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Norm inequalities for composition of the Dirac and Green's operators

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Abstract
We first prove a norm inequality for the composition of the Dirac operator and Green's operator. Then, we estimate for the Lipschitz and BMO norms of the composite operator in terms of the L^s norm of a differential form.
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1 Introduction

The purpose of this paper is to derive the norm inequalities for the composite operator $D \circ G$ of the Hodge-Dirac operator D and Green's operator G on differential forms. Specifically, we will develop the upper bounds for norms of the composite operator $D \circ G$ applied to differential form u in terms of the norm of u. We all know that there are different versions of Dirac operators, such as the Hodge-Dirac operator associated to a Riemannian manifold and the euclidean Dirac operator arising in Clifford analysis. The Dirac operator studied in this paper is the Hodge-Dirac operator defined by $D = d + d^\star$, where d is the exterior derivative, and $d^\star$ is the Hodge codifferential, which is the formal adjoint to d. Both the Dirac operator D and Green's operator G are widely studied and used in mathematics and physics. Since it was initiated by Paul Dirac in order to get a form of quantum theory compatible with special relativity, Dirac operators have been playing an important role in many fields of mathematics and physics, such as quantum mechanics, Clifford analysis and PDEs. Green's operator is a key operator, which has been very well used in several areas of mathematics. In many situations, the process of studying solutions to PDEs involves estimating the various norms of the operators and their compositions. Hence, we are motivated to establish the upper bounds for the composite operators in this paper. See [1–8] for recent work on the Dirac operator, Green's operator and their applications.

Let M be a bounded domain and B be a ball in $\mathbb{R}^n$, $n \geq 2$, throughout this paper. We use σB to express the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$, $\sigma > 0$. We do not distinguish the balls from cubes in this paper. We use $|E|$ to denote the Lebesgue measure of a set $E \subset \mathbb{R}^n$. We call w a weight if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e. Let $e_1, e_2, \ldots, e_n$ be the standard unit basis of $\mathbb{R}^n$, and let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the linear space of l-vectors, which is spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, $l = 0, 1, \ldots, n$. The Grassman algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For any $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is defined by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$, with summation over all l-tuples $I = (i_1, i_2, \dots, i_l)$ and all integers $l = 0, 1, \dots, n$. The Hodge star operator $\star: \wedge \to \wedge$ is defined by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle(\star 1)$ for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbb{R}$. The Hodge star is an isometric isomorphism on Λ with $\star : \wedge^l \to \wedge^{n-l}$ and $\star\star\,(-1)^{l(n-l)} : \wedge^l \to \wedge^l$.

A differential l-form ω on M is a de Rham current (see [9, Chapter III]) on M with values in $\wedge^l(\mathbb{R}^n)$. Differential forms are extensions of functions. For example, in $\mathbb{R}^n$, the function $u(x_1, x_2, \dots, x_n)$ is called a 0-form. Moreover, if $u(x_1, x_2, \dots, x_n)$ is differentiable, then it is called a differential 0-form. The 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \dots, x_n)\,dx_i$. If the coefficient functions $u_i(x_1, x_2, \dots, x_n)$, $i = 1, 2, \dots, n$, are differentiable, then $u(x)$ is called a differential 1-form. Similarly, a differential k-form $u(x)$ is generated by $\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\}$, $k = 1, 2, \dots, n$, that is, $u(x) = \sum_I \omega_I(x)\,dx_I = \sum \omega_{i_1 i_2 \cdots i_k}(x)\,dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where $I = (i_1, i_2, \dots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let $D'(M, \wedge^l)$ be the space of all differential l-forms on M, and let $L^p(M, \wedge^l)$ be the l-forms $\omega(x) = \sum_I \omega_I(x)\,dx_I$ on M satisfying $\int_M |\omega_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \dots, n$. We denote the exterior derivative by $d : D'(M, \wedge^l) \to D'(M, \wedge^{l+1})$ for $l = 0, 1, \dots, n-1$. The Hodge codifferential operator $d^\star : D'(M, \wedge^{l+1}) \to D'(M, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(M, \wedge^{l+1})$, $l = 0, 1, \dots, n-1$. The Dirac operator D involved in this paper is defined by $D = d + d^\star$. It is easy to check that $D^2 = \Delta$, where $\Delta = dd^\star + d^\star d$ is the Laplace-Beltrami operator. Let $\wedge^l M$ be the lth exterior power of the cotangent bundle, $C^\infty(\wedge^l M)$ be the space of smooth l-forms on M and $\mathcal{W}(\wedge^l M) = \{u \in L^1_{\mathrm{loc}}(\wedge^l M) : u \text{ has generalized gradient}\}$. The harmonic l-fields are defined by $\mathcal{H}(\wedge^l M) = \{u \in \mathcal{W}(\wedge^l M) : du = d^\star u = 0, u \in L^p \text{ for some } 1 < p < \infty\}$. The orthogonal complement of $\mathcal{H}$ in L^1 is defined by $\mathcal{H}^\perp = \{u \in L^1 : \langle u, h \rangle = 0 \text{ for all } h \in \mathcal{H}\}$. Then the Green's operator G is defined as $G : C^\infty(\wedge^l M) \to \mathcal{H}^\perp \cap C^\infty(\wedge^l M)$ by assigning $G(u)$ to be the unique element of $\mathcal{H}^\perp \cap C^\infty(\wedge^l M)$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^\infty(\wedge^l M)$ onto $\mathcal{H}$ so that $H(u)$ is the harmonic part of u. See [10] for more properties of these operators. We write $\|u\|_{s,M} = (\int_M |u|^s)^{1/s}$ and $\|u\|_{s,M,w} = (\int_M |u|^s w(x)\,dx)^{1/s}$, where $w(x)$ is a weight.

Let $\omega \in L^1_{\mathrm{loc}}(M, \wedge^l)$, $l = 0, 1, \dots, n$. We write $\omega \in \mathrm{locLip}_k(M, \wedge^l)$, $0 \le k \le 1$, if

$$\|\omega\|_{\mathrm{locLip}_k, M} = \sup_{\sigma Q \subset M} |Q|^{-(n+k)/n} \|\omega - \omega_Q\|_{1,Q} < \infty \tag{1.1}$$

for some $\sigma \ge 1$. Further, we write $\mathrm{Lip}_k(M, \wedge^l)$ for those forms, whose coefficients are in the usual Lipschitz space with exponent k and write $\|\omega\|_{\mathrm{Lip}_k, M}$ for this norm. Similarly, for $\omega \in L^1_{\mathrm{loc}}(M, \wedge^l)$, $l = 0, 1, \dots, n$, we write $\omega \in \mathrm{BMO}(M, \wedge^l)$ if

$$\|\omega\|_{\star, M} = \sup_{\sigma Q \subset M} |Q|^{-1} \|\omega - \omega_Q\|_{1,Q} < \infty \tag{1.2}$$

for some $\sigma \ge 1$. When ω is a 0-form, (1.2) reduces to the classical definition of $\mathrm{BMO}(M)$. The definitions of Lipschitz and BMO norms above appeared in [11].

2 L^s norm inequalities

In this section, we will develop Poincaré-type inequality with L^s norm for the composite operator $D \circ G$. This inequality will be used to prove other results in this paper. Using the same way in the proof of Propositions 5.15 and 5.17 in [12], we can prove that for any closed ball $\overline{B} = B \cup \partial B$, we have

$$\|dd^*G(u)\|_{s,\overline{B}} + \|d^*dG(u)\|_{s,\overline{B}} + \|dG(u)\|_{s,\overline{B}} + \|d^*G(u)\|_{s,\overline{B}} + \|G(u)\|_{s,\overline{B}} \le C(s)\|u\|_{s,\overline{B}}.$$

Note that for any Lebesgue measurable function f defined on a Lebesgue measurable set E with $|E| = 0$, we have $\int_E f\,dx = 0$. Thus, $\|u\|_{s,\partial B} = 0$ and $\|dd^*G(u)\|_{s,\partial B} + \|d^*dG(u)\|_{s,\partial B} + \|dG(u)\|_{s,\partial B} + \|d^*G(u)\|_{s,\partial B} + \|G(u)\|_{s,\partial B} = 0$ since $|\partial B| = 0$. Therefore, we obtain

$$\begin{aligned}
&\|dd^*G(u)\|_{s,B} + \|d^*dG(u)\|_{s,B} + \|dG(u)\|_{s,B} + \|d^*G(u)\|_{s,MB} + \|G(u)\|_{s,B} \\
&\quad = \|dd^*G(u)\|_{s,\overline{B}} + \|d^*dG(u)\|_{s,\overline{B}} + \|dG(u)\|_{s,\overline{B}} + \|d^*G(u)\|_{s,\overline{B}} + \|G(u)\|_{s,\overline{B}} \\
&\quad \le C(s)\|u\|_{s,\overline{B}} \\
&\quad = C(s)\|u\|_{s,B}.
\end{aligned}$$

Hence, we have the following lemma.

Lemma 2.1 *Let u be a smooth differential form defined in M and $1 < s < \infty$. Then there exists a positive constant $C = C(s)$, independent of u, such that*

$$\|dd^*G(u)\|_{s,B} + \|d^*dG(u)\|_{s,B} + \|dG(u)\|_{s,B} + \|d^*G(u)\|_{s,B} + \|G(u)\|_{s,B} \le C(s)\|u\|_{s,B}$$

for any ball $B \subset M$.

The following results about the homotopy operator T can be found in [13].

Lemma 2.2 *Let $u \in L^s_{\mathrm{loc}}(D, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form in a bounded and convex domain $D \subset \mathbb{R}^n$, and let T be the homotopy operator defined on differential forms. Then there is also a decomposition $u = Td(u) + dT(u)$ and*

$$\|Tu\|_{s,D} \le C|D| \operatorname{diam}(D)\|u\|_{s,D}.$$

Using the notation above, we can define the l-form $\omega_D \in D'(D, \wedge^l)$ by

$$\omega_D = |D|^{-1}\int_D \omega(y)\,dy, \quad l = 0, \quad \text{and} \quad \omega_D = d(T\omega), \quad l = 1, 2, \ldots, n$$

for all $\omega \in L^p(D, \wedge^l)$, $1 \le p < \infty$.

We will use the following generalized Hölder's inequality repeatedly in this paper.

Lemma 2.3 *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbb{R}^n$, then*

$$\|fg\|_{s,E} \le \|f\|_{\alpha,E} \cdot \|g\|_{\beta,E}$$

for any $E \subset \mathbb{R}^n$.

We now prove the following L^s norm inequality for the composite operator $D \circ G$ of the Dirac operator D and Green's operator G applied to differential forms.

Lemma 2.4 *Let $u \in L^s_{\mathrm{loc}}(M, \wedge^l)$, $l = 0, 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form in a domain M, D be the Dirac operator and G be Green's operator. Then there exists a constant C, independent of u, such that*

$$\|D(G(u))\|_{s,B} \le C\|u\|_{s,B} \tag{2.1}$$

for all balls $B \subset M$.

Proof Since the Dirac operator D can be expressed as $D = d + d^\star$, using Lemma 2.1, we have

$$\begin{aligned} \|D(G(u))\|_{s,B} &= \|(d + d^\star)G(u)\|_{s,B} \\ &\le \|(dG(u)\|_{s,B} + \|d^\star G(u)\|_{s,B} \\ &\le C\|u\|_{s,B}. \end{aligned} \tag{2.2}$$

We have completed the proof of Lemma 2.4.

Next, we prove the Poincaré-type inequality for the composition of the Dirac operator and Green's operator, which forms the foundation of this paper. □

Theorem 2.5 *Let $u \in L^s_{\mathrm{loc}}(M, \wedge^l)$, $l = 0, 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form in a domain M, D be the Dirac operator and G be Green's operator. Then there exists a constant C, independent of u, such that*

$$\|D(G(u)) - (D(G(u)))_B\|_{s,B} \le C|B| \operatorname{diam}(B)\|u\|_{s,B} \tag{2.3}$$

for all balls $B \subset M$.

Proof Applying the decomposition of differential forms described in Lemma 2.2 to the form $D(G(u))$ yields

$$D(G(u)) = d(T(D(G(u)))) + T(d(D(G(u)))) = (D(G(u)))_B + T(d(D(G(u)))), \tag{2.4}$$

where T is the homotopy operator appearing in Lemma 2.2. From Lemma 2.2, for any differential form v, we have

$$\|T(v)\|_{s,B} \le C_1|B| \operatorname{diam}(B)\|v\|_{s,B}, \tag{2.5}$$

where C_1 is a constant independent of v. Replacing v by $d(D(G(u)))$ in (2.5) yields

$$\|T(d(D(G(u))))\|_{s,B} \le C_2|B| \operatorname{diam}(B)\|d(D(G(u)))\|_{s,B}. \tag{2.6}$$

Noticing that $(D(G(u)))_B = d(T(D(G(u))))$ and using (2.6) and Lemma 2.1, we obtain

$$\begin{aligned} \|D(G(u)) - (D(G(u)))_B\|_{s,B} &= \|T(d(D(G(u))))\|_{s,B} \\ &\le C_2|B| \operatorname{diam}(B)\|d(D(G(u)))\|_{s,B} \end{aligned}$$

$$\begin{aligned}
&\leq C_2|B|\operatorname{diam}(B)\big\|d\big((d+d^\star)G(u)\big)\big\|_{s,B} \\
&\leq C_2|B|\operatorname{diam}(B)\big\|dd^\star G(u)\big\|_{s,B} \\
&\leq C_3|B|\operatorname{diam}(B)\|u\|_{s,B},
\end{aligned} \tag{2.7}$$

that is,

$$\big\|D\big(G(u)\big)-\big(D\big(G(u)\big)\big)_B\big\|_{s,B} \leq C_3|B|\operatorname{diam}(B)\|u\|_{s,B}.$$

We have completed the proof of Theorem 2.5. □

3 Upper bounds for Lipschitz and BMO norms

In this section, we establish the upper bounds for Lipschitz norms and BMO norms in terms of L^s norms. Using Theorem 2.5, we now obtain the upper bounds for Lipschitz norm of the composite operator $D \circ G$.

Theorem 3.1 *Let $u \in L^s_{\mathrm{loc}}(M, \wedge^l)$, $l = 0,1,2,\ldots,n$, $1 < s < \infty$, be a differential form in a domain M, D be the Dirac operator and G be Green's operator. Then there exists a constant C, independent of u, such that*

$$\big\|D\big(G(u)\big)\big\|_{\mathrm{locLip}_k,M} \leq C\|u\|_{s,M}, \tag{3.1}$$

where k is a constant with $0 \leq k \leq 1$.

Proof From Theorem 2.5, we find that

$$\big\|D\big(G(u)\big)-\big(D\big(G(u)\big)\big)_B\big\|_{s,B} \leq C_1|B|\operatorname{diam}(B)\|u\|_{s,B} \tag{3.2}$$

for all balls $B \subset M$. Using the Hölder inequality with $1 = 1/s + (s-1)/s$, we find that

$$\begin{aligned}
&\big\|D\big(G(u)\big)-\big(D\big(G(u)\big)\big)_B\big\|_{1,B} \\
&\quad = \int_B \big|D\big(G(u)\big)-\big(D\big(G(u)\big)\big)_B\big|\,dx \\
&\quad \leq \left(\int_B \big|D\big(G(u)\big)-\big(D\big(G(u)\big)\big)_B\big|^s\,dx\right)^{1/s}\left(\int_B 1^{s/(s-1)}\,dx\right)^{(s-1)/s} \\
&\quad = |B|^{(s-1)/s}\big\|D\big(G(u)\big)-\big(D\big(G(u)\big)\big)_B\big\|_{s,B} \\
&\quad = |B|^{1-1/s}\big\|D\big(G(u)\big)-\big(D\big(G(u)\big)\big)_B\big\|_{s,B} \\
&\quad \leq |B|^{1-1/s}\big(C_1|B|\operatorname{diam}(B)\|u\|_{s,B}\big) \leq C_2|B|^{2-1/s+1/n}\|u\|_{s,B}.
\end{aligned} \tag{3.3}$$

Hence, using the definition of the Lipschitz norm, (3.3), and $2 - 1/s + 1/n - 1 - k/n = 1 - 1/s + 1/n - k/n > 0$, we have

$$\begin{aligned}
\big\|D\big(G(u)\big)\big\|_{\mathrm{locLip}_k,M} &= \sup_{\sigma B\subset M} |B|^{-(n+k)/n}\big\|D\big(G(u)\big)-\big(D\big(G(u)\big)\big)_B\big\|_{1,B} \\
&= \sup_{\sigma B\subset M} |B|^{-1-k/n}\big\|D\big(G(u)\big)-\big(D\big(G(u)\big)\big)_B\big\|_{1,B}
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{\sigma B\subset M} |B|^{-1-k/n} C_2 |B|^{2-1/s+1/n}\|u\|_{s,B} \\
&= \sup_{\sigma B\subset M} C_2 |B|^{1-1/s+1/n-k/n}\|u\|_{s,B} \\
&\le \sup_{\sigma B\subset M} C_2 |M|^{1-1/s+1/n-k/n}\|u\|_{s,B} \\
&\le C_3 \sup_{\sigma B\subset M} \|u\|_{s,B} \\
&\le C_3 \|u\|_{s,M}.
\end{aligned}
\tag{3.4}
$$

The proof of Theorem 3.1 has been completed.

We have proved an estimate for the Lipschitz norm $\|\cdot\|_{\mathrm{locLip}_k,M}$ in Theorem 3.1. Now, we develop the estimates for the BMO norm $\|\cdot\|_{\star,M}$. Let $u \in \mathrm{locLip}_k(M,\wedge^l)$, $l = 0,1,\dots,n$, $0 \le k \le 1$ and M be a bounded domain. Then from the definitions of the Lipschitz and BMO norms, we know that

$$
\begin{aligned}
\|u\|_{\star,M} &= \sup_{\sigma B\subset M} |B|^{-1}\|u-u_B\|_{1,B} \\
&= \sup_{\sigma B\subset M} |B|^{k/n}|B|^{-(n+k)/n}\|u-u_B\|_{1,B} \\
&\le \sup_{\sigma B\subset M} |M|^{k/n}|B|^{-(n+k)/n}\|u-u_B\|_{1,B} \\
&\le |M|^{k/n} \sup_{\sigma B\subset M} |B|^{-(n+k)/n}\|u-u_B\|_{1,B} \\
&\le C_1 \sup_{\sigma B\subset M} |B|^{-(n+k)/n}\|u-u_B\|_{1,B} \\
&\le C_1 \|u\|_{\mathrm{locLip}_k,M},
\end{aligned}
$$

where C_1 is a positive constant. Hence, we have the following inequality between the Lipschitz norm and the BMO norm. □

Lemma 3.2 *If a differential form $u \in \mathrm{locLip}_k(M,\wedge^l)$, $l = 0,1,\dots,n$, $0 \le k \le 1$, in a bounded domain M, then $u \in \mathrm{BMO}(M,\wedge^l)$ and*

$$\|u\|_{\star,M} \le C\|u\|_{\mathrm{locLip}_k,M}, \tag{3.5}$$

where C is a constant.

Combining Theorems 3.1 and Lemma 3.2, we obtain the following inequality between the BMO norm and the L^s norm.

Theorem 3.3 *Let $u \in L^s(M,\wedge^l)$, $l = 0,1,2,\dots,n$, $1 < s < \infty$, be a differential form in a domain M, D be the Dirac operator and G be Green's operator. Then there exists a constant C, independent of u, such that*

$$\big\|D\big(G(u)\big)\big\|_{\star,M} \le C\|u\|_{s,M}. \tag{3.6}$$

Proof Since inequality (3.5) holds for any differential form, we may replace u by $D(G(u))$ in inequality (3.5) and obtain

$$\|D(G(u))\|_{\star,M} \le C_1 \|D(G(u))\|_{\mathrm{locLip}_k,M}, \tag{3.7}$$

where k is a constant with $0 \le k \le 1$. On the other hand, from Theorem 3.1, we have

$$\|D(G(u))\|_{\mathrm{locLip}_k,M} \le C_2 \|u\|_{s,M}. \tag{3.8}$$

Combining (3.7) and (3.8) gives $\|D(G(u))\|_{\star,M} \le C_3 \|u\|_{s,M}$. The proof of Theorem 3.3 has been completed. □

We will need the following lemma that appeared in [14].

Lemma 3.4 *Let φ be a strictly increasing convex function on $[0,\infty)$ with $\varphi(0) = 0$, and let E be a bounded domain in $\mathbb{R}^n$. Assume that u is a smooth differential form in E such that $\varphi(k(|u| + |u_E|)) \in L^1(E;\mu)$ for any real number $k > 0$ and $\mu(\{x \in E : |u - u_E| > 0\}) > 0$, where μ is a Radon measure defined by $d\mu = w(x)\,dx$ for a weight $w(x)$. Then for any positive constant a, we have*

$$\int_E \varphi(a|u|)\,d\mu \le C \int_E \varphi(2a|u - u_E|)\,d\mu,$$

where C is a positive constant.

The following *WRH*-class of differential forms was introduced in [15].

Definition 3.5 We say a differential form $u \in \wedge^l(E)$ belongs to the $WRH(\wedge^l, E)$-class and write $u \in WRH(\wedge^l, E)$, $l = 0, 1, 2, \ldots, n$, if for any constants $0 < s, t < \infty$, the inequality

$$\|u\|_{s,B} \le C|B|^{(t-s)/st} \|u\|_{t,\sigma B} \tag{3.9}$$

holds for any ball B with $\sigma B \subset E$, where $\sigma > 1$ and $C > 0$ are constants.

It is well known that any solutions of A-harmonic equations belong to *WRH*-class, see [16–20] for example. Hence, the *WRH*-class is a large set of differential forms.

Theorem 3.6 *Let $u \in L^s_{\mathrm{loc}}(M, \wedge^1)$, $l = 0, 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form such that $u - u_B \in WRH(\wedge^l, M)$-class and the Lebesgue measure $|\{x \in B : |u - u_B| > 0\}| > 0$ for any ball $B \subset M$. Assume that D is the Dirac operator, and G is Green's operator. Then there exists a constant C, independent of u, such that*

$$\|D(G(u))\|_{\mathrm{locLip}_k,M} \le C\|u\|_{\star,M}, \tag{3.10}$$

where k is a constant with $0 \le k \le 1$.

Proof Using Lemma 3.4 with $\varphi(t) = t^s$, $w(x) = 1$ over the ball B, we have

$$\|u\|_{s,B} \le C_1 \|u - u_B\|_{s,B}. \tag{3.11}$$

From Theorem 2.5 and (3.11), we obtain

$$\begin{aligned}\|D(G(u)) - (D(G(u)))_B\|_{s,B} &\le C_2|B|\operatorname{diam}(B)\|u\|_{s,B} \\ &\le C_3|B|\operatorname{diam}(B)\|u-u_B\|_{s,B}.\end{aligned} \tag{3.12}$$

From the definition of the Lipschitz norm, the Hölder inequality with $1 = 1/s + (s-1)/s$ and (3.12), for any ball B with $B \subset M$, we find that

$$\begin{aligned}\|D(G(u)) - (D(G(u)))_B\|_{1,B} &= \int_B |D(G(u)) - (D(G(u)))_B|\,dx \\ &\le \left(\int_B |D(G(u)) - (D(G(u)))_B|^s\,dx\right)^{1/s}\left(\int_B 1^{\frac{s}{s-1}}\,dx\right)^{(s-1)/s} \\ &= |B|^{(s-1)/s}\|D(G(u)) - (D(G(u)))_B\|_{s,B} \\ &= |B|^{1-1/s}\|D(G(u)) - (D(G(u)))_B\|_{s,B} \\ &\le C_4|B|^{2-1/s+1/n}\|u-u_B\|_{s,B}.\end{aligned} \tag{3.13}$$

Next, since $u - u_B \in WRH(\wedge^l, M)$-class, we have

$$\|u-u_B\|_{s,B} \le C_5|B|^{(1-s)/s}\|u-u_B\|_{1,\sigma_1 B} \tag{3.14}$$

for some constant $\sigma_1 > 1$. Combination of (3.13) and (3.14) gives

$$\begin{aligned}\|D(G(u)) - (D(G(u)))_B\|_{1,B} &\le C_4|B|^{2-1/s+1/n}\|u-u_B\|_{s,B} \\ &\le C_6|B|^{1+1/n}\|u-u_B\|_{1,\sigma_1 B}.\end{aligned} \tag{3.15}$$

Hence, we obtain

$$\begin{aligned}|B|^{-(n+k)/n}\|D(G(u)) - (D(G(u)))_B\|_{1,B} &\le C_6|B|^{1/n-k/n}\|u-u_B\|_{1,\sigma_1 B} \\ &= C_6|B|^{1+1/n-k/n}|B|^{-1}\|u-u_B\|_{1,\sigma_1 B} \\ &\le C_7|B|^{1+1/n-k/n}|\sigma_1 B|^{-1}\|u-u_B\|_{1,\sigma_1 B} \\ &\le C_7|M|^{1+1/n-k/n}|\sigma_1 B|^{-1}\|u-u_B\|_{1,\sigma_1 B} \\ &\le C_8|\sigma_1 B|^{-1}\|u-u_B\|_{1,\sigma_1 B}.\end{aligned} \tag{3.16}$$

Thus, taking the supremum on both sides of (3.16) over all balls $\sigma_2 B \subset M$ with $\sigma_2 > \sigma_1$ and using the definitions of the Lipschitz and BMO norms, we find that

$$\begin{aligned}\|D(G(u))\|_{\mathrm{locLip}_k,M} &= \sup_{\sigma_2 B\subset M}|B|^{-(n+k)/n}\|D(G(u)) - (D(G(u)))_B\|_{1,B} \\ &\le C_7\sup_{\sigma_2 B\subset M}|\sigma_1 B|^{-1}\|u-u_B\|_{1,\sigma_1 B} \\ &\le C_7\|u\|_{\star,M},\end{aligned} \tag{3.17}$$

that is,

$$\|D(G(u))\|_{\text{locLip}_k,M} \le C\|u\|_{\star,M}. \tag{3.18}$$

The proof of Theorem 3.6 has been completed. □

Replacing u by $D(G(u))$ in Lemma 3.2, we obtain the following comparison inequality between the Lipschitz norm and the BMO norm.

Corollary 3.7 *Let $u \in L^s(M, \wedge^l)$, $l = 0, 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form in a domain M, D be the Dirac operator and G be Green's operator. Then there exists a constant C, independent of u, such that*

$$\|D(G(u))\|_{\star,M} \le C\|D(G(u))\|_{\text{locLip}_k,M}. \tag{3.19}$$

4 Weighted inequalities

In this section, we establish the weighted norm comparison inequalities for the composition of the Dirac operator and Green's operator applied to differential form defined in a domain $M \subset \mathbb{R}^n$. For $\omega \in L^1_{\text{loc}}(M, \wedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we write $\omega \in \text{locLip}_k(M, \wedge^l, w^\alpha)$, $0 \le k \le 1$ if

$$\|\omega\|_{\text{locLip}_k,M,w^\alpha} = \sup_{\sigma Q \subset M} \left(\mu(Q)\right)^{-(n+k)/n} \|\omega - \omega_Q\|_{1,Q,w^\alpha} < \infty \tag{4.1}$$

for some $\sigma > 1$, where M is a bounded domain, the measure μ is defined by $d\mu = w(x)\,dx$, w is a weight. For convenience, we use the following simple notation $\text{locLip}_k(M, \wedge^l)$ for $\text{locLip}_k(M, \wedge^l, w)$. Similarly, for $\omega \in L^1_{\text{loc}}(M, \wedge^l, w)$, $l = 0, 1, \ldots, n$, we will write $\omega \in \text{BMO}(M, \wedge^l, w)$ if

$$\|\omega\|_{\star,M,w} = \sup_{\sigma Q \subset M} \left(\mu(Q)\right)^{-1} \|\omega - \omega_Q\|_{1,Q,w} < \infty \tag{4.2}$$

for some $\sigma > 1$, where the measure μ is defined by $d\mu = w(x)\,dx$, w is a weight. Again, we write $\text{BMO}(\Omega, \wedge^l)$ to replace $\text{BMO}(M, \wedge^l, w)$ when it is clear that the integral is weighted.

Definition 4.1 We say the weight $w(x)$ satisfies the $A_r(M)$ condition, $r > 1$, write $w \in A_r(M)$, if $w(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty$$

for any ball $B \subset M$.

For $u \in WRH(\wedge^l, M)$, using the Hölder inequality, we extend inequality (2.3) into the following weighted version

$$\|D(G(u)) - (D(G(u)))_B\|_{s,B,w} \le C_3|B|\operatorname{diam}(B)\|u\|_{s,\sigma B,w} \tag{4.3}$$

for all balls B with $\sigma B \subset M$, where $\sigma > 1$ is a constant.

Theorem 4.2 *Let $u \in L^s(M, \wedge^1, \mu)$, $l = 0, 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form in a bounded domain M such that $u \in WRH(\wedge^l, M)$-class. Assume that D is the Dirac operator, and G is Green's operator, where the measure μ is defined by $d\mu = w\,dx$ and $w \in A_r(M)$ for some $r > 1$ with $w(x) \geq \varepsilon > 0$ for any $x \in M$. Then there exists a constant C, independent of u, such that*

$$\|D(G(u))\|_{\mathrm{locLip}_k, M, w} \leq C\|u\|_{s,M,w}, \tag{4.4}$$

where k is a constant with $0 \leq k \leq 1$.

Proof Since $w(x) \geq \varepsilon > 0$, we have

$$\mu(B) = \int_B w\,dx \geq \int_B \varepsilon\,dx = C_1|B|,$$

which gives

$$\frac{1}{\mu(B)} \leq \frac{C_2}{|B|} \tag{4.5}$$

for any ball B. Using (4.3) and the Hölder inequality with $1 = 1/s + (s-1)/s$, we find that

$$\begin{aligned}
&\|D(G(u)) - (D(G(u)))_B\|_{1,B,w} \\
&\quad = \int_B |D(G(u)) - (D(G(u)))_B|\,d\mu \\
&\quad \leq \left(\int_B |D(G(u)) - (D(G(u)))_B|^s\,d\mu\right)^{1/s} \left(\int_B 1^{s/(s-1)}\,d\mu\right)^{(s-1)/s} \\
&\quad = (\mu(B))^{(s-1)/s} \|D(G(u)) - (D(G(u)))_B\|_{s,B,w} \\
&\quad = (\mu(B))^{1-1/s} \|D(G(u)) - (D(G(u)))_B\|_{s,B,w} \\
&\quad \leq (\mu(B))^{1-1/s} \left(C_3|B|\operatorname{diam}(B)\|u\|_{s,\sigma B,w}\right) \\
&\quad \leq C_4(\mu(B))^{1-1/s}|B|^{1+1/n}\|u\|_{s,\sigma B,w}.
\end{aligned} \tag{4.6}$$

From the definition of the weighted Lipschitz norm, (4.5) and (4.6), we obtain

$$\begin{aligned}
&\|D(G(u))\|_{\mathrm{locLip}_k, M, w} \\
&\quad = \sup_{\sigma B \subset M} (\mu(B))^{-(n+k)/n} \|D(G(u)) - (D(G(u)))_B\|_{1,B,w} \\
&\quad = \sup_{\sigma B \subset M} (\mu(B))^{-1-k/n} \|D(G(u)) - (D(G(u)))_B\|_{1,B,w} \\
&\quad \leq C_5 \sup_{\sigma B \subset M} (\mu(B))^{-1/s-k/n} |B|^{1+1/n} \|u\|_{s,\sigma B,w} \\
&\quad \leq C_6 \sup_{\sigma B \subset M} |B|^{-1/s-k/n+1+1/n} \|u\|_{s,\sigma B,w} \\
&\quad \leq C_6 \sup_{\sigma B \subset M} |M|^{-1/s-k/n+1+1/n} \|u\|_{s,\sigma B,w}
\end{aligned}$$

$$\begin{aligned}&\le C_6|M|^{-1/s-k/n+1+1/n}\sup_{\sigma B\subset M}\|u\|_{s,\sigma B,w}\\&\le C_7\|u\|_{s,M,w}\end{aligned}\tag{4.7}$$

since $-1/s-k/n+1+1/n>0$ and $|M|<\infty$. We have completed the proof of Theorem 4.2.

Next, we estimate the BMO norm in terms of the L^s norm. Let $u\in \operatorname{locLip}_k(M,\wedge^l)$, $l=0,1,\dots,n$, $0\le k\le 1$, in a bounded domain M. From the definitions of the weighted Lipschitz and the weighted BMO norms, we have

$$\begin{aligned}\|u\|_{\star,M,w}&=\sup_{\sigma B\subset M}\big(\mu(B)\big)^{-1}\|u-u_B\|_{1,B,w}\\&=\sup_{\sigma B\subset M}\big(\mu(B)\big)^{k/n}\big(\mu(B)\big)^{-(n+k)/n}\|u-u_B\|_{1,B,w}\\&\le\sup_{\sigma B\subset M}\big(\mu(M)\big)^{k/n}\big(\mu(B)\big)^{-(n+k)/n}\|u-u_B\|_{1,B,w}\\&\le\big(\mu(M)\big)^{k/n}\sup_{\sigma B\subset M}\big(\mu(B)\big)^{-(n+k)/n}\|u-u_B\|_{1,B,w}\\&\le C_1\sup_{\sigma B\subset M}\big(\mu(B)\big)^{-(n+k)/n}\|u-u_B\|_{1,B,w}\\&\le C_1\|u\|_{\operatorname{locLip}_k,M,w},\end{aligned}\tag{4.8}$$

where C_1 is a positive constant. Thus, we have obtained the following result. □

Theorem 4.3 *Let $u\in\operatorname{locLip}_k(M,\wedge^l,\mu)$, $l=0,1,\dots,n$, $0\le k\le 1$, be any differential form in a bounded domain M, where the measure μ is defined by $d\mu=w\,dx$ and $w\in A_r(M)$ for some $r>1$. Then $u\in\operatorname{BMO}(\Omega,\wedge^l,w)$ and*

$$\|u\|_{\star,\Omega,w}\le C\|u\|_{\operatorname{locLip}_k,\Omega,w},\tag{4.9}$$

where C is a constant.

Theorem 4.4 *Let $u\in L^s(M,\wedge^1,\mu)$, $l=0,1,2,\dots,n$, $1<s<\infty$, be a differential form in a bounded domain M such that $u\in WRH(\wedge^l,M)$-class. Assume that D is the Dirac operator, and G is Green's operator, where the measure μ is defined by $d\mu=w\,dx$ and $w\in A_r(M)$ for some $r>1$ with $w(x)\ge\varepsilon>0$ for any $x\in M$. Then there exists a constant C, independent of u, such that*

$$\big\|D\big(G(u)\big)\big\|_{\star,M,w}\le C\|u\|_{s,M,w}\tag{4.10}$$

holds for any bounded domain M.

Proof Replacing u by $D(G(u))$ in Theorem 4.4, we have

$$\big\|D\big(G(u)\big)\big\|_{\star,M,w}\le C_1\big\|D\big(G(u)\big)\big\|_{\operatorname{locLip}_k,M,w},\tag{4.11}$$

where k is a constant with $0\le k\le 1$. Using Theorem 4.3, we obtain

$$\big\|D\big(G(u)\big)\big\|_{\operatorname{locLip}_k,M,w}\le C_2\|u\|_{s,M,w}.\tag{4.12}$$

Substituting (4.12) into (4.11), we obtain

$$\left\|D\big(G(u)\big)\right\|_{\star,M,w} \le C_3\|u\|_{s,M,w}.$$

This ends the proof of Theorem 4.4. □

5 Applications

As applications of the results proved in this paper, we consider the following examples.

Example 5.1 Let u be a differential 1-form in $\mathbb{R}^3 - \{(0,0,0)\}$ defined by

$$u(x,y,z) = \frac{x\,dx}{\sqrt{x^2+y^2+z^2}} + \frac{y\,dy}{\sqrt{x^2+y^2+z^2}} + \frac{z\,dz}{\sqrt{x^2+y^2+z^2}}, \tag{5.1}$$

which can be considered as a vector field in $\mathbb{R}^3$. Let $B \subset \mathbb{R}^3$ be a ball with radius r such that $(0,0,0) \notin \overline{B}$. It is easy to see that

$$\|u\|_{s,B} = \left(\int_B |u|^s\,dx\wedge dy\wedge dz\right)^{1/s} = \left(\int_B 1\,dx\wedge dy\wedge dz\right)^{1/s} = |B|^{1/s}.$$

Using Theorem 2.5, we obtain an upper bound $C|B|^{1+1/s}\operatorname{diam}(B)$ for the complicated operator norm $\|D(G(u)) - (D(G(u)))_B\|_{s,B}$. Specifically, we have

$$\left\|D\big(G(u)\big) - \big(D\big(G(u)\big)\big)_B\right\|_{s,B} \le C|B|^{1+1/s}\operatorname{diam}(B) = 2rC\left(\frac{4}{3}\pi r^3\right)^{1+1/s}.$$

Choosing $M = B$ and u be the 1-form discussed in Example 5.1, using Theorem 3.3, we obtain an upper bound for $\|D(G(u))\|_{\star,M}$ as follows

$$\left\|D\big(G(u)\big)\right\|_{\star,M} \le C\left(\frac{4}{3}\pi r^3\right)^{1/s}.$$

In fact, it would be very hard to estimate $\|D(G(u))\|_{\star,M}$ directly from calculation of the operator norm.

Example 5.2 Let $u(x,y,z)$ be a 1-form defined in $\mathbb{R}^3$ by

$$u(x,y,z) = \frac{2}{3\pi}\left(\arctan\frac{y}{x-1}\,dx + \arctan\frac{y}{x+1}\,dy + \arctan\frac{z}{x^2+1}\,dz\right). \tag{5.2}$$

Let $r > 0$ be a constant, (x_0,y_0,z_0) be a fixed point with $x_0 > 2r$, $y_0 > 2r$, $z_0 > 2r$ and

$$M = \left\{(x,y,z) : (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \le r^2\right\}.$$

It would be very complicated for us to obtain the upper bound for the Lipschitz norm $\|D(G(u))\|_{\mathrm{locLip}_k,M}$ if we evaluated

$$\left\|D\big(G(u)\big)\right\|_{\mathrm{locLip}_k,M} = \sup_{\sigma Q\subset M} |Q|^{-(n+k)/n}\left\|D\big(G(u)\big) - \big(D\big(G(u)\big)\big)_Q\right\|_{1,Q}$$

directly. However, using Theorem 3.1, we can easily obtain the upper bound of the norm $\|D(G(u))\|_{\mathrm{locLip}_k,M}$ as follows. First, we know that $|M| = \frac{4}{3}\pi r^3$ and

$$\begin{aligned} \bigl|u(x,y,z)\bigr| &\le \frac{2}{3\pi}\left(\left|\arctan\frac{y}{x-1}\right| + \left|\arctan\frac{y}{x+1}\right| + \left|\arctan\frac{z}{x^2+1}\right|\right) \\ &\le \frac{2}{3\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}\right) \\ &= 1. \end{aligned} \tag{5.3}$$

Applying (3.1), we have

$$\begin{aligned} \bigl\|D\bigl(G(u)\bigr)\bigr\|_{\mathrm{locLip}_k,M} &\le C\|u\|_{s,M} \\ &= C\left(\int_M \bigl|u(x,y,z)\bigr|^s\,dx\wedge dy\wedge dz\right)^{1/s} \\ &\le C\left(\int_M 1\,dx\wedge dy\wedge dz\right)^{1/s} \\ &= C|M|^{1/s} \\ &= C\left(\frac{4}{3}\pi r^3\right)^{1/s}. \end{aligned}$$

Remark (i) The Poincaré-type inequalities for the composition of the Dirac operator and Green's operator presented in (2.3) and (4.3) can be extended into the global case. (ii) It should be noticed that the domains involved in this paper are general bounded domains, which largely increases the flexibility and applicability of our results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DS had the first drift of the paper and BL added some contents and re-organized the paper.

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