Ren and Yang *Journal of Inequalities and Applications* 2013, 2013:572
http://www.journalofinequalitiesandapplications.com/content/2013/1/572

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Growth estimates for modified Neumann integrals in a half-space

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Abstract
Our aim in this paper is to deal with the growth properties for modified Neumann integrals in a half-space of $\mathbf{R}^n$. As an application, the solutions of Neumann problems in it for a slowly growing continuous function are also given.

Keywords: Dirichlet problem; harmonic function; half-space

1 Introduction and main results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbf{R}^n$ ($n \geq 3$) denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set Ω of $\mathbf{R}^n$ are denoted by $\partial\Omega$ and $\overline{\Omega}$, respectively. For $x \in \mathbf{R}^n$ and $r > 0$, let $B_n(x, r)$ denote the open ball with center at x and radius r in $\mathbf{R}^n$.

The upper half-space is the set $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is ∂H. For a set F, $F \subset \mathbf{R}_+ \cup \{0\}$, we denote $\{x \in H; |x| \in F\}$ and $\{x \in \partial H; |x| \in F\}$ by HF and ∂HF, respectively. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$. Let θ be the angle between x and $\hat{e}_n$, *i.e.*, $x_n = |x|\cos\theta$ and $0 \leq \theta < \pi/2$, where $\hat{e}_n$ is the ith unit coordinate vector and $\hat{e}_n$ is normal to ∂H.

We shall say that a set $E \subset H$ has a covering $\{r_j, R_j\}$ if there exists a sequence of balls $\{B_j\}$ with centers in H such that $E \subset \bigcup_{j=0}^{\infty} B_j$, where r_j is the radius of B_j and R_j is the distance between the origin and the center of B_j.

For positive functions g_1 and g_2, we say that $g_1 \lesssim g_2$ if $g_1 \leq Mg_2$ for some positive constant M. Throughout this paper, let M denote various constants independent of the variables in question. Further, we use the standard notations, $[d]$ is the integer part of d and $d = [d] + \{d\}$, where d is a positive real number.

Given a continuous function f in ∂H, we say that h is a solution of the Neumann problem in H with f, if h is a harmonic function in H and

$$\lim_{x \in H, x \to y'} \frac{\partial}{\partial x_n} h(x) = f(y')$$

for every point $y' \in \partial H$.

For $x \in \mathbf{R}^n$ and $y' \in \mathbf{R}^{n-1}$, consider the kernel function

$$K_n(x, y') = -\frac{\beta_n}{|x - y'|^{n-2}},$$

where $\beta_n = 2/(n-2)\sigma_n$ and σ_n is the surface area of the n-dimensional unit sphere. It has the expression

$$K_n(x,y') = \sum_{k=0}^{\infty} \frac{|x|^k}{|y|^{n+k-2}} C_k^{\frac{n-2}{2}}\left(\frac{x\cdot y'}{|x||y'|}\right),$$

where $C_k^{\frac{n}{2}}(t)$ is the ultraspherical (Gegenbauer) polynomials [1]. The series converges for $|y'| > |x|$, and each term in it is a harmonic function of x.

The Neumann integral is defined by

$$N[f](x) = \int_{\partial H} K_n(x,y') f(y')\, dy',$$

where f is a continuous function on ∂H, $\alpha_n = 2/n\sigma_n$ and $\sigma_n = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2})$ is the volume of the unit n-ball.

The Neumann integral $N[f](x)$ is a solution of the Neumann problem on H with f if (see [2, Theorem 1 and Remarks])

$$\int_{\partial H} \frac{f(y')}{(1+|y'|)^{n-2}}\, dy' < \infty.$$

In this paper, we consider functions f satisfying

$$\int_{\partial H} \frac{|f(y')|^p}{(1+|y'|)^{n+\alpha-2}}\, dy' < \infty \tag{1.1}$$

for $1 \leq p < \infty$ and $\alpha \in \mathbf{R}$.

For this p and α, we define the positive measure μ on $\mathbf{R}^n$ by

$$d\mu(y') = \begin{cases} |f(y')|^p |y'|^{-n-\alpha+2}\, dy', & y' \in \partial H(1,+\infty), \\ 0, & Q \in \mathbf{R}^n - \partial H(1,+\infty). \end{cases}$$

If f is a measurable function on ∂H satisfying (1.1), we remark that the total mass of μ is finite.

Let $\epsilon > 0$ and $\delta \geq 0$. For each $x \in \mathbf{R}^n$, the maximal function $M(x;\mu,\delta)$ is defined by

$$M(x;\mu,\delta) = \sup_{0<\rho<\frac{|x|}{2}} \frac{\mu(B_n(x,r))}{\rho^\delta}.$$

The set $\{x \in \mathbf{R}^n; M(x;\mu,\delta) > \epsilon\}$ is denoted by $E(\epsilon;\mu,\delta)$.

To obtain the Neumann solution for the boundary data f, as in [3–6], we use the following modified kernel function defined by

$$L_{n,m}(x,y') = \begin{cases} -\beta_n \sum_{k=0}^{m-1} \frac{|x|^k}{|y|^{n+k-2}} C_k^{\frac{n-2}{2}}(\frac{x\cdot y'}{|x||y'|}), & |y'| \geq 1\ m \geq 1, \\ 0, & |y'| < 1\ m \geq 1, \\ 0, & m = 0 \end{cases}$$

for a non-negative integer m.

For $x \in \mathbf{R}^n$ and $y' \in \mathbf{R}^{n-1}$, the generalized Neumann kernel is defined by

$$K_{n,m}(x,y') = K_n(x,y') - L_{n,m}(x,y') \quad (m \geq 0).$$

Since $|x|^k C_k^{\frac{n-2}{2}}(\frac{x\cdot y'}{|x||y'|})$ $(k \geq 0)$ is harmonic in H (see [4]), $K_{n,m}(\cdot,y')$ is also harmonic in H for any fixed $y' \in \partial H$. Also, $K_{n,m}(x,y')$ will be of order $|y'|^{-(n+m-2)}$ as $y' \to \infty$ (see [7, Theorem D]).

Put

$$N_m[f](x) = \int_{\partial H} K_{n,m}(x,y') f(y')\,dy',$$

where f is a continuous function on ∂H. Here, note that $N_0[f](x)$ is nothing but the Neumann integral $N[f](x)$.

The following result is due to Siegel and Talvila (see [5, Corollary 2.1]). For similar results with respect to the Schrödinger operator in a half-space, we refer readers to papers by Su (see [8]).

Theorem A *If f is a continuous function on ∂H satisfying* (1.1) *with $p = 1$ and $\alpha = m$, then*

$$\lim_{|x|\to\infty, x\in H} N_m[f](x) = o\big(|x|^m \sec^{n-2}\theta\big). \tag{1.2}$$

The next result deals with a type of uniqueness of solutions for the Neumann problem on H (see [9, Theorem 3]).

Theorem B *Let l be a positive integer and m be a non-negative integer. If f is a continuous function on ∂H satisfying*

$$\int_{\partial H} \frac{|f(y')|}{(1+|y'|)^{n+m-2}}\,dy' < \infty,$$

and h is a solution of the Neumann problem on H with f such that

$$\lim_{|x|\to\infty, x\in H} h^+(x) = o\big(|x|^{l+m}\big),$$

then

$$h(x) = N_m[f](x) + \Pi(x') + \sum_{j=1}^{[\frac{l+m}{2}]} \frac{(-1)^j}{(2j)!} x_n^{2j} \Delta^j \Pi(x')$$

for any $x = (x', x_n) \in H$, where $h^+(x)$ is the positive part of h,

$$\Delta^j = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\right) \quad (j = 1, 2 \ldots)$$

and $\Pi(x')$ is a polynomial of $x' \in \mathbf{R}^{n-1}$ of degree less than $l + m$.

Our first aim is to be concerned with the growth property of $N_m[f]$ at infinity and establish the following theorem.

Theorem 1 *Let* $1 \le p < \infty$, $0 \le \beta \le (n-2)p$, $n+\alpha-2 > -(n-1)(p-1)$ *and*

$$1-\frac{1-\alpha}{p} < m < 2-\frac{1-\alpha}{p} \quad \text{if } p>1,$$
$$\alpha \le m < \alpha+1 \quad \text{if } p=1.$$

If f *is a measurable function on* ∂ *satisfying* (1.1), *then there exists a covering* $\{r_j, R_j\}$ *of* $E(\epsilon;\mu,(n-2)p-\beta)$ $(\subset H)$ *satisfying*

$$\sum_{j=0}^{\infty}\left(\frac{r_j}{R_j}\right)^{(n-2)p-\beta} < \infty \tag{1.3}$$

such that

$$\lim_{|x|\to\infty, x\in H-E(\epsilon;\mu,(n-2)p-\beta)} N_m[f](x) = o\left(|x|^{1+\frac{\alpha-1}{p}} \sec^{\frac{\beta}{p}}\theta\right). \tag{1.4}$$

Corollary 1 *Let* $1 < p < \infty$, $n+\alpha-2 > -(n-1)(p-1)$ *and*

$$1-\frac{1-\alpha}{p} < m < 2-\frac{1-\alpha}{p}.$$

If f *is a measurable function on* ∂H *satisfying* (1.1), *then*

$$\lim_{|x|\to\infty, x\in H} N_m[f](x) = o\left(|x|^{1+\frac{\alpha-1}{p}} \sec^{n-2}\theta\right). \tag{1.5}$$

As an application of Theorem 1, we now show the solution of the Neumann problem with continuous data on H.

Theorem 2 *Let* p, β, α *and* m *be defined as in Theorem* 1. *If* f *is a continuous function on* ∂H *satisfying* (1.1), *then the function* $N_m[f]$ *is a solution of the Neumann problem on* H *with* f *and* (1.4) *holds, where the exceptional set* $E(\epsilon;\mu,(n-2)p-\beta)$ $(\subset H)$ *has a covering* $\{r_j, R_j\}$ *satisfying* (1.3).

Remark In the case $p = 1$, $\alpha = m$ and $\beta = n-2$, then (1.3) is a finite sum and the set $E(\epsilon;\mu,0)$ is a bounded set. So (1.4) holds in H. That is to say, (1.2) holds. This is just the result of Theorem A.

Corollary 2 *Let* $1 \le p < \infty$, $n+\alpha-2 > -(n-1)(p-1)$ *and*

$$1-\frac{1-\alpha}{p} < m < 2-\frac{1-\alpha}{p} \quad \text{if } p>1,$$
$$\alpha \le m < \alpha+1 \quad \text{if } p=1.$$

If f *is a continuous function on* ∂H *satisfying* (1.1), *then the function* $N_m[f]$ *is a solution of the Neumann problem on* H *with* f *and* (1.5) *holds.*

The following result extends Theorem B, which is our result in the case $p = 1$ and $\alpha = m$.

Theorem 3 *Let $1 \leq p < \infty$, $\alpha > 1 - p$, l be a positive integer and*

$$1 - \frac{1-\alpha}{p} < m < 2 - \frac{1-\alpha}{p} \quad \text{if } p > 1,$$
$$\alpha \leq m < \alpha + 1 \quad \text{if } p = 1.$$

If f is a continuous function on ∂H satisfying (1.1) *and h is a solution of the Neumann problem on H with f such that*

$$\lim_{|x|\to\infty, x\in H} h^{+}(x) = o\bigl(|x|^{l+[1+\frac{\alpha-1}{p}]}\bigr), \tag{1.6}$$

then

$$h(x) = N_m[f](x) + \Pi\bigl(x'\bigr) + \sum_{j=1}^{[\frac{l+[1+\frac{\alpha-1}{p}]}{2}]} \frac{(-1)^j}{(2j)!} x_n^{2j} \Delta^j \Pi\bigl(x'\bigr) \tag{1.7}$$

for any $x = (x', x_n) \in H$ and $\Pi(x')$ is a polynomial of $x' \in \mathbf{R}^{n-1}$ of degree less than $l + [1 + \frac{\alpha-1}{p}]$.

2 Lemmas

In our discussions, the following estimates for the kernel function $K_{n,m}(x, y')$ are fundamental (see [10, Lemma 4.2] and [4, Lemmas 2.1 and 2.4]).

Lemma 1

(1) *If $1 \leq |y'| \leq \frac{|x|}{2}$, then $|K_{n,m}(x,y')| \lesssim |x|^{m-1}|y'|^{-n-m+3}$.*
(2) *If $\frac{|x|}{2} < |y'| \leq \frac{3}{2}|x|$, then $|K_{n,m}(x,y')| \lesssim |x-y'|^{2-n}$.*
(3) *If $\frac{3}{2}|x| < |y'| \leq 2|x|$, then $|K_{n,m}(x,y')| \lesssim x_n^{2-n}$.*
(4) *If $|y'| \geq 2|x|$ and $|y'| \geq 1$, then $|K_{n,m}(x,y')| \lesssim |x|^m|y'|^{2-n-m}$.*

The following lemma is due to Qiao (see [4]).

Lemma 2 *If $\epsilon > 0$, $\eta \geq 0$ and λ is a positive measure in $\mathbf{R}^n$ satisfying $\lambda(\mathbf{R}^n) < \infty$, then $E(\epsilon; \lambda, \eta)$ has a covering $\{r_j, R_j\}$ $(j = 1, 2, \ldots)$ such that*

$$\sum_{j=1}^{\infty} \left(\frac{r_j}{R_j}\right)^{\eta} < \infty.$$

Lemma 3 ([9, Lemma 4]) *Let p, β, α and m be defined as in Theorem* 1. *If f is a locally integral and upper semi-continuous function on ∂H satisfying* (1.1), *then*

$$\limsup_{x\in H, x\to y'} \frac{\partial}{\partial x_n} N_m[f](x) \leq f\bigl(y'\bigr)$$

for any fixed $y' \in \partial H$.

Lemma 4 ([2, Lemma 1]) *If $h(x)$ is a harmonic polynomial of $x = (x', x_n) \in H$ of degree m and $\partial h/\partial x_n$ vanishes on ∂H, then there exists a polynomial $\Pi(x')$ of degree m such that*

$$h(x) = \begin{cases} \Pi(x') + \sum_{j=1}^{[\frac{m}{2}]} \frac{(-1)^j}{(2j)!} x_n^{2j} \Delta^j \Pi(x'), & m \geq 2, \\ \Pi(x'), & m = 0, 1. \end{cases}$$

3 Proof of Theorem 1

For any $\epsilon > 0$, there exists $R_\epsilon > 1$ such that

$$\int_{\partial H(R_\epsilon,\infty)} \frac{|f(y')|^p}{(1+|y'|)^{n+\alpha-2}}\, dy' < \epsilon. \tag{3.1}$$

Take any point $x \in H(R_\epsilon,\infty) - E(\epsilon;\mu,(n-2)p-\beta)$ such that $|x| > 2R_\epsilon$, and write

$$\begin{aligned} N_m[f](x) &= \left(\int_{G_1} + \int_{G_2} + \int_{G_3} + \int_{G_4} + \int_{G_5}\right) K_{n,m}(x,y') f(y')\, dy' \\ &= U_1(x) + U_2(x) + U_3(x) + U_4(x) + U_5(x), \end{aligned}$$

where

$$\begin{aligned} &G_1 = \{y' \in \partial H : |y'| \leq 1\}, \qquad G_2 = \left\{y' \in \partial H : 1 < |y'| \leq \frac{|x|}{2}\right\}, \\ &G_3 = \left\{y' \in \partial H : \frac{|x|}{2} < |y'| \leq \frac{3}{2}|x|\right\}, \qquad G_4 = \left\{y' \in \partial H : \frac{3}{2}|x| < |y'| \leq 2|x|\right\} \\ &G_5 = \{y' \in \partial H : |y'| \geq 2|x|\}. \end{aligned}$$

First note that

$$\begin{aligned} |U_1(x)| &\lesssim \int_{G_1} \frac{|f(y')|}{|x-y'|^{n-2}}\, dy' \\ &\lesssim |x|^{2-n} \int_{G_1} |f(y')|\, dy', \end{aligned}$$

so that

$$\lim_{|x|\to\infty, x\in H} |x|^{-1+\frac{1-\alpha}{p}} U_1(x) = 0. \tag{3.2}$$

If $m < 2 - \frac{1-\alpha}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $(3 - n - m + \frac{n+\alpha-2}{p})q + n - 1 > 0$. By Lemma 1(1), (3.1) and the Hölder inequality, we have

$$\begin{aligned} |U_2(x)| &\lesssim |x|^{m-1} \int_{G_2} |y'|^{-n-m+3} |f(y')|\, dy' \\ &\lesssim |x|^{m-1} \left(\int_{G_2} \frac{|f(y')|^p}{|y'|^{n+\alpha-2}}\, dy'\right)^{\frac{1}{p}} \left(\int_{G_2} |y'|^{(-n-m+3+\frac{n+\alpha-2}{p})q}\, dy'\right)^{\frac{1}{q}} \\ &\lesssim |x|^{1-\frac{1-\alpha}{p}} \left(\int_{G_2} \frac{|f(y')|^p}{|y'|^{n+\alpha-2}}\, dy'\right)^{\frac{1}{p}}. \end{aligned} \tag{3.3}$$

Put

$$U_2(x) = U_{21}(x) + U_{22}(x),$$

where

$$U_{21}(x) = \int_{G_2\cap B_{n-1}(R_\epsilon)} K_{n,m}(x,y')f(y')\,dy',$$
$$U_{22}(x) = \int_{G_2\setminus B_{n-1}(R_\epsilon)} K_{n,m}(x,y')f(y')\,dy'.$$

If $|x| \geq 2R_\epsilon$, then

$$|U_{21}(x)| \lesssim R_\epsilon^{2-m-\frac{1-\alpha}{p}}|x|^{m-1}.$$

Moreover, by (3.1) and (3.3), we get

$$|U_{22}(x)| \lesssim \epsilon|x|^{1-\frac{1-\alpha}{p}}.$$

That is,

$$|U_2(x)| \lesssim \epsilon|x|^{1-\frac{1-\alpha}{p}}. \tag{3.4}$$

By Lemma 1(3), (3.1) and the Hölder inequality, we have

$$|U_4(x)| \lesssim \epsilon x_n^{2-n}|x|^{n-1-\frac{1-\alpha}{p}}. \tag{3.5}$$

If $m > 1 - \frac{1-\alpha}{p}$, then $(2-n-m+\frac{n+\alpha-2}{p})q + n - 1 < 0$. We obtain, by Lemma 1(4), (3.1) and the Hölder inequality,

$$\begin{aligned} |U_5(x)| &\lesssim |x|^m \int_{G_5} |y'|^{-n-m+2}|f(y')|\,dy' \\ &\lesssim |x|^m \left(\int_{G_5} \frac{|f(y')|^p}{|y'|^{n+\alpha-2}}\,dy'\right)^{\frac{1}{p}} \left(\int_{G_5} |y'|^{(-n-m+2+\frac{n+\alpha-2}{p})q}\,dy'\right)^{\frac{1}{q}} \\ &\lesssim \epsilon|x|^{1-\frac{1-\alpha}{p}}. \end{aligned} \tag{3.6}$$

Finally, we shall estimate $U_3(x)$. Take a sufficiently small positive number b such that $\partial H[\frac{|x|}{2}, \frac{3}{2}|x|] \subset B(x, \frac{|x|}{2})$ for any $x \in \Pi(b)$, where

$$\Pi(b) = \left\{x \in H;\ \inf_{y'\in\partial H}\left|\frac{x}{|x|} - \frac{y'}{|y'|}\right| < b\right\}$$

and divide H into two sets $\Pi(b)$ and $H - \Pi(b)$.

If $x \in H - \Pi(b)$, then there exists a positive number b' such that $|x - y'| \geq b'|x|$ for any $y' \in \partial H$, and hence

$$\begin{aligned}\left|U_3(x)\right| &\lesssim \int_{G_3} \left|y'\right|^{2-n} \left|f(y')\right| dy' \\ &\lesssim |x|^m \int_{G_3} \left|y'\right|^{2-n-m} \left|f(y')\right| dy' \\ &\lesssim \epsilon |x|^{1-\frac{1-\alpha}{p}},\end{aligned}$$

which is similar to the estimate of $U_5(x)$.

We shall consider the case $x \in \Pi(b)$. Now put

$$H_i(x) = \left\{y' \in \partial H\left[\frac{|x|}{2}, \frac{3}{2}|x|\right]; 2^{i-1}\delta(x) \leq \left|x - y'\right| < 2^i\delta(x)\right\},$$

where $\delta(x) = \inf_{y' \in H} |x - y'|$.

Since $\partial H \cap \{y' \in \mathbf{R}^{n-1} : |x - y'| < \delta(x)\} = \varnothing$, we have

$$U_3(x) = \sum_{i=1}^{i(x)} \int_{H_i(x)} \frac{|g(y')|}{|x - y'|^{n-2}}\, dy',$$

where $i(x)$ is a positive integer satisfying $2^{i(x)-1}\delta(x) \leq \frac{|x|}{2} < 2^{i(x)}\delta(x)$.

Similar to the estimate of $U_5(x)$, we obtain

$$\begin{aligned}&\int_{H_i(x)} \frac{|g(y')|}{|x - y'|^{n-2}}\, dy' \\ &\quad\lesssim \int_{H_i(x)} \frac{|g(y')|}{\{2^{i-1}\delta(x)\}^{n-2}}\, dy' \\ &\quad\lesssim \delta(x)^{\frac{\beta-(n-2)p}{p}} \int_{H_i(x)} \delta(x)^{\frac{(n-2)p-\beta}{p}-n+2} \left|g(y')\right| dy' \\ &\quad\lesssim \cos^{-\frac{\beta}{p}}\theta\,\delta(x)^{\frac{\beta-(n-2)p}{p}} \int_{H_i(x)} |x|^{-\frac{\beta}{p}} \left|g(y')\right| dy' \\ &\quad\lesssim |x|^{n-2-\frac{\beta}{p}} \cos^{-\frac{\beta}{p}}\theta\,\delta(x)^{\frac{\beta-(n-2)p}{p}} \int_{H_i(x)} \left|y'\right|^{2-n} \left|g(y')\right| dy' \\ &\quad\lesssim |x|^{n-1+\frac{\alpha-\beta-1}{p}} \cos^{-\frac{\beta}{p}}\theta \left(\frac{\mu(H_i(x))}{2^i\delta(x)^{(n-2)p-\beta}}\right)^{\frac{1}{p}}\end{aligned}$$

for $i = 0, 1, 2, \ldots, i(x)$.

Since $x \notin E(\epsilon; \mu, (n-2)p - \beta)$, we have

$$\frac{\mu(H_i(x))}{\{2^i\delta(x)\}^{(n-2)p-\beta}} \lesssim \frac{\mu(B_{n-1}(x, 2^i\delta(x)))}{\{2^i\delta(x)\}^{(n-2)p-\beta}} \lesssim M\bigl(x; \mu, (n-2)p - \beta\bigr) \lesssim \epsilon |x|^{\beta-(n-2)p}$$

for $i = 0, 1, 2, \ldots, i(x) - 1$ and

$$\frac{\mu(H_{i(x)}(x))}{\{2^i\delta(x)\}^{(n-2)p-\beta}} \lesssim \frac{\mu(B_{n-1}(x, \frac{|x|}{2}))}{(\frac{|x|}{2})^{(n-2)p-\beta}} \lesssim \epsilon |x|^{\beta-(n-2)p}.$$

So

$$\left|U_3(x)\right| \lesssim \epsilon |x|^{1+\frac{\alpha-1}{p}} \sec^{\frac{\beta}{p}} \theta. \tag{3.7}$$

Combining (3.2), (3.4)-(3.7), we obtain that if R_ϵ is sufficiently large and ϵ is a sufficiently small number, then $N_m[f](x) = o(|x|^{1+\frac{\alpha-1}{p}} \sec^{\frac{\beta}{p}} \theta)$ as $|x| \to \infty$, where $x \in H(R_\epsilon, +\infty) - E(\epsilon; \mu, (n-2)p-\beta)$. Finally, there exists an additional finite ball B_0 covering $H(0, R_\epsilon]$, which together with Lemma 2, gives the conclusion of Theorem 1.

4 Proof of Theorem 2

For any fixed $x \in H$, take a number R satisfying $R > \max\{1, 2|x|\}$. If $m > \frac{1-\alpha}{p}$, then $(2 - n - m + \frac{n+\alpha-2}{p})q + n - 1 < 0$. By (1.1), Lemma 1(4) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\partial H(R,\infty)} \left|K_{n,m}(x,y')\right|\left|f(y')\right| dy' \\
&\quad\lesssim |x|^m \int_{\partial H(R,\infty)} \left|y'\right|^{2-n-m} \left|f(y')\right| dy' \\
&\quad\lesssim |x|^m \left(\int_{\partial H(R,\infty)} \frac{|f(y')|^p}{|y'|^{n+\alpha-2}} dy'\right)^{\frac{1}{p}} \left(\int_{\partial H(R,\infty)} \left|y'\right|^{(-n-m+2+\frac{n+\alpha-2}{p})q} dy'\right)^{\frac{1}{q}} \\
&\quad< \infty.
\end{aligned}$$

Hence $N_m[f](x)$ is absolutely convergent and finite for any $x \in H$. Thus $N_m[f](x)$ is harmonic on H.

To prove

$$\lim_{x \to y', x \in H} \frac{\partial}{\partial x_n} N_m[f](x) = f(y')$$

for any point $y' \in \partial H$, we only need to apply Lemma 3 to $f(y)$ and $-f(y)$.

We complete the proof of Theorem 2.

5 Proof of Theorem 3

Consider the function $h'(x) = h(x) - N_m[f](x)$. Then it follows from Theorems 2 and 3 that $h'(x)$ is a solution of the Neumann problem on H with f and it is an even function of x_n (see [2, p.92]).

Since

$$0 \le \left\{h - N_m[f]\right\}^+(x) \le h^+(x) + \left\{N_m[f]\right\}^-(x)$$

for any $x \in H$, and

$$\lim_{|x| \to \infty, x \in H} N_m[f](x) = o\left(|x|^{1+\frac{\alpha-1}{p}}\right)$$

from Theorem 2.

Moreover, (1.6) gives that

$$\lim_{|x| \to \infty, x \in H} \left(h - N_m[f]\right)(x) = o\left(|x|^{l+[1+\frac{\alpha-1}{p}]}\right).$$

This implies that $h'(x)$ is a polynomial of degree less than $l+[1+\frac{\alpha-1}{p}]$ (see [11, Appendix]), which gives the conclusion of Theorem 3 from Lemma 4.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

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Acknowledgements
The authors are thankful to the referees for their helpful suggestions and necessary corrections in the completion of this paper.

Received: 13 August 2013 Accepted: 23 October 2013 Published: 04 Dec 2013

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10.1186/1029-242X-2013-572
Cite this article as: Ren and Yang: **Growth estimates for modified Neumann integrals in a half-space.** *Journal of Inequalities and Applications* 2013, 2013:572