POSITIVE PERIODIC SOLUTIONS OF FUNCTIONAL DISCRETE SYSTEMS AND POPULATION MODELS

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We apply a cone-theoretic fixed point theorem to study the existence of positive periodic solutions of the nonlinear system of functional difference equations $x(n+1) = A(n)x(n) + f(n,x_n)$.

1. Introduction

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, $\mathbb{Z}_-$ the negative integers, and $\mathbb{Z}^+$ the nonnegative integers. In this paper we explore the existence of positive periodic solutions of the nonlinear nonautonomous system of difference equations

$$x(n+1) = A(n)x(n) + f(n,x_n), \tag{1.1}$$

where, $A(n) = \operatorname{diag}[a_1(n), a_2(n), \dots, a_k(n)]$, a_j is ω-periodic, $f(n,x) : \mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous in x and $f(n,x)$ is ω-periodic in n and x, whenever x is ω-periodic, $\omega \geq 1$ is an integer. Let $\mathscr{X}$ be the set of all real ω-periodic sequences $\phi : \mathbb{Z} \to \mathbb{R}^k$. Endowed with the maximum norm $\|\phi\| = \max_{\theta \in \mathbb{Z}} \sum_{j=1}^{k} |\phi_j(\theta)|$ where $\phi = (\phi_1, \phi_2, \dots, \phi_k)^t$, $\mathscr{X}$ is a Banach space. Here t stands for the transpose. If $x \in \mathscr{X}$, then $x_n \in \mathscr{X}$ for any $n \in \mathbb{Z}$ is defined by $x_n(\theta) = x(n+\theta)$ for $\theta \in \mathbb{Z}$.

The existence of multiple positive periodic solutions of nonlinear functional differential equations has been studied extensively in recent years. Some appropriate references are [1, 14]. We are particularly motivated by the work in [8] on functional differential equations and the work of the first author in [4, 11, 12] on boundary value problems involving functional difference equations.

When working with certain boundary value problems whether in differential or difference equations, it is customary to display the desired solution in terms of a suitable Green's function and then apply cone theory [2, 4, 5, 6, 7, 10, 13]. Since our equation (1.1) is not this type of boundary value, we obtain a variation of parameters formula and then try to find a lower and upper estimates for the kernel inside the summation. Once those estimates are found we use Krasnoselskii's fixed point theorem to show the existence of a positive periodic solution. In [11], the first author studied the existence of periodic solutions of an equation similar to (1.1) using Schauder's second fixed point theorem.

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Throughout this paper, we denote the product of $y(n)$ from $n = a$ to $n = b$ by $\prod_{n=a}^{b} y(n)$ with the understanding that $\prod_{n=a}^{b} y(n) = 1$ for all $a > b$.

In [12], the first author considered the scalar difference equation

$$x(n+1) = a(n)x(n) + h(n)f(x(n-\tau(n))), \tag{1.2}$$

where $a(n)$, $h(n)$, and $\tau(n)$ are ω-periodic for ω an integer with $\omega \geq 1$. Under the assumptions that $a(n)$, $f(x)$, and $h(n)$ are nonnegative with $0 < a(n) < 1$ for all $n \in [0, \omega - 1]$, it was shown that (1.2) possesses a positive periodic solution. In this paper we generalize (1.2) to systems with infinite delay and address the existence of positive periodic solutions of (1.1) in the case $a(n) > 1$.

Let $\mathbb{R}_+ = [0, +\infty)$, for each $x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n$, the norm of x is defined as $|x| = \sum_{j=1}^{n} |x_j|$. $\mathbb{R}_+^n = \{(x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n : x_j \geq 0,\ j = 1, 2, \ldots, n\}$. Also, we denote $f = (f_1, f_2, \ldots, f_k)^t$, where t stands for transpose.

Now we list the following conditions.

(H1) $a(n) \neq 0$ for all $n \in [0, \omega - 1]$ with $\prod_{s=0}^{\omega-1} a_j(s) \neq 1$ for $j = 1, 2, \ldots, k$.

(H2) If $0 < a(n) < 1$ for all $n \in [0, \omega - 1]$ then, $f_j(n, \phi_n) \geq 0$ for all $n \in \mathbb{Z}$ and $\phi : \mathbb{Z} \to \mathbb{R}_+^n$, $j = 1, 2, \ldots, k$ where $\mathbb{R}_+ = [0, +\infty)$.

(H3) If $a(n) > 1$ for all $n \in [0, \omega - 1]$ then, $f_j(n, \phi_n) \leq 0$ for all $n \in \mathbb{Z}$ and $\phi : \mathbb{Z} \to \mathbb{R}_+^n$, $j = 1, 2, \ldots, k$ where $\mathbb{R}_+ = [0, +\infty)$.

(H4) For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $[\phi, \psi \in \mathscr{X},\ \|\phi\| \leq L,\ \|\psi\| \leq L,\ \|\phi - \psi\| < \delta,\ 0 \leq s \leq \omega]$ imply

$$|f(s, \phi_s) - f(s, \psi_s)| < \varepsilon. \tag{1.3}$$

2. Preliminaries

In this section we state some preliminaries in the form of definitions and lemmas that are essential to the proofs of our main results. We start with the following definition.

Definition 2.1. Let X be a Banach space and K be a closed, nonempty subset of X. The set K is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$

(ii) $u, -u \in K$ imply $u = 0$.

We now state the Krasnosel'skii fixed point theorem [9].

Theorem 2.2 (Krasnosel'skii). *Let $\mathscr{B}$ be a Banach space, and let $\mathscr{P}$ be a cone in $\mathscr{B}$. Suppose Ω_1 and Ω_2 are open subsets of $\mathscr{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and suppose that*

$$T : \mathscr{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow \mathscr{P} \tag{2.1}$$

is a completely continuous operator such that

(i) $\|Tu\| \leq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_2$; or

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathscr{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

For the next lemma we consider

$$x_j(n+1) = a_j x_j(n) + f_j(n, x_n), \quad j = 1, 2, \ldots, k. \tag{2.2}$$

The proof of the next lemma can be easily deduced from [11] and hence we omit it.

LEMMA 2.3. *Suppose (H1) holds. Then $x_j(n) \in \mathcal{X}$ is a solution of (2.2) if and only if*

$$x_j(n) = \sum_{u=n}^{n+\omega-1} G_j(n,u) f_j(u, x_u), \quad j = 1, 2, \ldots, k, \tag{2.3}$$

where

$$G_j(n,u) = \frac{\prod_{s=u+1}^{n+\omega-1} a_j(s)}{1 - \prod_{s=n}^{n+\omega-1} a_j(s)}, \quad u \in [n, n+\omega-1],\ j = 1, 2, \ldots, k. \tag{2.4}$$

Set

$$G(n,u) = \operatorname{diag}\big[G_1(n,u), G_2(n,u), \ldots, G_k(n,u)\big]. \tag{2.5}$$

It is clear that $G(n,u) = G(n+\omega, u+\omega)$ for all $(n,u) \in \mathbb{Z}^2$. Also, if either (H2) or (H3) holds, then (2.4) implies that

$$G_j(n,u) f_j(u, \phi_u) \geq 0 \tag{2.6}$$

for $(n,u) \in \mathbb{Z}^2$ and $u \in \mathbb{Z}$, $\phi : \mathbb{Z} \to \mathbb{R}^n_+$. To define the desired cone, we observe that if (H2) holds, then

$$\frac{\prod_{s=0}^{\omega-1} a_j(s)}{1 - \prod_{s=n}^{n+\omega-1} a_j(s)} \leq \left| G_j(n,u) \right| \leq \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1 - \prod_{s=n}^{n+\omega-1} a_j(s)} \tag{2.7}$$

for all $u \in [n, n+\omega-1]$. Also, if (H3) holds then

$$\frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{\left|1 - \prod_{s=n}^{n+\omega-1} a_j(s)\right|} \leq \left| G_j(n,u) \right| \leq \frac{\prod_{s=0}^{\omega-1} a_j(s)}{\left|1 - \prod_{s=n}^{n+\omega-1} a_j(s)\right|} \tag{2.8}$$

for all $u \in [n, n+\omega-1]$. For all $(n,s) \in \mathbb{Z}^2$, $j = 1, 2, \ldots, k$, we define

$$\begin{aligned} \sigma_2 &:= \min\left\{\left(\prod_{s=0}^{\omega-1} a_j(s)\right)^2,\ j = 1, 2, \ldots, n\right\}, \\ \sigma_3 &:= \min\left\{\left(\prod_{s=0}^{\omega-1} a_j^{-1}(s)\right)^2,\ j = 1, 2, \ldots, n\right\}. \end{aligned} \tag{2.9}$$

We note that if $0 < a(n) < 1$ for all $n \in [0, \omega-1]$, then $\sigma_2 \in (0,1)$. Also, if $a(n) > 1$ for all $n \in [0, \omega-1]$, then $\sigma_3 \in (0,1)$. Conditions (H2) and (H3) will have to be handled separately. That is, we define two cones; namely, $\mathcal{P}2$ and $\mathcal{P}3$. Thus, for each $y \in \mathcal{X}$ set

$$\begin{aligned} \mathcal{P}2 &= \{y \in \mathcal{X} : y(n) \geq 0,\ n \in \mathbb{Z},\ \text{and}\ y(n) \geq \sigma_2 \|y\|\}, \\ \mathcal{P}3 &= \{y \in \mathcal{X} : y(n) \geq 0,\ n \in \mathbb{Z},\ \text{and}\ y(n) \geq \sigma_3 \|y\|\}. \end{aligned} \tag{2.10}$$

Define a mapping $T : \mathcal{X} \to \mathcal{X}$ by

$$(Tx)(n) = \sum_{u=n}^{n+\omega-1} G(n,u) f(u,x_u), \tag{2.11}$$

where $G(n,u)$ is defined following (2.4). We denote

$$(Tx) = (T_1x, T_2x, \dots, T_nx)^t. \tag{2.12}$$

It is clear that $(Tx)(n+\omega) = (Tx)(n)$.

Lemma 2.4. *If (H1) and (H2) hold, then the operator $T\mathcal{P}2 \subset \mathcal{P}2$. If (H1) and (H3) hold, then $T\mathcal{P}3 \subset \mathcal{P}3$.*

Proof. Suppose (H1) and (H2) hold. Then for any $x \in \mathcal{P}2$ we have

$$(T_j x(n)) \geq 0, \quad j = 1, 2, \dots, k. \tag{2.13}$$

Also, for $x \in \mathcal{P}2$ by using (2.4), (2.7), and (2.11) we have that

$$\begin{aligned} (T_j x)(n) &\leq \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1 - \prod_{s=n}^{n+\omega-1} a_j(s)} \sum_{u=n}^{n+\omega-1} |f_j(u,x_u)|, \\ ||T_j x|| = \max_{n \in [0,\omega-1]} |T_j x(n)| &\leq \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1 - \prod_{s=n}^{n+\omega-1} a_j(s)} \sum_{u=n}^{n+\omega-1} |f_j(u,x_u)|. \end{aligned} \tag{2.14}$$

Therefore,

$$\begin{aligned} (T_j x)(n) &= \sum_{u=n}^{n+\omega-1} G_j(n,u) f_j(u,x_u) \\ &\geq \frac{\prod_{s=0}^{\omega-1} a_j(s)}{1 - \prod_{s=n}^{n+\omega-1} a_j(s)} \sum_{u=n}^{n+\omega-1} |f_j(u,x_u)| \\ &\geq \left(\prod_{s=0}^{\omega-1} a_j(s) \right)^2 ||T_j x|| \geq \sigma_2 ||T_j x||. \end{aligned} \tag{2.15}$$

That is, $T\mathcal{P}2$ is contained in $\mathcal{P}2$. The proof of the other part follows in the same manner by simply using (2.8), and hence we omit it. This completes the proof. □

To simplify notation, we state the following notation:

$$A_2 = \min_{1\le j\le k} \frac{\prod_{s=0}^{\omega-1} a_j(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)}, \tag{2.16}$$

$$B_2 = \max_{1\le j\le k} \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)}, \tag{2.17}$$

$$A_3 = \min_{1\le j\le k} \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{\left|1-\prod_{s=n}^{n+\omega-1} a_j(s)\right|}, \tag{2.18}$$

$$B_3 = \max_{1\le j\le k} \frac{\prod_{s=0}^{\omega-1} a_j(s)}{\left|1-\prod_{s=n}^{n+\omega-1} a_j(s)\right|}, \tag{2.19}$$

where k is defined in the introduction.

Lemma 2.5. *If (H1), (H2), and (H4) hold, then the operator $T:\mathcal{P}2\to\mathcal{P}2$ is completely continuous. Similarly, if (H1), (H3), and (H4) hold, then the operator $T:\mathcal{P}3\to\mathcal{P}3$ is completely continuous.*

Proof. Suppose (H1), (H2), and (H4) hold. First show that T is continuous. By (H4), for any $L>0$ and $\varepsilon>0$, there exists a $\delta>0$ such that $[\phi,\psi\in\mathscr{X},\ \|\phi\|\le L,\ \|\psi\|\le L,\ \|\phi-\psi\|<\delta]$ imply

$$\max_{0\le s\le\omega-1} \left| f(s,\phi_s)-f(s,\psi_s)\right| < \frac{\varepsilon}{B_2\omega}, \tag{2.20}$$

where B_2 is given by (2.17). If $x,y\in\mathcal{P}2$ with $\|x\|\le L$, $\|y\|\le L$, and $\|x-y\|<\delta$, then

$$\begin{aligned}\left|(Tx)(n)-(Ty)(n)\right| &\le \sum_{u=n}^{n+\omega-1} \left|G(n,u)\right| \left|f(u,x_u)-f(u,y_u)\right| \\ &\le B_2\sum_{u=0}^{\omega-1}\left|f(u,x_u)-f(u,y_u)\right| < \varepsilon\end{aligned} \tag{2.21}$$

for all $n\in[0,\omega-1]$, where $|G(n,u)| = \max_{1\le j\le n}|G_j(n,u)|$, $j=1,2,\dots,k$. This yields $\|(Tx)-(Ty)\|<\varepsilon$. Thus, T is continuous. Next we show that T maps bounded subsets into compact subsets. Let $\varepsilon=1$. By (H4), for any $\mu>0$ there exists $\delta>0$ such that $[x,y\in\mathscr{X},\ \|x\|\le\mu,\ \|y\|\le\mu,\ \|x-y\|<\delta]$ imply

$$\left|f(s,x_s)-f(s,y_s)\right| < 1. \tag{2.22}$$

We choose a positive integer N so that $\delta>\mu/N$. For $x\in\mathscr{X}$, define $x^i(n)=ix(n)/N$, for $i=0,1,2,\dots,N$. For $\|x\|\le\mu$,

$$\begin{aligned}\left\|x^i-x^{i-1}\right\| &= \max_{n\in\mathbb{Z}}\left|\frac{ix(n)}{N}-\frac{(i-1)x(n)}{N}\right| \\ &\le \frac{\|x\|}{N}\le\frac{\mu}{N}<\delta.\end{aligned} \tag{2.23}$$

Thus, $|f(s,x^i) - f(s,x^{i-1})| < 1$. As a consequence, we have

$$f(s,x_s) - f(s,0) = \sum_{i=1}^{N} \left(f(s,x^i) - f(s,x^{i-1})\right), \tag{2.24}$$

which implies that

$$\begin{aligned} |f(s,x_s)| &\le \sum_{i=1}^{N} |f(s,x_s^i) - f(s,x_s^{i-1})| + |f(s,0)| \\ &< N + |f(s,0)|. \end{aligned} \tag{2.25}$$

Thus, f maps bounded sets into bounded sets. It follows from the above inequality and (2.11), that

$$\begin{aligned} \|(Tx)(n)\| &\le B_2 \sum_{j=1}^{k} \left(\sum_{u=n}^{n+T-1} |f_j(u,x_u)| \right) \\ &\le B_2 \omega\left(N + |f(s,0)|\right). \end{aligned} \tag{2.26}$$

If we define $S = \{x \in \mathscr{X} : \|x\| \le \mu\}$ and $Q = \{(Tx)(n) : x \in S\}$, then S is a subset of $\mathbb{R}^{\omega k}$ which is closed and bounded and thus compact. As T is continuous in x, it maps compact sets into compact sets. Therefore, $Q = T(S)$ is compact. The proof for the other case is similar by simply invoking (2.19). This completes the proof. □

3. Main results

In this section we state two theorems and two corollaries. Our theorems and corollaries are stated in a way that unify both cases; $0 < a(n) < 1$ and $a(n) > 1$ for all $n \in [0, \omega - 1]$.

Theorem 3.1. *Assume that (H1) holds.*

(a) *Suppose (H2) and (H4) hold and that there exist two positive numbers R_1 and R_2 with $R_1 < R_2$ such that*

$$\sup_{\|\phi\|=R_1,\ \phi\in\mathcal{P}2} |f(s,x_s)| \le \frac{R_1}{\omega B_2}, \tag{3.1}$$

$$\inf_{\|\phi\|=R_2,\ \phi\in\mathcal{P}2} |f(s,x_s)| \ge \frac{R_2}{\omega A_2}, \tag{3.2}$$

where A_2 and B_2 are given by (2.16) and (2.17), respectively. Then, there exists $\overline{x} \in \mathcal{P}2$ which is a fixed point of T and satisfies $R_1 \le \|\overline{x}\| \le R_2$.

(b) *Suppose (H3) and (H4) hold and that there exist two positive numbers R_1 and R_2 with $R_1 < R_2$ such that*

$$\begin{aligned} \sup_{\|\phi\|=R_1,\ \phi\in\mathcal{P}3} |f(s,x_s)| &\le \frac{R_1}{\omega B_3}, \\ \inf_{\|\phi\|=R_2,\ \phi\in\mathcal{P}3} |f(s,x_s)| &\ge \frac{R_2}{\omega A_3}, \end{aligned} \tag{3.3}$$

where A_3 and B_3 are given by (2.18) and (2.19), respectively. Then, there exists $\overline{x} \in \mathcal{P}3$ which is a fixed point of T and satisfies $R_1 \leq \|\overline{x}\| \leq R_2$.

Proof. Suppose (H1), (H2), and (H4) hold. Let $\Omega_\xi = \{x \in \mathcal{P}2 \mid \|x\| < \xi\}$. Let $x \in \mathcal{P}2$ which satisfies $\|x\| = R_1$, in view of (3.1), we have

$$\begin{aligned} |(Tx)(n)| &\leq \sum_{u=n}^{n+\omega-1} |G(n,u)| \, |f(u,x_u)| \\ &\leq B_2\omega\frac{R_1}{\omega B_2} = R_1. \end{aligned} \tag{3.4}$$

That is, $\|Tx\| \leq \|x\|$ for $x \in \mathcal{P}2 \cap \partial\Omega_{R_1}$. let $x \in \mathcal{P}2$ which satisfies $\|x\| = R_2$ we have, in view of (3.2),

$$|(Tx)(n)| \geq A_2 \sum_{u=n}^{n+\omega-1} |f(u,x_u)| \geq A_2\omega\frac{R_2}{\omega A_2} = R_2. \tag{3.5}$$

That is, $\|Tx\| \geq \|x\|$ for $x \in \mathcal{P}2 \cap \partial\Omega_{R_2}$. In view of Theorem 2.2, T has a fixed point in $\mathcal{P}2 \cap (\bar{\Omega}_2 \setminus \Omega_1)$. It follows from Lemma 2.4 that (1.1) has an ω-periodic solution $\overline{x}$ with $R_1 \leq \|\overline{x}\| \leq R_2$. The proof of (b) follows in a similar manner by simply invoking conditions (3.3). □

As a consequence of Theorem 3.1, we state a corollary omitting its proof.

Corollary 3.2. *Assume that (H1) holds.*

(a) *Suppose (H2) and (H4) hold and*

$$\begin{aligned} \lim_{\phi\in\mathcal{P}2,\ \|\phi\|\to 0} \frac{|f(s,\phi_s)|}{\|\phi\|} &= 0, \\ \lim_{\phi\in\mathcal{P}2,\ \|\phi\|\to \infty} \frac{|f(s,\phi_s)|}{\|\phi\|} &= \infty. \end{aligned} \tag{3.6}$$

Then (1.1) has a positive periodic solution.

(b) *Suppose (H3) and (H4) hold and*

$$\begin{aligned} \lim_{\phi\in\mathcal{P}3,\ \|\phi\|\to 0} \frac{|f(s,\phi_s)|}{\|\phi\|} &= 0, \\ \lim_{\phi\in\mathcal{P}3,\ \|\phi\|\to \infty} \frac{|f(s,\phi_s)|}{\|\phi\|} &= \infty. \end{aligned} \tag{3.7}$$

Then (1.1) has a positive periodic solution.

Theorem 3.3. *Suppose that (H1) holds.*

(a) *Suppose (H2) and (H4) hold and that there exist two positive numbers R_1 and R_2 with $R_1 < R_2$ such that*

$$\inf_{\|\phi\|=R_1,\ \phi\in\mathcal{P}2} |f(s,x_s)| \geq \frac{R_1}{\omega B_2}, \qquad \sup_{\|\phi\|=R_2,\ \phi\in\mathcal{P}2} |f(s,x_s)| \leq \frac{R_2}{\omega A_2}, \tag{3.8}$$

where A_2 and B_2 are given by (2.16) and (2.17), respectively. Then, there exists $\overline{x} \in \mathcal{P}2$ which is a fixed point of T and satisfies $R_1 \leq \|\overline{x}\| \leq R_2$.

(b) *Suppose (H3) and (H4) hold and that there exist two positive numbers R_1 and R_2 with $R_1 < R_2$ such that*

$$\inf_{\|\phi\|=R_1,\ \phi\in\mathcal{P}3} |f(s,x_s)| \geq \frac{R_1}{\omega B_3}, \qquad \sup_{\|\phi\|=R_2,\ \phi\in\mathcal{P}3} |f(s,x_s)| \leq \frac{R_2}{\omega A_3}, \tag{3.9}$$

where A_3 and B_3 are given by (2.18) and (2.19), respectively. Then, there exists $\overline{x} \in \mathcal{P}3$ which is a fixed point of T and satisfies $R_1 \leq \|\overline{x}\| \leq R_2$.

The proof is similar to the proof of Theorem 3.1 and hence we omit it. As a consequence of Theorem 3.3, we have the following corollary.

Corollary 3.4. *Assume that (H1) holds.*

(a) *Suppose (H2) and (H4) hold and*

$$\lim_{\phi\in\mathcal{P}2,\ \|\phi\|\to 0} \frac{|f(s,\phi_s)|}{\|\phi\|} = \infty, \qquad \lim_{\phi\in\mathcal{P}2,\ \|\phi\|\to \infty} \frac{|f(s,\phi_s)|}{\|\phi\|} = 0. \tag{3.10}$$

Then (1.1) has a positive periodic solution.

(b) *Suppose (H3) and (H4) hold and*

$$\lim_{\phi\in\mathcal{P}3,\ \|\phi\|\to 0} \frac{|f(s,\phi_s)|}{\|\phi\|} = \infty, \qquad \lim_{\phi\in\mathcal{P}3,\ \|\phi\|\to \infty} \frac{|f(s,\phi_s)|}{\|\phi\|} = 0. \tag{3.11}$$

Then (1.1) has a positive periodic solution.

4. Applications to population dynamics

In this section, we apply our results from the previous section and show that some population models admit the existence of a positive periodic solution. We start by considering the scalar discrete model that governs the growth of population $N(n)$ of a single species whose members compete among themselves for the limited amount of food that is available to sustain the population. Thus, we consider the scalar equation

$$N(n+1) = \alpha(n)N(n)\left[1 - \frac{1}{N_0(n)}\sum_{s=-\infty}^{0} B(s)N(n+s)\right], \quad n \in \mathbb{Z}. \tag{4.1}$$

We note that (4.1) is a generalization of the known logistic model

$$N(n+1) = \alpha N(n)\left[1 - \frac{N(n)}{N_0}\right], \tag{4.2}$$

where α is the intrinsic per capita growth rate and N_0 is the total carrying capacity. For more biological information on (4.1), we refer the reader to [3]. We remark that in (4.1), the term $\sum_{s=-\infty}^{0} B(s)N(n+s)$, is equivalent to $\sum_{u=-\infty}^{n} B(u-s)N(u)$. We chose to write (4.1) that way so that it can be put in the form of $x(n+1) = a(n)x(n) + f(n,x_n)$. Before we state our results in the form of a theorem, we assume that

(P1) $\alpha(n) > 1$, $N_0(n) > 0$ for all $n \in \mathbb{Z}$ with $\alpha(n)$, $N_0(n)$ are ω-periodic and

(P2) $B(n)$ is nonnegative on $\mathbb{Z}_-$ with $\sum_{n=-\infty}^{0} B(n) < \infty$.

Theorem 4.1. *Under assumptions (P1) and (P2), (4.1) has at least one positive ω-periodic solution.*

Proof. Let $a(n) = \alpha(n)$ and

$$f(n,x_n) = -\frac{x(n)\alpha(n)}{N_0(n)}\sum_{s=-\infty}^{0} B(s)x(n+s). \tag{4.3}$$

It is clear that $f(n,x_n)$ is ω-periodic whenever x is ω-periodic and (H1) and (H3) hold since $f(n,\phi_n) \le 0$ for all $(n,\phi) \in \mathbb{Z} \times (\mathbb{Z},\mathbb{R}_+)$. To verify (H4), we let $x,y : \mathbb{Z} \to \mathbb{R}_+$ with $\|x\| \le L$, $\|y\| \le L$ for some $L > 0$. Then

$$\begin{aligned}
&|f(n,x_n) - f(n,y_n)| \\
&\quad = \left|\frac{x(n)\alpha(n)}{N_0(n)}\sum_{s=-\infty}^{0} B(s)x(n+s) - \frac{y(n)\alpha(n)}{N_0(n)}\sum_{s=-\infty}^{0} B(s)y(n+s)\right| \\
&\quad \le \left|\frac{x(n)\alpha(n)}{N_0(n)}\right|\sum_{s=-\infty}^{0} B(s)\,|x(n+s) - y(n+s)|\,ds \\
&\qquad + \left|\frac{(x(n)-y(n))\alpha(n)}{N_0(n)}\right|\sum_{s=-\infty}^{0} B(s)\,|y(n+s)| \\
&\quad \le \frac{L\|\alpha\|}{N_{0*}}\max_{s\in\mathbb{Z}_-}|x(n+s) - y(n+s)| + \frac{|x(n)-y(n)|\,\|\alpha\|L}{N_{0*}},
\end{aligned} \tag{4.4}$$

where $N_{0*} = \min\{N_0(s) : 0 \le s \le \omega - 1\}$. For any $\varepsilon > 0$, choose $\delta = \varepsilon N_{0*}/(2L\|a\|)$. If $\|x - y\| < \delta$, then

$$|f(n,x_n) - f(n,y_n)| < L\|\alpha\|\delta/N_{0*} + \delta\|\alpha\|L/N_{0*} = 2L\|\alpha\|\delta/N_{0*} = \varepsilon. \tag{4.5}$$

This implies that (H4) holds. We now show that (3.7) hold. For $\phi \in \mathcal{P}3$, we have $\phi(n) \geq \sigma_3\|\phi\|$ for all $n \in [0, \omega - 1]$. This yields

$$\frac{|f(n,\phi)|}{\|\phi\|} \leq \max_{\tau\in[0,\omega-1]} \frac{\alpha(\tau)}{N_0(\tau)} \sum_{s=-\infty}^{0} B(s)\|\phi\| \longrightarrow 0 \tag{4.6}$$

as $\|\phi\| \to 0$ and

$$\frac{|f(n,\phi)|}{\|\phi\|} \geq \min_{\tau\in[0,\omega-1]} \frac{\alpha(\tau)}{N_0(\tau)} \sum_{s=-\infty}^{0} B(s)\sigma_3{}^2\|\phi\| \longrightarrow +\infty \tag{4.7}$$

as $\|\phi\| \to \infty$. Thus, (3.7) are satisfied. By (b) of Corollary 3.2, (4.1) has a positive ω-periodic solution. This completes the proof. □

Next we consider the Volterra discrete system

$$x_i(n+1) = x_i(n)\left[a_i(n) - \sum_{j=1}^{k} b_{ij}(n)x_j(n) - \sum_{j=1}^{k}\sum_{s=-\infty}^{n} C_{ij}(n,s)g_{ij}(x_j(s))\right], \tag{4.8}$$

where $x_i(n)$ is the population of the ith species, $a_i, b_{ij} : \mathbb{Z} \to \mathbb{R}$ are ω-periodic and $C_{ij}(n,s) : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$ is ω-periodic.

Theorem 4.2. *Suppose that the following conditions hold for $i, j = 1,2,\ldots,k$.*

(i) *$a_i(n) > 1$, for all $n \in [0,\omega-1]$, and $a_i(n)$ is ω-periodic,*

(ii) *$b_{ij}(n) \geq 0$, $C_{ij}(n,s) \geq 0$ for all $(n,s) \in \mathbb{Z}^2$,*

(iii) *$g_{ij} : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous in x and increasing with $g_{ij}(0) = 0$,*

(iv) *$b_{ii}(s) \neq 0$, for $s \in [0,\omega-1]$,*

(v) *$C_{ij}(n+\omega, s+\omega) = C_{ij}(n,s)$ for all $(n,s) \in \mathbb{Z}^2$ with $\max_{n\in\mathbb{Z}} \sum_{s=-\infty}^{n} |C_{ij}(n,s)| < +\infty$.*

Then (4.8) has a positive ω-periodic solution.

Proof. For $x = (x_1, x_2, \ldots, x_n)^T$, define

$$f_i(n,x_n) = -x_i(t)\sum_{j=1}^{k} b_{ij}(n)x_j(n) - \sum_{j=1}^{k}\sum_{s=-\infty}^{n} C_{ij}(n,s)g_{ij}(x_j(s)) \tag{4.9}$$

for $i = 1,2,\ldots,k$ and set $f = (f_1, f_2, \ldots, f_n)^t$. Then by some manipulation of conditions (i)–(v), the conditions (H1) and (H2) are satisfied. Also, it is clear that f satisfies (H4). Define

$$\begin{gathered} b^* = \max\{||b_{ij}|| : i, j = 1,2,\ldots,k\}, \\ C^* = \max\left\{\sup_{n\in\mathbb{Z}} \sum_{j=1}^{n}\sum_{s=-\infty}^{n} |C_{ij}(n,s)| : i = 1,2,\ldots,k\right\}, \\ g^*(u) = \max\{g_{ij}(u) : i, j = 1,2,\ldots,k\}. \end{gathered} \tag{4.10}$$

Let $x \in \mathcal{P}3$. Since g is increasing in x, we arrive at

$$|f_i(n,x_n)| \le |x_i(n)| \left[b^* \|x\| + \sum_{j=1}^{n} \sum_{s=-\infty}^{n} |C_{ij}(n,s)| g_{ij}(\|x_j\|) \right]. \tag{4.11}$$

Thus

$$|f(n,x_n)| \le \|x\| [b^* \|x\| + C^* g^*(\|x\|)], \tag{4.12}$$

which implies

$$\frac{|f(n,x_s)|}{\|x\|} \le [b^* \|x\| + C^* g^*(\|x\|)] \longrightarrow 0 \tag{4.13}$$

as $\|x\| \to 0$. For $x \in \mathcal{P}3$, $x_i(n) \ge \sigma_3 \|x_i\|$ for all $n \in \mathbb{Z}$. Also, from (ii), $b_{ij}(t)$, $C_{ij}(t,s)$ have the same sign. Thus, using condition (iii) we have

$$\begin{aligned} |f_i(n,x_n)| &= \sum_{j=1}^{n} x_i(n) |b_{ij}(n)| x_j(n) + \sum_{j=1}^{k} \sum_{s=-\infty}^{n} |C_{ij}(n,s)| g_{ij}(x_j(s)) \\ &\ge |b_{ii}(n) \mid x_i(n)|^2 \ge \sigma_3{}^2 \|x_i\|^2 |b_{ii}(n)|, \\ |f(n,x_s)| &\ge \sigma_3{}^2 \sum_{i=1}^{k} \|x_i\|^2 \min_{1\le i\le k} |b_{ii}(n)| \ge \frac{\sigma_3{}^2}{k} \|x\|^2 \min_{1\le i\le k} |b_{ii}(n)|. \end{aligned} \tag{4.14}$$

Here we have applied the inequality $(\sum_{i=1}^{k} \|x_i\|)^2 \le k \sum_{i=1}^{k} \|x_i\|^2$. Thus,

$$\frac{|f(n,x_s)|}{\|x\|} \longrightarrow +\infty \quad \text{as } \|x\| \longrightarrow +\infty. \tag{4.15}$$

By (b) of Corollary 3.2, (4.8) has a positive ω-periodic solution. This completes the proof. □

THEOREM 4.3. *Suppose that the following conditions hold for $i, j = 1, 2, \dots, k$.*

(i) $0 < a_i(n) < 1$, *for all $n \in [0, \omega - 1]$, and $a_i(n)$ is ω-periodic,*

(ii) $b_{ij}(n) \le 0$, $C_{ij}(n,s) \le 0$ *for all $(n,s) \in \mathbb{Z}^2$,*

(iii) $g_{ij} : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous in x and increasing with $g_{ij}(0) = 0$,*

(iv) $b_{ii}(s) \ne 0$, *for $s \in [0, \omega - 1]$,*

(v) $C_{ij}(n+\omega, s+\omega) = C_{ij}(n,s)$ *for all $(n,s) \in \mathbb{Z}^2$ with* $\max_{n\in\mathbb{Z}} \sum_{s=-\infty}^{n} |C_{ij}(n,s)| < +\infty$.

Then (4.8) has a positive ω-periodic solution. The proof follows from part (a) of Corollary 3.2.

Remark 4.4. In the statements of Theorems 4.2 and 4.3 condition (iv) can be replaced by

(iv*) $\sum_{j=1}^{k} \sum_{s=-\infty}^{n} |C_{ij}(n,s)| \ne 0$ and $g_{ii}(x) \to +\infty$ as $x \to +\infty$.

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