k+1}) + \overline{\eta}_2(c_k) + 1)$.

Remark 2.3 Note that if $\eta_2 \le 0$ in $[0,1] \times R$, then for all $s \in R$, the functional $\chi_2(\cdot, s)$ is non-decreasing on $[0,1]$. Hence, in case (1) of the above theorem, we have $c_k \le \overline{c}_k$.

3 Proof of Theorem 1.1

Now, we consider the family of functionals $J(\theta, u) : [0,1] \times H^1_{0,r}(\Omega) \to R$ defined by

$$
J(\theta, u) = J_0(u) - \theta \int_{\Omega} gu \, dx,
\tag{19}
$$

with

$$
J_0(u) = \frac{1}{2} \int_{\Omega} \left(|\nabla u|^2 + \left[m^2 - \phi_0(\varphi_u + \phi_0) \right] u^2 \right) dx - \int_{\Omega} F(|x|, u) \, dx.
$$

Clearly, J is a C^1 functional such that $J(0, u) = J_0(u)$ and $J(1, u) = K(u)$. Notice that the functional $J_0(u)$ is even by assumption (f_2). According to (14),

$$
\frac{\partial J}{\partial \theta}(\theta, u) = - \int_{\Omega} gu \, dx,
\tag{20}
$$

$$
\frac{\partial J}{\partial u}(\theta, u)[v] = \int_{\Omega} \nabla v \nabla u \, dx + \int_{\Omega} \left[m^2 - (\phi_0 + \varphi_u)^2 \right] uv \, dx
$$
$$
\qquad - \int_{\Omega} f(|x|, u) v \, dx - \theta \int_{\Omega} gv \, dx
\tag{21}
$$

for all $\theta \in [0,1]$ and $u, v \in H^1_{0,r}(\Omega)$.

The following lemma allows us to prove that functional J verifies the assumption of Bolle's abstract theorem.

Lemma 3.1 *There exist three strictly positive constants α_1, α_2 and α_3 such that for all $(\theta, u) \in [0, 1] \times H^1_{0,r}(\Omega)$,*

$$\|\nabla\|_2^2 + \|u\|_s^s \leq \alpha_1 J(\theta, u) - \alpha_2 \frac{\partial J}{\partial u}(\theta, u)[u] + \alpha_3. \tag{22}$$

Proof For any $\delta > 0$,

$$J(\theta, u) - \delta \frac{\partial J}{\partial u}(\theta, u)[u] = \left(\frac{1}{2} - \delta\right) \int_\Omega |\nabla u|^2 \, dx + \left(\frac{1}{2} - \delta\right) m^2 \int_\Omega u^2 \, dx$$
$$+ I_1 + I_2 - (1 - \delta)\theta \int_\Omega gu \, dx \tag{23}$$

with

$$I_1 = -\int_\Omega \left[\frac{1}{2} \phi_0(\varphi_u + \phi_0) - \delta(\varphi_u + \phi_0)^2\right] u^2 \, dx,$$

$$I_2 = \int_\Omega \delta f(|x|, u) u \, dx - \int_\Omega F(|x|, u) \, dx.$$

By Remark 2.2, we get $\phi_0 \leq \varphi_u + \phi_0 \leq 0$ and $-\frac{1}{2}\phi_0 \leq (\frac{1}{2} - \delta)\phi_0 - \delta\varphi_u \leq (\frac{1}{2} - \delta)\phi_0$. Hence,

$$I_1 = -\int_\Omega (\varphi_u + \phi_0)\left(\left(\frac{1}{2} - \delta\right)\phi_0 - \delta\varphi_u\right) u^2 \, dx$$
$$\geq -\frac{1}{2} \int_\Omega \phi_0^2 u^2 \, dx. \tag{24}$$

By assumption (f$_3$), there exist two constants $b_1, b_2 > 0$ such that for every $t \in R$,

$$F(r, u) \geq b_1 |u|^s - b_2. \tag{25}$$

Hence, we get

$$I_2 = \int_\Omega \delta f(|x|, u) u \, dx - \int_\Omega F(|x|, u) \, dx$$
$$\geq (s\delta - 1) \int_\Omega F(|x|, u) \, dx$$
$$\geq (s\delta - 1)\left(b_1 \|u\|_s^s - m(\Omega) b_2\right). \tag{26}$$

According to Remark 2.1, there exists $\varepsilon_0 > 0$ such that

$$\|\phi_0\|_\infty^2 \leq \left(\frac{s-2}{s} - \varepsilon_0\right) m^2.$$

Using the Cauchy-Schwarz inequality, we get

$$\int_\Omega gu \, dx \leq \frac{\varepsilon_0 m^2}{2} \|u\|_2^2 + \frac{1}{2\varepsilon_0 m^2} \|g\|_2^2.$$

Then equation (23) becomes

$$
J(\theta, u) - \delta \frac{\partial J}{\partial u}(\theta, u)[u]
$$

$$
\geq \left(\frac{1}{2} - \delta\right) \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} \left[\left(\frac{1}{2} - \delta\right) m^2 - \frac{1}{2}\phi_0^2 - \frac{\varepsilon_0}{2} m^2\right] u^2 \, dx
$$

$$
+ (s\delta - 1)\left(b_1 \|u\|_s^s - m(\Omega) b_2\right) - \frac{1}{2\varepsilon_0 m^2} \|g\|_2^2.
$$

Now, we choose δ satisfying $\frac{1}{s} < \delta < \frac{1}{s} + \varepsilon_0$. Then $s\delta - 1 > 0$ and $(\frac{1}{2} - \frac{\varepsilon_0}{2} - \delta)m^2 - \frac{1}{2}\phi_0^2 > 0$, which implies equation (22). □

In the sequel, C_i denotes some suitable positive constants. Now, we give the following simple fact.

Lemma 3.2 *For any $0 \leq \rho < +\infty$, if the boundary condition ζ is radial on $\partial\Omega$ in equation (7), then ϕ_0 is also radial in Ω.*

Proof (1) Case $\rho = 0$, *i.e.*, $\Omega = \Omega_0$ is a ball. We denote $h = \zeta(x)$ for $|x| = 1$. Then it is obvious that

$$
\phi_0 \equiv qh - \omega
$$

is the unique solution of the equation

$$
\begin{cases}
\Delta \phi_0 = 0 & \text{in } \Omega, \\
\phi_0 = q\zeta - \omega & \text{on } \partial\Omega.
\end{cases}
\tag{27}
$$

(2) Case $\rho > 0$, *i.e.*, $\Omega = \Omega_\rho$ is an *annulus*. We denote $h_1 = \zeta(x)$ for $|x| = \rho$ and $h_2 = \zeta(x)$ for $|x| = \rho + 1$. By calculating using general ordinary differential equation theory and maximal principle, we obtain that the radial function

$$
\phi_0(x) = \frac{q(h_1 - h_2)\rho^{N-2}(\rho+1)^{N-2}}{(\rho+1)^{N-2} - \rho^{N-2}} |x|^{N-2} + \left[qh_1 - \omega - \frac{q(h_1 - h_2)(\rho+1)^{N-2}}{(\rho+1)^{N-2} - \rho^{N-2}}\right]
$$

is the unique solution of equation (27). □

Proof of Theorem 1.1 We first prove that any critical point $u \in H_{0,r}^1(\Omega)$ of $K|_{H_{0,r}^1(\Omega)}$ is also a critical point of K in $H_0^1(\Omega)$.

Let $O(N) = \{A_{N \times N} : \text{orthogonal matrices}\}$. Consider the $O(N)$ group action T_g on $L^2(\Omega)$ defined by

$$
T_g u(x) := u(gx) \quad \text{for any } g \in O(N) \text{ and } u \in L^2(\Omega).
\tag{28}
$$

Since for each $u \in H_0^1(\Omega)$, φ_u is the unique solution of

$$
\begin{cases}
\Delta \varphi_u = (\varphi_u + \phi_0) u_0^2 & \text{in } \Omega, \\
\varphi_u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{29}
$$

so, for any $g \in O(N)$, we have $T_g(\Delta \varphi_u) = T_g(\varphi_u)T_g(u^2) + T_g(\phi_0)T_g(u_0^2)$ in Ω. By radial symmetry of ϕ_0 (Lemma 3.2), we get

$$\begin{cases} \Delta(T_g\varphi_u) = T_g(\varphi_u)(T_gu)^2 + \phi_0 T_g(u_0)^2 & \text{in } \Omega, \\ T_g\varphi_u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, by the definition of $\Phi : u \to \varphi_u$, we obtain

$$T_g\big(\Phi(u)\big) = \Phi(T_gu). \tag{30}$$

Using (30) and the T_g invariance of the norm in $H_r^1(\Omega)$, $L^p(\Omega)$, we deduce that K is $O(N)$-invariant, *i.e.*,

$$K(T_gu) = K(u), \quad \forall u \in H_0^1(\Omega), g \in O(N).$$

Then, according to the principle of symmetric criticality [18], any critical point $u \in H_{0,r}^1(\Omega)$ of $K|_{H_{0,r}^1(\Omega)}$ is also a critical point of K in $H_0^1(\Omega)$. Then we want to apply Bolle's method to K restricted to $H_{0,r}^1(\Omega)$.

Firstly, we prove that J satisfies assumption (H$_1$) of Theorem 2.1. So let us consider a sequence $\{(\theta_n, u_n)\} \subset [0,1] \times H_{0,r}^1(\Omega)$ such that

$$\big\{J(\theta_n, u_n)\big\}_n \text{ is bounded} \quad \text{and} \quad \lim_{n\to\infty} \frac{\partial J}{\partial u}(\theta_n, u_n) = 0. \tag{31}$$

Let $\varphi_n = \Phi[u_n]$. Then the expression of $\frac{\partial J}{\partial u}$ and (15) implies

$$\varepsilon_n = \frac{\partial J}{\partial u}(\theta_n, u_n) = -\Delta u_n + \big[m^2 - (\varphi_n + \phi_0)^2\big]u_n - f\big(|x|, u_n\big) - \theta g \tag{32}$$

with $\varepsilon \to 0$. Clearly, Lemma 3.1 implies $\{\|\nabla u_n\|\}_n$ is bounded. By assumption (f$_1$) and $L^p(\Omega) \hookrightarrow H_{0,r}^1(\Omega)$, we get

$$\int_\Omega \big|f(r, u_n)\big|^{\frac{p}{p-1}} dx \le C_1\big(\|u_n\|_p^p + 1\big) \le C_2\big(\|\nabla u\|_2^p + 1\big).$$

That is, $\{f(r, u_n)\}_n$ are bounded in $L^{\frac{p}{p-1}}(\Omega)$. Moreover, by Remark 2.2, $\{[m^2 - (\varphi_n + \phi_0)^2]u_n\}_n$ are bounded in $L^2(\Omega)$. Hence these two sequences $\{f(r, u_n)\}_n$ and $\{[m^2 - (\varphi_n + \phi_0)^2]u_n\}_n$, up to subsequences, converge in $H_r^{-1}(\Omega)$ (the dual space of $H_{0,r}^1(\Omega)$). Thus, as $-\Delta$ is an isomorphism from $H_{0,r}^1(\Omega)$ to its dual space $H_r^{-1}(\Omega)$, there exists a subsequence of $\{u_n\}$ converging strongly in $H_{0,r}^1(\Omega)$ by the standard argument.

By equation (20), for all $(\theta, u) \in [0,1] \times H_{0,r}^1(\Omega)$,

$$\left|\frac{\partial J}{\partial \theta}(\theta, u)\right| \le \|g\|_2\|u\|_2 \le \|g\|_2\|\nabla u\|_2. \tag{33}$$

Hence Lemma 3.1 implies that J satisfies assumption (H$_2$) of Theorem 2.1.

If $(\theta, u) \in [0,1] \times H_{0,r}^1(\Omega)$ satisfies $\frac{\partial J}{\partial u}(\theta, u) = 0$, then

$$\|u\|_s^s \le \alpha_1\big|J(\theta, u)\big| + \alpha_3 \le \alpha_4\big(1 + J^2(\theta, u)\big)^{\frac{1}{2}},$$

with $\alpha_4 = \sqrt{(\alpha_1 + \alpha_3)\max\{\alpha_1, \alpha_3\}}$. So, there exists a positive constant C_3 such that

$$\left|\frac{\partial J}{\partial \theta}(\theta, u)\right| \leq C_3\left(1 + J^2(\theta, u)\right)^{\frac{1}{2s}}$$

by (33). Hence, J verifies assumption (H$_3$) of Theorem 2.1 with $\eta_2(\theta, t) = -\eta_1(\theta, t) = C_3(1 + t^2)^{\frac{1}{2s}}$. Notice that $\overline{\eta}_1(t) = \overline{\eta}_2(t) = C_3(1 + t^2)^{\frac{1}{2s}}$. So, by the definition of J,

$$
\begin{aligned}
J(\theta, u) &\leq \frac{1}{2}\|\nabla u\|_2^2 + C_4\|u\|_2^2 - \int_\Omega F(|x|, u)\,dx - \theta\int_\Omega gu\,dx \\
&\leq \frac{1}{2}\|\nabla u\|_2^2 + C_5\|u\|_2^2 - b_1\|u\|_s^s - b_2 m(\Omega) - \|g\|_2^2,
\end{aligned}
\tag{34}
$$

using inequalities (25) and (33). Then, as in a finite dimensional space all the norms are equivalent and $s > 2$, we get that J satisfies assumption (H$_4$) of Theorem 2.1.

Now we apply Theorem 2.1 to J and assume that case (1) in the results of Theorem 2.1 occurs for all k large enough. Then, according to the definition of $\overline{\eta}_1$ and $\overline{\eta}_2$, we obtain

$$c_{k+1} - c_k \leq C_6\left(c_{k+1}^{\frac{1}{s}} + c_k^{\frac{1}{s}} + 1\right),\tag{35}$$

where c_k is the critical level of J defined by (17). Hence, using a similar argument as in Lemma 5.3 of [19], there exists an integer $k_0 \in N$ such that

$$c_k \leq C_7 k^{\frac{s}{s-1}} \quad \text{for all } k \geq k_0.\tag{36}$$

On the other hand, assumption (f$_1$) implies

$$J(0, u) \geq \frac{1}{2}\|\nabla u\|_2^2 - C_8\|u\|_p^p - C_9.\tag{37}$$

So, by applying the arguments developed in [20], the radial symmetry of the problem implies

$$c_k \geq C_{10} k^{\frac{p}{p-2}}, \quad \text{if } k \text{ is large enough},\tag{38}$$

which contradicts with (36).

That means case (1) in the result of Theorem 2.1 occurs for infinitely many indexes, and a sequence $\{\overline{c}_n\}$ of the critical level of $J(1, \cdot) = K(\cdot)$ exists such that $\overline{c}_k \to +\infty$ by Remark 2.3 and (38). Thus, if $\{u_n\}_n$ is the corresponding sequence of critical points, by (34) for $\theta = 1$, it follows that $\|\nabla u_n\|_2 \to +\infty$. We complete the proof of Theorem 1.1. □

Author details
[1]Department of Mathematics, Harbin University of Science and Technology, Harbin, 150080, PR China. [2]Department of Mathematics, Harbin Engineering University, Harbin, 150001, PR China.

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