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Imbedding inequalities with L^φ-norms for composite operators

Yuming Xing[1*] and Shusen Ding[2]

*Correspondence: xyuming@hit.edu.cn
[1]Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we prove imbedding inequalities with L^φ-norms for the composition of the potential operator and homotopy operator applied to differential forms. We also establish the global imbedding inequality in L^φ-averaging domains.

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1 Introduction

The theory about operators applied to functions has been very well developed. However, the study about operators applied to differential forms has just begun. The purpose of this paper is to establish the local and global imbedding inequalities with L^φ-norms for the composition of the homotopy operator T and the potential operator P applied to differential forms. Specifically, we estimate the upper bound of the Orlicz-Sobolev-norm $\|T(P(u)) - (T(P(u)))_B\|_{W^{1,\varphi}(B,\wedge^l)}$ in terms of the L^φ-norm $\|u\|_{L^\varphi(\sigma B)}$, where $\sigma > 1$ is a constant, B is a ball, and u is a differential form satisfying the A-harmonic equation. We also establish the global imbedding theorems in the L^φ-averaging domains and bounded domains, respectively. Differential forms and operators T and P are widely used not only in analysis and partial differential equations [1–7], but also in physics and potential analysis [8–11]. We all know that any differential form u can be decomposed as $u = d(Tu) + T(du)$, where d is the differential operator, and T is the homotopy operator. In many situations, we need to estimate the composition of the homotopy operator T and the potential operator P. For example, when we consider the decomposition of $P(u)$, we have to study the composition $T \circ P$ of the homotopy operator T and the potential operator P. Our main results are presented and proved in Theorem 2.6, Theorem 3.3 and Theorem 3.6, respectively.

We assume that Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, B and σB are the balls with the same center and $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$ throughout this paper. We do not distinguish the balls from cubes in this paper. We use $|E|$ to denote the n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$. For a function u, the average of u over B is defined by $u_B = \frac{1}{|B|}\int_B u\,dx$. All integrals involved in this paper are the Lebesgue integrals. Differential forms are extensions of differentiable functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2, \ldots, x_n)$ is called a 0-form. A differential 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \ldots, x_n)\,dx_i$, where the coefficient functions $u_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, are differentiable. Similarly, a differential k-form $u(x)$ can be expressed as

$$u(x) = \sum_I u_I(x)\,dx_I = \sum u_{i_1 i_2 \cdots i_k}(x)\,dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where $I = (i_1, i_2, \ldots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(\Omega, \wedge^l)$ be the space of all differential l-forms in Ω, and $L^p(\Omega, \wedge^l)$ be the l-forms $u(x) = \sum_I u_I(x)\,dx_I$ in Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by d and the Hodge star operator by $\star$. The Hodge codifferential operator $d^\star$ is given by $d^\star = (-1)^{nl+1} \star d\star$, $l = 1, 2, \ldots, n$. For $u \in D'(\Omega, \wedge^l)$ the vector-valued differential form

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$$

consists of differential forms $\frac{\partial u}{\partial x_i} \in D'(\Omega, \wedge^l)$, where the partial differentiation is applied to the coefficients of ω. The nonlinear partial differential equation

$$d^\star A(x, du) = B(x, du) \tag{1.1}$$

is called non-homogeneous A-harmonic equation, where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions

$$\left|A(x,\xi)\right| \le a|\xi|^{p-1}, \qquad A(x,\xi)\cdot\xi \ge |\xi|^p \quad \text{and} \quad \left|B(x,\xi)\right| \le b|\xi|^{p-1} \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants, and $1 < p < \infty$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi = 0 \tag{1.3}$$

for all $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ with compact support. If u is a function (0-form) in $\mathbb{R}^n$, equation (1.1) reduces to

$$\operatorname{div} A(x, \nabla u) = B(x, \nabla u). \tag{1.4}$$

If the operator $B = 0$, equation (1.1) becomes

$$d^\star A(x, du) = 0, \tag{1.5}$$

which is called the (homogeneous) A-harmonic equation. Let $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x,\xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then A satisfies the required conditions, and (1.5) becomes the p-harmonic equation $d^\star(du|du|^{p-2}) = 0$ for differential forms. See [1–3, 12–16] for recent results on the A-harmonic equations and related topics.

Assume that $D \subset \mathbb{R}^n$ is a bounded, convex domain. The following operator K_y with the case $y = 0$ was first introduced by Cartan in [8]. Then it was extended to the following general version in [6]. For each $y \in D$, a linear operator $K_y : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$ defined by $(K_y\omega)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1}\omega(tx+y-ty; x-y, \xi_1, \ldots, \xi_{l-1})\,dt$ and the decomposition $\omega = d(K_y\omega) + K_y(d\omega)$ correspond. A homotopy operator $T : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$ is defined by an averaging K_y over all points y in D

$$T\omega = \int_D \varphi(y) K_y\omega\,dy, \tag{1.6}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y)\,dy = 1$. For simplicity purpose, we write $\xi = (\xi_1, \ldots, \xi_{l-1})$. Then $T\omega(x; \xi) = \int_0^1 t^{l-1} \int_D \varphi(y)\omega(tx + y - ty; x - y, \xi)\,dy\,dt$. By substituting $z = tx + y - ty$ and $t = s/(1 + s)$, we have

$$T\omega(x; \xi) = \int_D \omega\bigl(z, \zeta(z, x - z), \xi\bigr)\,dz, \tag{1.7}$$

where the vector function $\zeta : D \times \mathbb{R}^n \to \mathbb{R}^n$ is given by $\zeta(z, h) = h \int_0^\infty s^{l-1}(1 + s)^{n-1}\varphi(z - sh)\,ds$. The integral (1.7) defines a bounded operator $T : L^s(D, \Lambda^l) \to W^{1,s}(D, \Lambda^{l-1})$, $l = 1, 2, \ldots, n$, and the decomposition

$$u = d(Tu) + T(du) \tag{1.8}$$

holds for any differential form u. The l-form $\omega_D \in D'(D, \Lambda^l)$ is defined by

$$\begin{aligned} &\omega_D = \fint_D \omega(y)\,dy = |D|^{-1} \int_D \omega(y)\,dy, \quad l = 0, \quad \text{and} \\ &\omega_D = d(T\omega), \quad l = 1, 2, \ldots, n, \end{aligned} \tag{1.9}$$

for all $\omega \in L^p(D, \Lambda^l)$, $1 \le p < \infty$. Also, for any differential form u, we have

$$\bigl\|\nabla(Tu)\bigr\|_{p,D} \le C|D|\|u\|_{p,D}, \quad \text{and} \quad \|Tu\|_{p,D} \le C|D| \operatorname{diam}(D)\|u\|_{p,D}. \tag{1.10}$$

From [17, p.16], we know that any open subset Ω in $\mathbb{R}^n$ is the union of a sequence of cubes Q_k, whose sides are parallel to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from F. Specifically,

(i) $\Omega = \bigcup_{k=1}^\infty Q_k$,
(ii) $Q_j^0 \cap Q_k^0 = \phi$ if $j \neq k$,
(iii) there exist two constants $c_1, c_2 > 0$ (we can take $c_1 = 1$, and $c_2 = 4$), so that

$$c_1 \operatorname{diam}(Q_k) \le \operatorname{distance}(Q_k, F) \le c_2 \operatorname{diam}(Q_k). \tag{1.11}$$

Thus, the definition of the homotopy operator T can be generalized to any domain Ω in $\mathbb{R}^n$: For any $x \in \Omega$, $x \in Q_k$ for some k. Let T_{Q_k} be the homotopy operator defined on Q_k (each cube is bounded and convex). Thus, we can define the homotopy operator T_Ω on any domain Ω by

$$T_\Omega = \sum_{k=1}^\infty T_{Q_k} \chi_{Q_k(x)}. \tag{1.12}$$

Recently, Hui Bi extended the definition of the potential operator to the case of differential forms, see [3]. For any differential l-form $u(x)$, the potential operator P is defined by

$$Pu(x) = \sum_I \int_E K(x,y)u_I(y)\,dy\,dx_I, \tag{1.13}$$

where the kernel $K(x,y)$ is a nonnegative measurable function defined for $x \neq y$, and the summation is over all ordered l-tuples I. The $l = 0$ case reduces to the usual potential operator,

$$Pf(x) = \int_E K(x,y)f(y)\,dy, \tag{1.14}$$

where $f(x)$ is a function defined on $E \subset R^n$. See [3] and [9] for more results about the potential operator. We say a kernel K on $\mathbb{R}^n \times \mathbb{R}^n$ satisfies the standard estimates if there exist δ, $0 < \delta \le 1$ and a constant C such that for all distinct points x and y in $\mathbb{R}^n$, and all z with $|x-z| < \frac{1}{2}|x-y|$, the kernel K satisfies

(i) $K(x,y) \le C|x-y|^{-n}$;
(ii) $|K(x,y) - K(z,y)| \le C|x-z|^{\delta}|x-y|^{-n-\delta}$;
(iii) $|K(y,x) - K(y,z)| \le C|x-z|^{\delta}|x-y|^{-n-\delta}$.

In this paper, we always assume that P is the potential operator defined in (1.13) with the kernel $K(x,y)$ satisfying condition (i) of the standard estimates. Recently, Hui Bi in [3] proved the following inequality for the potential operator.

$$\|P(u)\|_{p,E} \le C\|u\|_{p,E}, \tag{1.15}$$

where $u \in D'(E, \wedge^l)$, $l = 0, 1, \ldots, n-1$, is a differential form defined in a bounded and convex domain E, and $p > 1$ is a constant.

2 Local imbedding inequalities

In this section, we prove the local L^{φ} imbedding inequalities for $T \circ P$ applied to solutions of the non-homogeneous A-harmonic equation in a bounded domain. We will need the following definitions and a notation. A continuously increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, is called an Orlicz function. The Orlicz space $L^{\varphi}(\Omega)$ consists of all measurable functions f on Ω such that $\int_\Omega \varphi(\frac{|f|}{\lambda})\,dx < \infty$ for some $\lambda = \lambda(f) > 0$. $L^{\varphi}(\Omega)$ is equipped with the nonlinear Luxemburg functional

$$\|f\|_{L^{\varphi}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) dx \le 1\right\}.$$

A convex Orlicz function φ is often called a Young function. If φ is a Young function, then $\|\cdot\|_{L^{\varphi}(\Omega)}$ defines a norm in $L^{\varphi}(\Omega)$, which is called the Luxemburg norm or Orlicz norm. For any subset $E \subset \mathbb{R}^n$, we use $W^{1,\varphi}(E, \wedge^l)$ to denote the Orlicz-Sobolev space of l-forms, which equals $L^{\varphi}(E, \wedge^l) \cap L_1^{\varphi}(E, \wedge^l)$ with the norm

$$\|u\|_{W^{1,\varphi}(E)} = \|u\|_{W^{1,\varphi}(E,\wedge^l)} = \operatorname{diam}(E)^{-1}\|u\|_{L^{\varphi}(E)} + \|\nabla u\|_{L^{\varphi}(E)}. \tag{2.1}$$

If we choose $\varphi(t) = t^p$, $p > 1$ in (2.1), we obtain the usual L^p norm for $W^{1,p}(E, \wedge^l)$

$$\|u\|_{W^{1,p}(E)} = \|u\|_{W^{1,p}(E,\wedge^l)} = \operatorname{diam}(E)^{-1}\|u\|_{p,E} + \|\nabla u\|_{p,E}. \tag{2.1'}$$

Definition 2.1 [18] We say a Young function φ lies in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, if (i) $1/C \le \varphi(t^{1/p})/g(t) \le C$ and (ii) $1/C \le \varphi(t^{1/q})/h(t) \le C$ for all $t > 0$, where g is a convex increasing function, and h is a concave increasing function on $[0,\infty)$.

From [18], each of φ, g and h in above definition is doubling in the sense that its values at t and $2t$ are uniformly comparable for all $t > 0$, and the consequent fact that

$$C_1 t^q \le h^{-1}\bigl(\varphi(t)\bigr) \le C_2 t^q, \qquad C_1 t^p \le g^{-1}\bigl(\varphi(t)\bigr) \le C_2 t^p, \tag{2.2}$$

where C_1 and C_2 are constants. Also, for all $1 \le p_1 < p < p_2$ and $\alpha \in \mathbb{R}$, the function $\varphi(t) = t^p \log_+^\alpha t$ belongs to $G(p_1, p_2, C)$ for some constant $C = C(p, \alpha, p_1, p_2)$. Here $\log_+(t)$ is defined by $\log_+(t) = 1$ for $t \le e$; and $\log_+(t) = \log(t)$ for $t > e$. Particularly, if $\alpha = 0$, we see that $\varphi(t) = t^p$ lies in $G(p_1, p_2, C)$, $1 \le p_1 < p < p_2$. We will need the following reverse Hölder inequality.

Lemma 2.2 [19] *Let u be a solution of the non-homogeneous A-harmonic equation* (1.1) *in a domain Ω and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \le C|B|^{(t-s)/st}\|u\|_{t,\sigma B} \tag{2.3}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

We first prove the following local inequality for the composition $T \circ P$ with the L^φ-norm.

Theorem 2.3 *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, let Ω be a bounded and convex domain, let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator defined in* (1.6), *and let P be the potential operator defined in* (1.13) *with the kernel $K(x,y)$ satisfying condition* (i) *of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$, and u is a solution of the non-homogeneous A-harmonic equation* (1.1) *in Ω. Then there exists a constant C, independent of u, such that*

$$\bigl\|T\bigl(P(u)\bigr) - \bigl(T\bigl(P(u)\bigr)\bigr)_B\bigr\|_{L^\varphi(B)} \le C \operatorname{diam}(B)\|u\|_{L^\varphi(\sigma B)} \tag{2.4}$$

for all balls B with $\sigma B \subset \Omega$.

Proof Since $u = d(Tu) + T(du)$ and $d(Tu) = u_B$ hold for any differential form u, we have

$$u - u_B = T(du). \tag{2.5}$$

Using (1.15) and noticing $\|u_B\|_{q,B} \le C_1\|u\|_{q,B}$ for any differential form, it follows that

$$\bigl\|dT\bigl(P(u)\bigr)\bigr\|_{q,B} = \bigl\|\bigl(P(u)\bigr)_B\bigr\|_{q,B} \le C_2\bigl\|\bigl(P(u)\bigr)\bigr\|_{q,B} \le C_3\|u\|_{q,B} \tag{2.6}$$

for $q > 1$, Replacing u by $T(P(u))$ in (2.5) and using (1.10) and (2.6), we obtain

$$\begin{aligned}\|T(P(u)) - (T(P(u)))_B\|_{q,B} &= \|Td(T(P(u)))\|_{q,B} \\ &\le C_4|B|\operatorname{diam}(B)\|dT(P(u))\|_{q,B} \\ &\le C_5|B|\operatorname{diam}(B)\|u\|_{q,B} \end{aligned} \tag{2.7}$$

for any differential form u and all balls B with $B \subset \Omega$. From Lemma 2.2, for any positive numbers p and q, it follows that

$$\left(\int_B |u|^q\,dx\right)^{1/q} \le C_6|B|^{(p-q)/pq}\left(\int_{\sigma B}|u|^p\,dx\right)^{1/p}, \tag{2.8}$$

where σ is a constant $\sigma > 1$. Using Jensen's inequality for h^{-1}, (2.2), (2.7), (2.8), (i) in Definition 2.1, and noticing the fact that φ and h are doubling, and φ is an increasing function, we obtain

$$\begin{aligned}&\int_B \varphi\big(\big|T(P(u)) - (T(P(u)))_B\big|\big)\,dx \\ &\quad= h\left(h^{-1}\left(\int_B \varphi\big(\big|T(P(u)) - (T(P(u)))_B\big|\big)\,dx\right)\right) \\ &\quad\le h\left(\int_B h^{-1}\big(\varphi\big(\big|T(P(u)) - (T(P(u)))_B\big|\big)\big)\,dx\right) \\ &\quad\le h\left(C_7\int_B \big|T(P(u)) - (T(P(u)))_B\big|^q\,dx\right) \\ &\quad\le C_8\varphi\left(\left(C_7\int_B \big|T(P(u)) - (T(P(u)))_B\big|^q\,dx\right)^{1/q}\right) \\ &\quad\le C_8\varphi\left(C_9|B|^{1+1/n}\left(\int_B |u|^q\,dx\right)^{1/q}\right) \\ &\quad\le C_8\varphi\left(C_{10}|B|^{1+1/n+(p-q)/pq}\left(\int_{\sigma B}|u|^p\,dx\right)^{1/p}\right) \\ &\quad\le C_8\varphi\left(\left(C_{10}^p|B|^{p(1+1/n)+(p-q)/q}\int_{\sigma B}|u|^p\,dx\right)^{1/p}\right) \\ &\quad\le C_{11}g\left(C_{10}^p|B|^{p(1+1/n)+(p-q)/q}\int_{\sigma B}|u|^p\,dx\right) \\ &\quad= C_{11}g\left(\int_{\sigma B}C_{10}^p|B|^{p(1+1/n)+(p-q)/q}|u|^p\,dx\right) \\ &\quad\le C_{11}\int_{\sigma B}g\big(C_{10}^p|B|^{p(1+1/n)+(p-q)/q}|u|^p\big)\,dx \\ &\quad\le C_{12}\int_{\sigma B}\varphi\big(C_{10}|B|^{1+\frac{1}{n}+\frac{p-q}{pq}}|u|\big)\,dx. \end{aligned} \tag{2.9}$$

Since $p \geq 1$, then $1 + \frac{1}{n} + \frac{p-q}{pq} > \frac{1}{n}$. Hence, we have $|B|^{1+\frac{1}{n}+\frac{p-q}{pq}} \leq C_{13}|B|^{\frac{1}{n}}$. Note that φ is doubling, we obtain

$$\varphi\bigl(C_{10}|B|^{1+\frac{1}{n}+\frac{p-q}{pq}}|u|\bigr) \leq C_{14}|B|^{\frac{1}{n}}\varphi\bigl(|u|\bigr). \tag{2.10}$$

Combining (2.9) and (2.10) and using $|B|^{\frac{1}{n}} = C_{15}\,\mathrm{diam}(B)$ yields

$$\int_B \varphi\bigl(\bigl|T\bigl(P(u)\bigr) - \bigl(T\bigl(P(u)\bigr)\bigr)_B\bigr|\bigr)\,dx \leq C_{16}\ \mathrm{diam}(B)\int_{\sigma B}\varphi\bigl(|u|\bigr)\,dx. \tag{2.11}$$

Since each of φ, g and h in Definition 2.1 is doubling, from (2.11), we have

$$\int_B \varphi\left(\frac{|T(P(u)) - (T(P(u)))_B|}{\lambda}\right)dx \leq C\,\mathrm{diam}(B)\int_{\sigma B}\varphi\left(\frac{|u|}{\lambda}\right)dx$$

for all balls B with $\sigma B \subset \Omega$ and any constant $\lambda > 0$. From (2.1) and the last inequality, we have the following inequality with the Luxemburg norm

$$\bigl\|T\bigl(P(u)\bigr) - \bigl(T\bigl(P(u)\bigr)\bigr)_B\bigr\|_{L^{\varphi}(B)} \leq C\,\mathrm{diam}(B)\|u\|_{L^{\varphi}(\sigma B).}$$

The proof of Theorem 2.3 has been completed. □

In order to prove our main local imbedding theorem, we will need the following Theorems 2.4 and 2.5.

Theorem 2.4 *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$, let Ω be a bounded and convex domain, let T be the homotopy operator defined in* (1.6), *and let P be the potential operator defined in* (1.13) *with the kernel $K(x,y)$ satisfying condition* (i) *of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$, and u is a solution of the nonhomogeneous A-harmonic equation* (1.1) *in Ω. Then there exists a constant C, independent of u, such that*

$$\bigl\|Td\bigl(T\bigl(P(u)\bigr)\bigr)\bigr\|_{L^{\varphi}(B)} \leq C|B|\,\mathrm{diam}(B)\|u\|_{L^{\varphi}(\sigma B)} \tag{2.12}$$

for all balls B with $\sigma B \subset \Omega$.

Proof Using (1.10), we have

$$\bigl\|Td\bigl(T\bigl(P(u)\bigr)\bigr)\bigr\|_{q,B} \leq C_1|B|\,\mathrm{diam}(B)\bigl\|d\bigl(T\bigl(P(u)\bigr)\bigr)\bigr\|_{q,B} \tag{2.13}$$

for any differential form u and $q > 1$. Using (1.15) and the fact that $d(Tu) = u_B$, and noticing that

$$\|u_B\|_{q,B} \leq C_2|B|\|u\|_{q,B}$$

holds for any differential form u, we obtain

$$\bigl\|d\bigl(T\bigl(P(u)\bigr)\bigr)\bigr\|_{q,B} = \bigl\|P(u)_B\bigr\|_{q,B} \leq C_3|B|\bigl\|P(u)\bigr\|_{q,B} \leq C_4|B|\|u\|_{q,B} \tag{2.14}$$

for all balls B with $B \subset \Omega$. From (2.13) and (2.14), it follows that

$$\| Td(T(P(u))) \|_{q,B} \le C_5 |B|^2 \operatorname{diam}(B) \|u\|_{q,B}. \tag{2.15}$$

By Lemma 2.2, for any positive numbers p and q, it follows that

$$\left(\int_B |u|^q \, dx \right)^{1/q} \le C_6 |B|^{(p-q)/pq} \left(\int_{\sigma B} |u|^p \, dx \right)^{1/p}, \tag{2.16}$$

where σ is a constant $\sigma > 1$. Using Jensen's inequality for h^{-1}, (2.2), (2.15), (2.16), (i) in Definition 2.1, and noticing the fact that φ and h are doubling, and φ is an increasing function, we obtain

$$\begin{aligned}
\int_B \varphi\big(\big|Td(T(P(u)))\big|\big)\, dx &= h\left(h^{-1}\left(\int_B \varphi\big(\big|Td(T(P(u)))\big|\big)\, dx\right)\right) \\
&\le h\left(\int_B h^{-1}\big(\varphi\big(\big|Td(T(P(u)))\big|\big)\big)\, dx\right) \\
&\le h\left(C_7 \int_B \big|Td(T(P(u)))\big|^q \, dx\right) \\
&\le C_8 \varphi\left(\left(C_7 \int_B \big|Td(T(P(u)))\big|^q \, dx\right)^{1/q}\right) \\
&\le C_8 \varphi\left(C_9 |B|^2 \operatorname{diam}(B) \left(\int_B |u|^q \, dx\right)^{1/q}\right) \\
&\le C_8 \varphi\left(C_{10} |B|^{2+(p-q)/pq} \operatorname{diam}(B) \left(\int_{\sigma B} |u|^p \, dx\right)^{1/p}\right) \\
&\le C_{12} \varphi\left(\left(C_{11}^p |B|^{2p+(p-q)/q} \big(\operatorname{diam}(B)\big)^p \int_{\sigma B} |u|^p \, dx\right)^{1/p}\right) \\
&\le C_{13} g\left(C_{11}^p |B|^{2p+(p-q)/q} \big(\operatorname{diam}(B)\big)^p \int_{\sigma B} |u|^p \, dx\right) \\
&= C_{11} g\left(\int_{\sigma B} C_{11}^p |B|^{2p+(p-q)/q} \big(\operatorname{diam}(B)\big)^p |u|^p \, dx\right) \\
&\le C_{12} \int_{\sigma B} g\big(C_{11}^p |B|^{2p+(p-q)/q} \big(\operatorname{diam}(B)\big)^p |u|^p\big)\, dx \\
&\le C_{13} \int_{\sigma B} \varphi\big(C_{11} |B|^{2+(p-q)/pq} \operatorname{diam}(B) |u|\big)\, dx.
\end{aligned} \tag{2.17}$$

Since $p \ge 1$, then $1 + \frac{p-q}{pq} > 0$. Hence, we have

$$|B|^{2+\frac{p-q}{pq}} = |B||B|^{1+\frac{p-q}{pq}} = |B||B|^{1+1/q-1/p} \le C_{14}|B|.$$

Note that φ is doubling, we obtain

$$\varphi\big(C_{11}|B|^{2+(p-q)/pq} \operatorname{diam}(B)|u|\big) \le C_{15}|B| \operatorname{diam}(B) \varphi\big(|u|\big). \tag{2.18}$$

Combining (2.17) and (2.18) yields

$$\int_B \varphi\big(\big|d\big(T\big(P(u)\big)\big)\big|\big)\,dx \le C_{16}|B|\operatorname{diam}(B)\int_{\sigma B}\varphi\big(|u|\big)\,dx. \tag{2.19}$$

Since each of φ, g and h in Definition 2.1 is doubling, from (2.19), we have

$$\int_B \varphi\left(\frac{|Td(T(P(u)))|}{\lambda}\right)dx \le C_{17}|B|\operatorname{diam}(B)\int_{\sigma B}\varphi\left(\frac{|u|}{\lambda}\right)dx \tag{2.20}$$

for all balls B with $\sigma B \subset \Omega$ and any constant $\lambda > 0$. From (2.1) and (2.20), we have the following inequality with the Luxemburg norm

$$\big\|Td\big(T\big(P(u)\big)\big)\big\|_{L^\varphi(B)} \le C_{18}|B|\operatorname{diam}(B)\|u\|_{L^\varphi(\sigma B).} \tag{2.21}$$

The proof of Theorem 2.4 has been completed. □

Theorem 2.5 *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, let Ω be a bounded and convex domain, let T be the homotopy operator defined in* (1.6), *and let P be the potential operator defined in* (1.13) *with the kernel $K(x,y)$ satisfying condition* (i) *of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$, and u is a solution of the nonhomogeneous A-harmonic equation* (1.1) *in Ω. Then there exists a constant C, independent of u, such that*

$$\big\|\nabla Td\big(T\big(P(u)\big)\big)\big\|_{L^\varphi(B)} \le C|B|\|u\|_{L^\varphi(\sigma B)} \tag{2.22}$$

for all balls B with $\sigma B \subset \Omega$.

Proof Replacing u by $d(T(P(u)))$ in the first inequality in (1.10), we find that

$$\big\|\nabla Td\big(T\big(P(u)\big)\big)\big\|_{q,B} \le C_1|B|\big\|d\big(T\big(P(u)\big)\big)\big\|_{q,B} \tag{2.23}$$

holds for any differential form u and $q > 1$. From (2.14), we have

$$\big\|d\big(T\big(P(u)\big)\big)\big\|_{q,B} =\le C_2|B|\|u\|_{q,B}. \tag{2.24}$$

Combining (2.23) and (2.24) yields

$$\big\|\nabla Td\big(T\big(P(u)\big)\big)\big\|_{q,B} \le C_3|B|^2\|u\|_{q,B} \tag{2.25}$$

for all balls B with $B \subset \Omega$. Starting with (2.25) and using the similar method as we did in the proof of Theorem 2.4, we can obtain

$$\big\|\nabla Td\big(T\big(P(u)\big)\big)\big\|_{L^\varphi(B)} \le C_4|B|\|u\|_{L^\varphi(\sigma B)}. \tag{2.26}$$

The proof of Theorem 2.5 has been completed. □

Now, we are ready to present and prove the main local theorem, the L^φ-imbedding theorem, as follows.

Theorem 2.6 *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, let Ω be a bounded and convex domain, let T be the homotopy operator defined in* (1.6), *and let P be the potential operator defined in* (1.13) *with the kernel $K(x,y)$ satisfying condition* (i) *of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$, and u is a solution of the non-homogeneous A-harmonic equation* (1.1) *in Ω. Then there exists a constant C, independent of u, such that*

$$\left\| T\big(P(u)\big) - \big(T\big(P(u)\big)\big)_B \right\|_{W^{1,\varphi}(B,\wedge^l)} \le C|B| \|u\|_{L^\varphi(\sigma B)} \tag{2.27}$$

for all balls B with $\sigma B \subset \Omega$.

Proof From (2.1), (2.12) and (2.22), we have

$$\begin{aligned}
&\left\| T\big(P(u)\big) - \big(T\big(P(u)\big)\big)_B \right\|_{W^{1,\varphi}(B,\wedge^l)} \\
&\quad = \left\| Td\big(T\big(P(u)\big)\big) \right\|_{W^{1,\varphi}(B,\wedge^l)} \\
&\quad = \big(\mathrm{diam}(B)\big)^{-1} \left\| Td\big(T\big(P(u)\big)\big) \right\|_{L^\varphi(B)} + \left\| \nabla Td\big(T\big(P(u)\big)\big) \right\|_{L^\varphi(B)} \\
&\quad \le \big(\mathrm{diam}(B)\big)^{-1} \big(C_1|B| \,\mathrm{diam}(B) \|u\|_{L^\varphi(\sigma_1 B)}\big) + C_2|B| \|u\|_{L^\varphi(\sigma_2 B)} \\
&\quad \le C_1|B| \|u\|_{L^\varphi(\sigma_1 B)} + C_2|B| \|u\|_{L^\varphi(\sigma_2 B)} \\
&\quad \le C_3|B| \|u\|_{L^\varphi(\sigma B)}
\end{aligned} \tag{2.28}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma = \max\{\sigma_1, \sigma_2\}$. The proof of Theorem 2.6 has been completed. □

The following version of local imbedding will be used in Section 3 to establish a global imbedding theorem which indicates that the operator $T \circ P$ is bounded.

Theorem 2.7 *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, let Ω be a bounded and convex domain, let T be the homotopy operator defined in* (1.6), *and let P be the potential operator defined in* (1.13) *with the kernel $K(x,y)$ satisfying condition* (i) *of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$, and u is a solution of the non-homogeneous A-harmonic equation* (1.1) *in Ω. Then there exists a constant C, independent of u, such that*

$$\left\| TP(u) \right\|_{W^{1,\varphi}(B,\wedge^l)} \le C|B| \|u\|_{L^\varphi(\sigma B)} \tag{2.29}$$

for all balls B with $\sigma B \subset \Omega$.

Proof Applying (1.10) to $P(u)$, then using (1.15), we find that

$$\left\| TP(u) \right\|_{q,B} \le C_1|B| \,\mathrm{diam}(B) \left\| P(u) \right\|_{q,B} \le C_2|B| \,\mathrm{diam}(B) \|u\|_{q,B} \tag{2.30}$$

and

$$\left\| \nabla TP(u) \right\|_{q,B} \le C_3|B| \,\mathrm{diam}(B) \left\| P(u) \right\|_{q,B} \le C_4|B| \,\mathrm{diam}(B) \|u\|_{q,B} \tag{2.31}$$

for any differential form u and all balls B with $B \subset \Omega$, where $q > 1$ is a constant. Starting with (2.30) and (2.31) and using the similar method developed in the proof of Theorem 2.5, we obtain

$$\|TP(u)\|_{L^\varphi(B)} \le C_5|B|\operatorname{diam}(B)\|u\|_{L^\varphi(\sigma_1 B)} \tag{2.32}$$

and

$$\|\nabla TP(u)\|_{L^\varphi(B)} \le C_6|B|\|u\|_{L^\varphi(\sigma_2 B)}, \tag{2.33}$$

respectively, where σ_1 and σ_2 are constants. From (2.1), (2.32) and (2.33), we have

$$\begin{aligned}
&\|TP(u)\|_{W^{1,\varphi}(B,\wedge^l)} \\
&\quad = (\operatorname{diam}(B))^{-1}\|TP(u)\|_{L^\varphi(B)} + \|\nabla TP(u)\|_{L^\varphi(B)} \\
&\quad = (\operatorname{diam}(B))^{-1}\big(C_5|B|\operatorname{diam}(B)\|u\|_{L^\varphi(\sigma_1 B)}\big) + C_6|B|\|u\|_{L^\varphi(\sigma_2 B)} \\
&\quad \le C_7|B|\|u\|_{L^\varphi(\sigma B)},
\end{aligned} \tag{2.34}$$

where $\sigma = \max\{\sigma_1, \sigma_2\}$. The proof of Theorem 2.7 has been completed. □

Note that if we choose $\varphi(t) = t^p \log_+^\alpha t$ or $\varphi(t) = t^p$ in Theorems 2.3, 2.4, 2.5, 2.6 and 2.7, we will obtain some $L^p(\log_+^\alpha L)$-norm or L^p-norm inequalities, respectively. For example, let $\varphi(t) = t^p \log_+^\alpha t$ in Theorem 2.6, we have the following imbedding inequalities for $T \circ P$ with the $L^p(\log_+^\alpha L)$-norms.

Corollary 2.8 *Let $\varphi(t) = t^p \log_+^\alpha t$, $p \ge 1$ and $\alpha \in \mathbb{R}$. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$, and u is a solution of the non-homogeneous A-harmonic equation* (1.1). *Then there exists a constant C, independent of u such that*

$$\|T(P(u)) - (T(P(u)))_B\|_{W^{1,t^p \log_+^\alpha t}(B,\wedge^l)} \le C|B|\|u\|_{L^p(\log_+^\alpha L)(\sigma B)} \tag{2.35}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant.

Selecting $\varphi(t) = t^p$ in Theorem 2.6, we obtain the usual imbedding inequalities $T \circ P$ with the L^p-norms.

$$\|T(P(u)) - (T(P(u)))_B\|_{W^{1,p}(B,\wedge^l)} \le C|B|\|u\|_{p,\sigma B} \tag{2.36}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant. Similarly, if we choose $\varphi(t) = t^p \log_+^\alpha t$ or $\varphi(t) = t^p$ in Theorems 2.3, 2.4, 2.5 and 2.7, respectively, we will obtain the corresponding special results.

3 Global imbedding theorem

We have established the local L^φ-norm and L^φ-imbedding inequalities for $T \circ P$ and some composite operators related to the imbedding theorem for $T \circ P$. In this section, we prove the global L^φ-imbedding theorem in the following L^φ-averaging domains.

Definition 3.1 [20] Let φ be an increasing convex function on $[0,\infty)$ with $\varphi(0) = 0$. We call a proper subdomain $\Omega \subset \mathbb{R}^n$ an L^φ-averaging domain if $|\Omega| < \infty$, and there exists a constant C such that

$$\int_\Omega \varphi\bigl(\tau|u - u_{B_0}|\bigr)\,dx \le C \sup_{B\subset\Omega} \int_B \varphi\bigl(\sigma|u - u_B|\bigr)\,dx \tag{3.1}$$

for some ball $B_0 \subset \Omega$ and all u such that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$, where τ, σ are constants with $0 < \tau < \infty$, $0 < \sigma < \infty$ and the supremum is over all balls $B \subset \Omega$.

From the definition above, we see that L^s-averaging domains are special L^φ-averaging domains when $\varphi(t) = t^s$ in Definition 3.1. Also, uniform domains and the John domains are very special L^φ-averaging domains, see [1] and [20] for more results about the averaging domains.

Lemma 3.2 [19] (Covering lemma) *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that $\bigcup_i Q_i = \Omega$, $\sum_{Q_i\in\mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q} \le N\chi_\Omega$ and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube need not be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$, which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \dots, Q_k = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_i$, $i = 0,1,2,\dots,k$, for some $\rho = \rho(n,\delta)$.*

Now, we are ready to prove another main theorem, the global imbedding theorem with the L^φ-norm, as follows.

Theorem 3.3 *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, let $C \ge 1$, Ω be any convex bounded L^φ-averaging domain, let T be the homotopy operator defined in* (1.6), *and let P be the potential operator defined in* (1.13) *with the kernel $K(x,y)$ satisfying condition* (i) *of the standard estimates. Assume that $\varphi(|v|) \in L^1(\Omega)$, and $v \in D'(\Omega,\wedge^1)$ is a solution of the non-homogeneous A-harmonic equation* (1.1) *in Ω. Then there exists a constant C, independent of v, such that*

$$\bigl\|T\bigl(P(v)\bigr) - \bigl(T\bigl(P(v)\bigr)\bigr)_{B_0}\bigr\|_{W^{1,\varphi}(\Omega)} \le C\|v\|_{L^\varphi(\Omega)}, \tag{3.2}$$

where $B_0 \subset \Omega$ is some fixed ball.

Proof Since $v \in \wedge^1$, it follows that $T(P(v)) \in \wedge^0$, and hence $\nabla((TP(v))_{B_0}) = d((TP(v))_{B_0})$. Note that $(TP(v))_{B_0}$ is a closed form, then $d((TP(v))_{B_0}) = 0$. Thus,

$$\bigl\|\nabla\bigl(\bigl(TP(v)\bigr)_{B_0}\bigr)\bigr\|_{L^\varphi(\Omega)} = \bigl\|d\bigl(\bigl(TP(v)\bigr)_{B_0}\bigr)\bigr\|_{L^\varphi(\Omega)} = 0. \tag{3.3}$$

Applying the first inequality in (1.10) to $P(v)$, we have

$$\bigl\|\nabla\bigl(T\bigl(P(v)\bigr)\bigr)\bigr\|_{q,B} \le C_1|B|\bigl\|(P(v)\bigr\|_{q,B} \tag{3.4}$$

for any ball B and $q > 1$. Starting from (3.4), and using the similar method to the proof of Theorem 2.4, we obtain

$$\bigl\|\nabla\bigl(T\bigl(P(v)\bigr)\bigr)\bigr\|_{L^\varphi(B)} \le C_2\|v\|_{L^\varphi(\sigma B)}, \tag{3.5}$$

where $\sigma > 1$ is a constant. From the covering lemma and (3.5), it follows that

$$\begin{aligned}
\big\|\nabla\big(T\big(P(v)\big)\big)\big\|_{L^{\varphi}(\Omega)} &\le \sum_{B\in\mathcal{V}}\big\|\nabla\big(T\big(P(v)\big)\big)\big\|_{L^{\varphi}(B)} \\
&\le \sum_{B\in\mathcal{V}}\big(C_2\|v\|_{L^{\varphi}(\sigma B)}\big) \\
&\le C_2 N\|v\|_{L^{\varphi}(\Omega)}, \\
&\le C_3\|v\|_{L^{\varphi}(\Omega)},
\end{aligned} \tag{3.6}$$

where N is a positive integer appearing in the covering lemma. Letting $u = T(P(v))$ and using (2.11), we find that

$$\begin{aligned}
&\big\|T\big(P(v)\big) - \big(T\big(P(v)\big)\big)_{B_0}\big\|_{L^{\varphi}(\Omega)} \\
&\quad\le C_4\int_{\Omega}\varphi\big(\big|T\big(P(v)\big) - \big(T\big(P(v)\big)\big)_{B_0}\big|\big)\,dx \\
&\quad\le C_5\int_{\Omega}\varphi\big(|u - u_{B_0}|\big)\,dx \\
&\quad\le C_6\sup_{B\subset\Omega}\int_{B}\varphi\big(|u - u_B|\big)\,dx \\
&\quad= C_6\sup_{B\subset\Omega}\int_{B}\varphi\big(\big|T\big(P(v)\big) - T\big(P(v)\big)_B\big|\big)\,dx \\
&\quad\le C_6\sup_{B\subset\Omega}\bigg(C_7\operatorname{diam}(B)\int_{B}\varphi\big(|v|\big)\,dx\bigg) \\
&\quad\le C_6\sup_{B\subset\Omega}\bigg(C_7\operatorname{diam}(\Omega)\int_{\Omega}\varphi\big(|v|\big)\,dx\bigg) \\
&\quad\le C_8\operatorname{diam}(\Omega)\int_{\Omega}\varphi\big(|v|\big)\,dx \\
&\quad\le C_9\operatorname{diam}(\Omega)\|v\|_{L^{\varphi}(\Omega)}.
\end{aligned} \tag{3.7}$$

From (2.1), (3.6) and (3.7), we have

$$\begin{aligned}
&\big\|T\big(P(v)\big) - \big(T\big(P(v)\big)\big)_{B_0}\big\|_{W^{1,\varphi}(\Omega)} \\
&\quad= \big(\operatorname{diam}(\Omega)\big)^{-1}\big\|T\big(P(v)\big) - \big(T\big(P(v)\big)\big)_{B_0}\big\|_{L^{\varphi}(\Omega)} + \big\|\nabla\big(T\big(P(v)\big) - \big(TP(v)\big)_{B_0}\big)\big\|_{L^{\varphi}(\Omega)} \\
&\quad\le \big(\operatorname{diam}(\Omega)\big)^{-1}\big\|T\big(P(v)\big) - \big(T\big(P(v)\big)\big)_{B_0}\big\|_{L^{\varphi}(\Omega)} + \big\|\nabla\big(T\big(P(v)\big)\big)\big\|_{L^{\varphi}(\Omega)} \\
&\qquad+ \big\|\nabla\big(\big(TP(v)\big)_{B_0}\big)\big\|_{L^{\varphi}(\Omega)} \\
&\quad= \big(\operatorname{diam}(\Omega)\big)^{-1}\big\|T\big(P(v)\big) - \big(T\big(P(v)\big)\big)_{B_0}\big\|_{L^{\varphi}(\Omega)} + \big\|\nabla\big(T\big(P(v)\big)\big)\big\|_{L^{\varphi}(\Omega)} \\
&\quad= \big(\operatorname{diam}(\Omega)\big)^{-1}\big(C_9\operatorname{diam}(\Omega)\|v\|_{L^{\varphi}(\Omega)}\big) + C_3\|v\|_{L^{\varphi}(\Omega)} \\
&\quad\le C_{10}\|v\|_{L^{\varphi}(\Omega)}
\end{aligned} \tag{3.8}$$

We have completed the proof of Theorem 3.3. □

It is well known that any John domain is a special L^φ-averaging domain [1]. Hence, we have the following global L^φ-imbedding theorem for the John domains.

Theorem 3.4 *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, let $C \ge 1$, Ω be any convex bounded John domain, let T be the homotopy operator defined in* (1.6)*, and let P be the potential operator defined in* (1.13) *with the kernel $K(x,y)$ satisfying condition* (i) *of the standard estimates. Assume that $\varphi(|v|) \in L^1(\Omega)$, and $v \in D'(\Omega, \wedge^1)$ is a solution of the non-homogeneous A-harmonic equation* (1.1) *in Ω. Then there exists a constant C, independent of v, such that*

$$\left\| T\big(P(v)\big) - \big(T\big(P(v)\big)\big)_{B_0} \right\|_{W^{1,\varphi}(\Omega)} \le C\|v\|_{L^\varphi(\Omega)}, \tag{3.9}$$

where $B_0 \subset \Omega$ is some fixed ball.

Choosing $\varphi(t) = t^p \log_+^\alpha t$ in Theorems 3.3, we obtain the following imbedding inequality with the $L^p(\log_+^\alpha L)$-norms.

Corollary 3.5 *Let $\varphi(t) = t^p \log_+^\alpha t$, $p \ge 1$, $\alpha \in \mathbb{R}$, Ω be any convex bounded L^φ-averaging domain, let T be the homotopy operator defined in* (1.6)*, and let P be the potential operator defined in* (1.13) *with the kernel $K(x,y)$ satisfying condition* (i) *of the standard estimates. Assume that $\varphi(|v|) \in L^1(\Omega)$, and $v \in D'(\Omega, \wedge^1)$ is a solution of the non-homogeneous* A-harmonic *equation* (1.1) *in Ω. Then there exists a constant C, independent of v, such that*

$$\left\| T\big(P(v)\big) - \big(T\big(P(v)\big)\big)_{B_0} \right\|_{W^{1,\varphi}(\Omega)} \le C\|v\|_{L^\varphi(\Omega),} \tag{3.10}$$

where $B_0 \subset \Omega$ is some fixed ball.

Next, let S be the set of all solutions of the non-homogeneous A-harmonic equation in Ω. We have the following version of imbedding theorem with L^φ norm for any bounded domain, which says that the composite operator $T \circ P$ maps $W^{1,\varphi}(\Omega, \wedge^1) \cap S$ continuously into $L^\varphi(\Omega)$.

Theorem 3.6 *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, let T be the homotopy operator defined in* (1.6)*, and let P be the potential operator defined in* (1.13) *with the kernel $K(x,y)$ satisfying condition* (i) *of the standard estimates. Assume that $\varphi(|v|) \in L^1(\Omega)$, and $v \in D'(\Omega, \wedge^1) \cap S$ in Ω. Then the composite operator $T \circ P$ maps $W^{1,\varphi}(\Omega, \wedge^1) \cap S$ continuously into $L^\varphi(\Omega)$. Furthermore, there exists a constant C, independent of v, such that*

$$\left\| TP(v) \right\|_{W^{1,\varphi}(\Omega)} \le C\|v\|_{L^\varphi(\Omega)} \tag{3.11}$$

holds for any bounded domain Ω.

Proof Let $v \in D'(\Omega, \wedge^1)$ be a solution of equation (1.1). Since the composite operator $T \circ P$ is continuous if and only if it is bounded, we only need to prove that (3.11) holds. Using

(2.29) and the Lemma 3.2, we obtain

$$\begin{aligned}\|TP(v)\|_{W^{1,\varphi}(\Omega)} &\le \sum_{B\in\mathcal{V}}\left(\|TP(v)\|_{W^{1,\varphi}(B)}\right)\\ &\le \sum_{B\in\mathcal{V}}\left(C_1|B|\|v\|_{L^{\varphi}(\sigma B)}\right)\\ &\le C_1N|\Omega|\|v\|_{L^{\varphi}(\Omega)}\\ &\le C_2\|v\|_{L^{\varphi}(\Omega)}. \end{aligned} \tag{3.12}$$

Hence, inequality (3.11) holds. We have completed the proof of Theorem 3.6. □

Selecting $\varphi(t) = t^p$ in Theorems 3.3, we have the following version of the imbedding inequality with L^p-norms.

Corollary 3.7 *Let $\varphi(t) = t^p$, $p \ge 1$, let T be the homotopy operator defined in* (1.6) *and P be the potential operator defined in* (1.13). *Assume that $\varphi(|v|) \in L^1(\Omega)$, and $v \in D'(\Omega, \wedge^1)$ is a solution of the non-homogeneous A-harmonic equation* (1.1) *in Ω. Then there exists a constant C, independent of v, such that*

$$\left\|TP(v) - \left(T\left(P(v)\right)\right)_{B_0}\right\|_{W^{1,p}(\Omega)} \le C\|v\|_{p,\Omega}$$

holds for any bounded domain Ω.

4 Examples

In this last section, we will present two examples to show applications of our imbedding theorems. All of our local and global inequalities work for these two examples. We should note that functions are 0-forms. Thus, all of our theorems proved in this paper will work for harmonic functions. For example, choose u to be a function (0-form) in the homogeneous A-harmonic equation (1.5), then (1.5) reduces to the following A-harmonic equation

$$\operatorname{div} A(x, \nabla u) = 0 \tag{4.1}$$

for functions. Assume that $A(x,\xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, the operator $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ satisfies the required conditions (1.2) and the equation (4.1) becomes the usual p-harmonic equation for functions

$$\operatorname{div}\left(\nabla u|\nabla u|^{p-2}\right) = 0, \tag{4.2}$$

which is equivalent to

$$(p-2)\sum_{k=1}^{n}\sum_{i=1}^{n} u_{x_k}u_{x_i}u_{x_kx_i} + |\nabla u|^2\Delta u = 0. \tag{4.3}$$

If we choose $p = 2$ in (4.2), we have the Laplace equation $\Delta u = 0$ for functions. Thus, from Theorem 3.3, we have the following inequality for harmonic functions.

Example 4.1 Let u be a solution of the usual A-harmonic equation (4.1) or the p-harmonic equation (4.2), let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, and let Ω be any bounded L^{φ}-averaging domain. If $\varphi(|u|) \in L^1(\Omega)$, then there exists a constant C, independent of u, such that

$$\big\| T\big(P(u)\big) - \big(T\big(P(u)\big)\big)_{B_0}\big\|_{W^{1,\varphi}(\Omega)} \le C\|u\|_{L^{\varphi}(\Omega)},$$

where $B_0 \subset \Omega$ is some fixed ball.

Example 4.2 Let $u(x,y)$ be a function (0-form) defined in $\mathbb{R}^2$ by

$$u(x,y) = \frac{1}{\pi}\left(\arctan\frac{y}{x-1} - \arctan\frac{y}{x+1}\right).$$

We can check that $u(x,y)$ satisfies the Laplace equation $u_{xx}(x,y) + u_{yy}(x,y) = 0$ in the upper half plane, that is, $u(x,y)$ is a harmonic function in the upper half plane. Let $r > 0$ be a constant, let (x_0,y_0) be a fixed point with $y_0 > r$ and $B = \{(x,y) : (x-x_0)^2 + (y-y_0)^2 \le r^2\}$. To obtain the upper bound for the Orlicz-Sobolev-norm $\|T(P(u)) - (T(P(u)))_B\|_{W^{1,\varphi}(B,\wedge^l)}$ directly, it would be very complicated. However, using Theorem 2.6 with $n = 2$, we can easily obtain the upper bound of the Orlicz-Sobolev-norm $\|T(P(u)) - (T(P(u)))_B\|_{W^{1,\varphi}(B,\wedge^l)}$ as follows. First, we know that $|B| = \pi r^2$ and

$$\begin{aligned}
\big|u(x,y)\big| &\le \frac{1}{\pi}\left|\arctan\frac{y}{x-1} - \arctan\frac{y}{x+1}\right| \\
&\le \frac{1}{\pi}\left|\arctan\frac{y}{x-1}\right| + \left|\arctan\frac{y}{x+1}\right| \\
&\le \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \\
&= 1.
\end{aligned} \tag{4.4}$$

Applying (2.27) and (4.4), we have

$$\begin{aligned}
&\big\| T\big(P(u)\big) - \big(T\big(P(u)\big)\big)_B\big\|_{W^{1,\varphi}(B,\wedge^l)} \\
&\quad\le C_1|B|\|u\|_{L^{\varphi}(\sigma B)} \\
&\quad\le C_2|B|\int_{\sigma B}\varphi\big(|u|\big)\,dx \\
&\quad\le C_2|B|\int_{\sigma B}\varphi(1)\,dx \\
&\quad\le C_2\pi r^2\varphi(1)|\sigma B| \\
&\quad\le C_2\pi r^2\varphi(1)\pi(\sigma r)^2 \\
&\quad\le C_3\varphi(1)r^4.
\end{aligned}$$

Remark

(i) We know that the L^s-averaging domains are the special L^{φ}-averaging domains. Thus, Theorem 3.3 also holds for the L^s-averaging domain;

(ii) Theorem 3.6 holds for any bounded domain in $\mathbb{R}^n$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Author details
[1]Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, P.R. China. [2]Department of Mathematics, Seattle University, Seattle, 98122, USA.

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