Eur. Phys. J. Special Topics **229**, 887–904 (2020)

https://doi.org/10.1140/epjst/e2020-900238-8

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Review

Fractional order chaotic systems: history, achievements, applications, and future challenges

Mohammad Saleh Tavazoei[a]

Electrical Engineering Department, Sharif University of Technology, Tehran, Iran

Received 27 October 2019 / Received in final form 26 November 2019
Published online 26 March 2020

Abstract. Motivated by the importance of study on the complex behaviors, which may be exhibited by fractional order differential equations, this review paper focuses on dynamical fractional order systems exhibiting chaotic behaviors. The review begins with a brief history on the first publications on the above-mentioned subject. Then, the review is continued by investigating the recent progresses relevant to fractional order chaotic systems. Furthermore, a summary on some applications for such systems, which have been reported in the literature, is presented. Finally, the paper is closed by discussing some open problems on the aforementioned research subject. These open problems, as future challenges for further study on fractional order chaotic systems, can specify some direction lines for continuing the research on that subject.

1 Introduction

More than three centuries after the first attempts to construct basic foundations of fractional calculus [1,2], today this mathematical tool has opened its way in various branches of science and engineering [3]. Benefiting from the generalized differentiation/integration operators introduced via fractional calculus, an extended framework for defining dynamical models has been engendered [4,5]. This extended framework is more capable to describe the phenomena of the real world, in comparison to the framework that is available from the traditional calculus (For some samples, see [6–9]). On the other hand, chaos theory, as another interesting mathematical branch, provides new insights to discover the world around us [10,11]. Fractional order chaotic dynamical systems can be considered as the intersection of the mathematical tool of fractional calculus and the mathematical branch of chaos theory. This paper aims to review the research works done on the subject of fractional order chaotic systems. This review includes expressing a brief history on the subject, describing the works done on the active relevant subfields, explaining the applications, and clarifying the challenges for future research works.

This review paper is organized as follows. In Section 2, firstly some useful definitions relevant to fractional order dynamical systems are restated. Then, a brief history on the first research works dealing with fractional order chaotic systems is

[a] e-mail: tavazoei@sharif.edu

presented in this section. In continuation of the review, Section 3 discusses on the need to differentiate between inherently chaotic fractional order systems and fractional order systems with chaotic integer order approximations. Some directions in recent researches on fractional order chaotic systems are reviewed in Section 4. Moreover, the applications of fractional order chaotic systems, which have been previously reported in literature, are summarized in Section 5. Furthermore, some open problems and future challenges in study of this kind of systems are discussed in Section 6. Finally, the paper is concluded in Section 7.

2 Preliminaries and a brief history

2.1 Basic definitions

To define the fractional differentiation operator, there are some commonly used definitions. As one of these popular definitions, the Caputo fractional derivative of order α $(0 < \alpha \notin N)$ for function $f(t)$ is defined by

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{-\alpha-1+m} f^{(m)}(\tau) d\tau \,, \tag{1} $$

where m denotes the first integer number greater than α, i.e. $m-1 < \alpha < m$ and $\Gamma(.)$ specifies the Gamma function [12]. For briefness, we skip to restate the other popular definitions of fractional differentiation operator (e.g. Riemann-Liouville and Grünwald–Letnikov definitions), and the interested readers are referred to [12,13] for details on different definitions for fractional order operators. Considering the definition of fractional derivative, a fractional order dynamical system can be described in the pseudo-state space form as

$$ \begin{array}{l} {}_0D_t^{\alpha_1} x_1 = f_1(x_1, x_2, \ldots, x_n) \\ {}_0D_t^{\alpha_2} x_2 = f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ {}_0D_t^{\alpha_n} x_n = f_n(x_1, x_2, \ldots, x_n) \end{array} , \tag{2} $$

where $x_i \in R$ and $\alpha_i \in (0, 1]$ for $i = 1, 2, \ldots, n$. In system (2), the summation of the involved derivative orders α_i, i.e. $\sum_{i=1}^{n} \alpha_i$, is called the effective dimension of the system. $x = [x_1, x_2, \ldots, x_n]^T \in R^n$ is the pseudo-state vector of system (2), and each x_i $(i = 1, 2, \ldots, b)$ is a pseudo-state variable for this system (To find out why instead of "state vector", x is called the "pseudo-state vector", see [14]). Also, (2) denotes a commensurate order system, where $\alpha_1 = \alpha_2 = \ldots = \alpha_n$. Otherwise, (2) will be an incommensurate order system. The integer order counterpart of system (2) is of the following common from

$$ \begin{array}{l} \dot{x}_1 = f_1(x_1, x_2, \ldots, x_n) \\ \dot{x}_2 = f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, \ldots, x_n) \end{array} . \tag{3} $$

2.2 The first research works on fractional order chaotic systems

The first attempts to introduce chaotic systems containing fractional order differentiation operators were made by T. T. Hartley, C. F. Lorenzo, and H. K. Qammer.

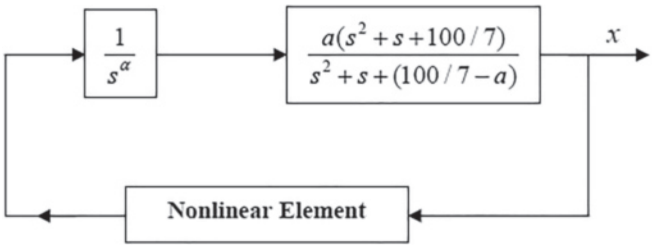

Fig. 1. A closed-loop equivalent form for system (4) [15].

In their research paper [15], published in August 1995, a fractional order version of the Chua system, defined by

$$\begin{array}{l} {}_0D_t^\alpha x = a\left(y + \frac{x-2x^3}{7}\right) \\ {}_0D_t^\alpha y = x - y + z \\ {}_0D_t^\alpha z = -\frac{100}{7}y \end{array}, \tag{4}$$

where α and a are positive constants, has been introduced. To analyze the chaotic behavior of system (4), firstly a closed-loop equivalent form for this system as that shown in Figure 1 has been considered in [15]. Then, the fractional integration operator $\frac{1}{s^\alpha}$ in this equivalent form has been replaced by some fifth order biproper transfer functions, as rational approximations for the operator $\frac{1}{s^\alpha}$. This means that in [15], approximations in the form

$$\frac{1}{s^\alpha} \approx \frac{\sum_{k=0}^{5} a_k s^k}{\sum_{i=0}^{5} b_i s^i}, \tag{5}$$

have been used to obtain the trajectories in system (4).

The numerical simulation based analysis of [15] revealed that system (4) can exhibit chaotic behaviors, where its effective dimension is less than 3. Reporting this result was interesting for the researchers working in the field on dynamical systems, because from the Poincaré–Bendixson theorem [16] it is known that the integer order systems in form $\dot{x} = f(x)$, where $f : R^n \to R^n$ is a continuously differentiable function, with effective dimensions less than 3 cannot exhibit chaotic behavior. It is worth mentioning that on the basis of the equivalent closed-loop form shown in Figure 1 for system (4) and using the results of [17], describing function based analysis to justify the existence of chaos in this system have been presented in [15].

After publishing [15], inspired from this paper, various research works with the subject of introducing and analyzing specific fractional order chaotic systems were published. Among these research works, it seems that [18] and [19] are the first two papers with the aforementioned subject which have been published after [15]. In [18], which has been published in 1998, a class of fractional order cellular neural networks with chaotic behaviors has been introduced. Also, [19], which has been published in 1999, deals with numerical analysis of the chaotic behavior of the following fractional order system.

$$\begin{array}{l} \dot{x} = 0.9 - y \\ \dot{y} = 0.4 + z \\ \dot{z} = {}_0D_t^{1-q}(xy - z) \end{array}. \tag{6}$$

In the special case $q = 1$, (6) specifies one of the simple chaotic systems introduced by Sprott in [20]. In addition to the above-mentioned two papers, various research works, numerically investigating fractional order counterparts of famous chaotic dynamical systems, were published within a decade after publishing the first paper on the subject of fractional order chaotic systems. Some of the chaotic fractional order systems studied in these works are as follows.

– Fractional order Lorenz system [21,22]
– Fractional order Rössler system [23]
– Fractional order Chen system [24]
– Fractional order Lü system [25]
– Fractional order Arneodo system [26]
– Fractional order Genesio–Tesi system [27]
– Fractional order Ikeda system [28]
– Fractional order Duffing system [29]

Also, [30–34] are some of the other primary works on fractional order chaotic systems.

3 Fractional order chaotic systems versus approximating integer high order chaotic ones

One of the items focused by the pioneering research work [15] and the most of the other papers following this work in the later years is to numerically find the minimum effective dimension for capability to exhibit chaotic behaviors in a specific fractional order system. Continuing these efforts, in research work [35], published in July 2007, a lower bound for such a minimum effective dimension in commensurate order systems, whose integer order counterpart has a double scroll attractor, is analytically derived (This result was later extended in [36] for the incommensurate order systems). A surprising consequence of the lower bound found in [35] was that this analytic bound was inconsistent with some results previously reported on the minimum effective dimensions in fractional order chaotic systems (Some samples of these inconsistencies have been investigated in [35,37,38]). A more in-depth analysis revealed that such inconsistencies originate from the numerical methods used for simulation of fractional order systems [39]. More specially, using approximations in the forms such as (5) for approximating the fractional operators can cause that the proximate system is inconsistent with the original one in the viewpoint of the specifications relevant to the equilibrium points (specifications such as the number, location, and stability/instability of equilibrium points) [40]. In fact, the aforementioned class of frequency-domain approximations converts the original fractional order system into an integer high order one. Due to the inherent differences between the original fractional order system and its integer high order approximation, their capabilities to generate chaos may be different, i.e. the original system may have represent regular behaviors, whereas its integer order approximation is chaotic [39]. Conversely, the original fractional order system may chaotically behave, whereas its approximation is a regular system. For more clarifying the point, let us recall a numerical example from [37]. Consider the fractional order Lorenz system

$$\begin{aligned} {}_0D_t^{0.8}x &= 10(x-y) \\ {}_0D_t^{0.8}y &= 28x - xz + 8z\,. \\ {}_0D_t^{0.8}z &= xy - \tfrac{8}{3}z \end{aligned} \tag{7}$$

It can be checked that $(x, y, z) = (\pm\sqrt{96}, \pm\sqrt{96}, 36)$ are two equilibrium points for system (7). By linearizing system (7) around these points, it is deduced that $(x, y, z) = (\pm\sqrt{96}, \pm\sqrt{96}, 36)$ are two stable equilibrium points in system (7). Consequently, asymptotically stable orbits around the aforementioned equilibrium points can be observed in numerical simulation results of system (7) (For a sample, see Figure 1 of [37]). But, if the approximation

$$\frac{1}{s^{0.8}} \approx \frac{5.235s^3+1453s^2+5306s+254.9}{s^4+658.1s^3+5700s^2+658.2s+1}, \tag{8}$$

is used for simulation of fractional order system (7), the obtained system for approximating (7) is an integer order one of order 12. Using the linearization technique, it is shown that all of the equilibrium points in the approximating integer order system are unstable. In this case, the existence of chaos in approximating integer order system is revealed from numerical simulation results, such that the original asymptotically stable orbits around the stable equilibrium points $(x, y, z) = (\pm\sqrt{96}, \pm\sqrt{96}, 36)$ are converted to chaotic ones (For more discussions, see [37]). This means that fake chaotic behaviors are exhibited by the approximating integer order system, whereas the original system (7) is not a chaotic one.

Considering the above-mentioned example, it is necessary to distinguish between chaotic oscillations inherently generated by fractional order systems and those created by their integer high order approximations (More analytical details on this point can be found in [40] and [41]).

4 More recent achievements

In this section, some active topics, which have been gained attention of the researches working on chaotic phenomena in fractional order systems, are summarized.

4.1 Study on specific fractional order chaotic systems and bifurcation analysis

In recent years, study on the properties of fractional order counterparts of the introduced integer order chaotic systems by using more advanced techniques has been still continued. Some of these studies deal with bifurcation analysis in fractional order Lorenz system [42], fractional order Rössler system [43], and fractional order Chua system [44]. Another related topic, which has been paid attention by the researchers, is to investigate the strange attractors produced by fractional order chaotic systems. For instance, fractional order chaotic systems with multiwing attractors have been studied in [45–49].

4.2 Chaos analysis in fractional order real-world inspired models

Investigating the chaotic behaviors of fractional order counterparts of real-world inspired models, such as fractional order electrical models, fractional order mechanical models, fractional order biological models, and fractional order financial models has been the subject of various research works published in recent years. For example, in [50] it has been shown that a fractional order electrical RLC circuit containing a Chua's diode [51] can generate chaos, where it is excited by sinusoidal inputs. Also, in some research works the chaotic behaviors of fractional order electrical circuits containing memristors [52], e.g. fractional order Chua circuit including a memristor [53], simple electrical circuits containing a fractional capacitor, a fractional inductor, and a memristor [54–56], electrical circuits consisting of a fractional capacitor and a delayed memristor [57], and fractional order memristor based neural networks [58], have been studied (It is worth mentioning that the circuitry implementation of fractional order chaotic systems has been also taken into consideration in recent years [59–64]).

Some samples of fractional order mechanical/electromechanical models exhibiting chaotic oscillations can be found in [65–68]. Moreover, the fractional order HIV model, the fractional order cancer model, the fractional order model of pancreatic β-cells, and the fractional order prey predator model are some of the biological models whose chaotic behaviors have been analyzed in [69–72], receptively. Furthermore, [73–76] are some samples of research works dealing with fractional order financial systems having chaotic behaviors.

4.3 Hidden attractors in fractional order systems

The existence of hidden attractors in some fractional order systems is another subject which has been taken into consideration in the literature of fractional order chaotic systems. For instance, in [77–85] some fractional order systems with no equilibrium point have been introduced which possess hidden attractors. In [86] a fractional order chaotic system with no equilibrium point and infinitely many hidden attractors has been analyzed. The existence of hidden chaotic attractors in a fractional order system with a stable equilibrium point has been investigated in [87]. In [88] a fractional order system with two equilibrium points and hidden attractors has been analyzed. A memristor based hyperchaotic fractional order system with a line of equilibrium points, which possesses hidden chaotic attractors, has been investigated in [89]. A fractional order chaotic system having infinite number of equilibrium points, which are placed on a line and a hyperbola, with the potential of generating hidden chaotic attractors has been studied in [90]. A fractional order chaotic system with circular equilibrium points and hidden chaotic attractors has been introduced in [91]. In [92], the hidden chaotic attractors of the fractional order counterpart of a generalized Lorenz system, fractional order version of a non-smooth Chua system, and fractional order Rabinovich-Fabrikant system have been studied.

4.4 Study on fractional order chaotic maps

The discrete version of fractional calculus, called as the fractional discrete calculus [93], provides a framework for describing the fractional difference operators, as generalizations for the simple difference operator. By using these operators, fractional order discrete-time dynamical system are defined. Considering this point, till now different fractional order chaotic maps have been analyzed in literature. Some of these maps are as follows.

- Fractional order logistic map [94,95]
- Fractional order sine map [96]
- Fractional order Ikeda map [97,98]
- Fractional order Henon map [98,99]
- Fractional order Lozi map [100]
- Fractional order Lorenz map [100]
- Fractional order flow map [100]
- Fractional order Grassi–Miller map [101]
- Fractional order unified map [102]

Some other fractional order chaotic maps have been introduced and analyzed in [103–106].

4.5 Chaotic systems defined via using more generalized differentiation operators

Although fractional order differentiation/integration operators are themselves the generalized versions of traditional differentiation/integration operators, more extended forms have been also introduced for these fractional order operators. Baed on these extended operators, which lead to more extended calculuses, more generalized models for describing dynamical systems can be proposed. For instance, variable order derivative of function $f(t)$, as a generalization for the fractional order derivative of this function, can be defined as

$$ {}_0D_t^{\alpha(t)} f(t) = \tfrac{1}{\Gamma(1-\alpha(t))} \int_0^t (t-\tau)^{-\alpha(t)} f'(\tau) d\tau \,, \tag{9}$$

where function $\alpha(t) \in (0,1)$ denotes the order of differentiation operator ${}_0D_t^{\alpha(t)}$ [107]. On the basis of variable order differentiation operators, some variable order chaotic systems have been introduced, and their dynamical behaviors have been analyzed in literature. Some samples of these chaotic systems can be found in [108–112]. It is worth mentioning that by introducing variable order difference operators, discrete-time variable order chaotic systems have been also investigated (For some samples, see [112] and [113]).

Another generalization for fractional order differentiation operator, called as the distributed order differentiator, is defined by

$$ {}_0D_t^{w(\alpha)} f(t) = \int_0^1 w(\alpha) {}_0D_t^{\alpha} f(t) d\alpha \,, \tag{10}$$

in which operator ${}_0D_t^{\alpha}$ is given by (1) and $w(\alpha)$ $(0 \leq \alpha \leq 1)$ specifies the weight function [114]. Using this generalized form of differentiation operator, distributed order dynamical systems can be defined. Considering this point, distributed order systems exhibiting chaotic behaviors have been studied in some research works (For example, [115] and [116]).

4.6 Lyapunov exponents in fractional order systems

Considering the importance of the concept of Lyapunov exponents for analysis of dynamical systems in the viewpoint of sensitivity to initial conditions and consequently its impact in justifying the existence of chaos, some studies on Lyapunov exponents in fractional order systems have been done in literature. For instance, by defining the Lyapunov exponents for fractional order systems in [117], this research work has obtained analytical upper and lower bounds for the values of Lyapunov exponents in fractional order dynamical systems. These bounds have been also simplified in [117] for the special class of linear time invariant fractional order systems. Moreover, in [117] the validity of the found upper bound has been numerically verified for the fractional order Chen system

$$\begin{array}{l} {}_0D_t^{0.92} x = 35(y-x) \\ {}_0D_t^{0.92} y = -8x - xz + 27y \,. \\ {}_0D_t^{0.92} z = xy - 3z \end{array} \tag{11}$$

Another research work dealing with Lyapunov exponents in fractional order systems is [118], in which a semi-analytical method has been proposed to calculate these exponents in fractional order systems. This method, which is also applicable for computation of the Lyapunov exponents in integer order systems, follows two steps:

transforming the original fractional/integer order system to a discrete map, and then finding the Lyapunov exponents of the obtained discrete map via a QR factorization procedure. The well-posedness of Lyapunov exponents in the class of piecewise continuous fractional order systems, described in the form

$$\dot{x} = g(x) + A(x)s(x)\,, \tag{12}$$

where $x \in R^n$, $g : R^n \longrightarrow R^n$ is a continuous function, $A(x)$ denotes a $n \times n$ matrix, and the elements of function $s : R^n \longrightarrow R^n$ are piecewise constant, has been proved in [119]. This obtained result has been verified in the fractional order Sprott system

$$\begin{array}{l} {}_0D_t^{0.99}x = y \\ {}_0D_t^{0.98}y = z \\ {}_0D_t^{0.97}z = -x - y - 0.5z + \mathrm{sign}(x) \end{array}, \tag{13}$$

the piecewise linear fractional order Chen system

$$\begin{array}{l} {}_0D_t^{0.99}x = 1.18(y - x) \\ {}_0D_t^{0.9}y = \mathrm{sign}(x)(0.02 - z) + 0.12y\,, \\ {}_0D_t^{0.999}z = \mathrm{sign}(y)x - 0.16z \end{array} \tag{14}$$

and the fractional order Shimizu–Morioka system

$$\begin{array}{l} {}_0D_t^{0.99}x = y \\ {}_0D_t^{0.97}y = (1 - z)\mathrm{sign}(x) - 0.75y\,. \\ {}_0D_t^{0.98}z = x^2 - 0.45z \end{array} \tag{15}$$

To facilitate the computation of Lyapunov exponents in fractional order systems, a Matlab code, constructed based on the Wolf algorithm [120], has been proposed in [121]. In the code proposed in [121], the part of numerical integration of integer order differential equations, available in the classical Wolf algorithm, has been replaced by the predictor-corrector Adams–Bashforth–Moulton method [122] for numerical integration of fractional order differential equations.

4.7 Chaos control in fractional order dynamics

Andrievskii and Fradkov in their review papers [123] and [124] on the subject of *chaos control*, have categorized three major classes of problems related to this subject:

– Eliminating chaotic oscillations
– Generating chaotic oscillations (Chaotization)
– Synchronization of chaotic systems

In the research field of fractional order chaotic systems, there are numerous works on the above-mentioned classes of research problems. For example, in the context of the first class, till now various control methods have been introduced for suppression of chaotic oscillation via stabilization of the unstable equilibrium points of fractional order chaotic systems. These methods have been originated from different powerful tools in control systems theory, e.g., linear control based methods [125–127], robust control based methods [128–130], adaptive control based methods [131–134], predictive control based methods [135,136], and intelligent control based methods [137–139]. Concerning the first class of problems, there are also some works attempting to reduce the chaotic oscillations in fractional order systems to regular ones (For example, [140–142]). Furthermore, fractional calculus based techniques have been taken into attention to be used for suppression of chaotic oscillations in integer order systems [143–145]. Similarly, the second class of above-mentioned problems has been considered by the researchers working in the field of fractional order chaotic systems (For some relevant works, see [146–148]).The research works have been done regarding the third class of above-mentioned problems are relatively more diverse in comparison to those dealing with the first and the second classes. Synchronization between two identical fractional order chaotic systems [149–153], synchronization between two similar fractional order chaotic systems in the presence of model uncertainties and external disturbances [154–158], synchronization between a fractional order chaotic system and an integer order one [159–161], and synchronization between two different fractional order chaotic systems [162–166] are some samples categories of the works done to solve the third problem.

4.8 Parameter identification in fractional order chaotic systems

Another related topic, which has been devoted attention of the researchers working on fractional order chaotic systems, is parameter identification in these systems. This issue, that can be formulated as an optimization problem, has been considered is some related works, and effective methods have been proposed for solving the optimization problem corresponding to parameter identification in fractional order chaotic systems (For some samples, see [167–174]).

5 Some applications

Till now, different applications have been reported for fractional order chaotic systems. Some of these applications are reviewed in this section.

5.1 Applications in image encryption, sound encryption and compression-encryption techniques

Generally, it has been known that fractional order dynamical systems exhibiting chaotic behaviors can be useful in design of appropriate algorithms for secure communication [175–178]. More specially, such dynamical systems have been used in image encryption algorithms (For some samples, see [179–183]), sound encryption algorithms (See [184] and [185]) and image compression-encryption techniques (For some examples, see [186] and [187]).

5.2 Applications in authentication algorithms

In [188], benefiting from the chaotic fractional order logistic map, an image authentication technique has been introduced. The proposed technique is a four-step one, which in its fourth step fractional order logistic map is applied to generate hash sequences. It is worth noting that fractional order differentiation operators along with integer order chaotic dynamics have been also applied in proposing authentication algorithms. For example, in [189] a fingerprint identification method has been proposed in which fractional order differentiation operators is used for edge detection, and a chaotic system, introduced by Sprott [190], is applied for classification.

5.3 Applications in modeling of the chaotic phenomena

Fractional order dynamical models are good candidate to describe the behavior of chaotic phenomena [191,192]. For instance, in [193], fractional order chaotic dynamics have been used for modeling the neuronal chaotic activities. Also, the chaotic vibrations in viscoelastic plates have been described in [194] by fractional order dynamical models. Moreover, in [195] it has been experimentally verified that the reason of the inconsistency between numerical simulation results and experimental results of a chaotic circuit may be the fractionality nature of the energy storage elements existing in the electrical circuit. Consequently, considering the fractionality nature in the model of the circuit, which yields in a fractional order chaotic model, results in more realistic models.

5.4 Applicability in detection methods for engineering applications

On the basis of synchronization of fractional order chaotic systems, till now different detection methods with the aim of applicability in specific engineering applications have been introduced. For example, in [196] a method of fault diagnosis to be used in ball bearing systems has been proposed, that can detect small changes occurring in signals of a ball bearing system by benefiting from a fractionalized version of the Chen–Lee chaotic system [197]. Also, a real-time method for monitoring the power quality in power systems has been suggested in [198] on the basis of using the fractional order Sprott chaotic system [190]. The suggested method is capable to detect the undesirable disturbances acting on power systems in a high precision framework. Moreover, a real-time intelligent method gaining from fractional order chaotic dynamics has been proposed in [199] for detection of islanding in solar power systems having grid connected topologies. Furthermore, relying on the synchronization of fractional order chaotic systems, an identification method to recognize the chattering caused by machine milling processing has been proposed in [200].

6 Some future challenges

In this section, some open problems in the research field of fractional order chaotic systems are discussed. As future challenges, these open problems invite further research works to enrich our knowledge about chaotic behaviors in fractional order dynamical systems.

6.1 Chaos in continuous-time dynamical systems with a lower number of pseudo-state variables

From the Poincaré–Bendixson theorem [16], it is known that the well-defined integer order systems in the form (3) with less than three state variables (i.e., where $n < 3$) cannot be chaotic. Even though, the placement of a memory element such as time-delay in state-space equations changes the story. In this new situation, chaos can occur in an integer order system of the form

$$\dot{x} = f(x, x_\tau), \tag{16}$$

where $x(t) \in R$ and $x_\tau = x(t - \tau)$ (For example, the simple integer order system $\dot{x} = sin(x(t - 5))$ [201]). Such a conclusion is also valid for the fractional order time-delay systems, i.e., a time-delay system in the form

$${}_0D_t^\alpha x = f(x, x_\tau), (x(t) \in R), \tag{17}$$

can be chaotic (For some samples, see [202] and [203]). Consequently, the existence of the time-delay, as a memory element, in first-order differential equations or fractional differential equations of order $\alpha \in (0,1)$ can cause the occurrence of chaos. On the other hand, a significant difference between fractional derivative and its traditional counterpart is that the first one is a non-local operator, whereas the second operator is a local operator [204]. This means that fractional order differentiation operators in (2) are inherently memory elements [205]. Now, this question is raised: Can the existence of these non-local operators in (2) with $\alpha_i \in (0,1]$ (similar as the influence of the existence of time-delay elements in differential equations) cause the occurrence of chaos in the cases that the number of pseudo-state variables of (2) is less than 3 (i.e., where $n < 3$)? This is an open problem, which can be considered in future research work. It is worth noting that assumption $\alpha_i \in (0,1]$ is necessary in the aforementioned question. Otherwise, chaotic models with less than three pseudo-state variables and the involved orders greater than one have been previously in literature (For example, in [18]).

6.2 Toward finding the minimum effective dimension for the occurrence of chaos

From the earliest research works on chaos analysis in specific fractional order dynamical systems to the most recent ones, finding the minimum effective dimension for the under-study system to generate chaotic oscillations is one of the major research topics. Such a minimum effective dimension, which is definitely related to the definition type of the fractional differentiation operator used in differential equations describing the system dynamics, has been often obtained via numerical searching manners. Due to the influence of numerical methods, which are used for simulation of fractional order systems, on generate fake chaotic oscillations or eliminate the actual chaotic oscillations, some of the reported minimum effective dimensions for the existence of chaos in fractional order systems are not correct (For more details, see [35,39–41]). In [206] by numerically investigating different fractional order systems, a conjecture on the non-existence of chaos, where the effective dimension (i.e., $\sum_{i=1}^{n} \alpha_i$ in (2)) is less than 3, has been stated. Proving/rejecting of this conjecture in an analytical manner, where functions f_i are continuously differentiable, is another interesting problem for future research works (If assumption on differentiability of functions f_i are removed, fractional order chaotic systems with very low effective dimension can be found. For some samples, see [207]). In [208], it has been verified that using different numerical methods for simulation of a fractional order system yield in different minimum effective dimensions for the existence of chaos. The minimum effective dimensions near 1 have been reported in [208] for the existence of chaotic oscillations in numerical simulation results of the fractional order simplified Lorenz system. But, it needs to specify whether the demonstrated chaotic oscillations are the actual ones (i.e., can exist in the actual response of the system) or not. The study done in [209] reveals that fractional order systems in the form (2) with continuously differentiable functions f_i can potentially generate (regular) oscillations, even if they have very low effective dimensions.

6.3 Rigorous proofs for the existence of chaotic attractors in fractional order systems

Most of the papers investigating chaotic behaviors in fractional order systems rely on analysis of numerical simulation results or experimental signals. Since the numerical methods applied for simulations and the measurement devices used in experiments can influence to demonstrate fake chaotic oscillations, analytical tools for proving the existence of chaos in dynamical systems is of great importance [210]. Such analytical tools have been introduced for the integer order systems (For instance, tools constructed on Shil'nikov Theorem [210] and Li-Yorke criterion [211], and the other powerful tools introduced in literature [212,213]). On the basis of these analytical tools, rigorous proofs can be provided for the existence of chaotic attractors in famous integer order chaotic systems [214–216]. But, to the best of the author knowledge, such analytical tools yielding sufficient conditions for the existence of chaos have not been extended for the class of fractional order dynamical systems. In addition to appropriate numerical simulation based methods [217], two analytical approaches are common for chaos detection in fractional order systems. The first approach is based on stability analysis of the equilibrium points in the under-study system (For examples, see [35,36,39–41,209]). Such an approach results in necessary conditions for the occurrence of chaos, or equivalently sufficient conditions for "no-chaos". The second analytical approach for predicting the occurrence of chaos in fractional order systems has been constructed on the basis of describing function (harmonic balance) method (For some samples, see [218–220]). Due to approximate nature of this method [221,222], using the second approach does not result in obtaining sufficient (or even necessary) conditions for the existence of chaos in fractional order systems. Consequently, it is required to establish analytical methods for proving the existence of chaotic attractors in fractional order systems, although it does not seem to be an easy task.

6.4 Finding fractional order chaotic systems whose integer order counterpart has a non-chaotic structure

It has been known that incommensurate order system with involved orders in the range $(0, 1)$ may have unstable equilibrium points, whereas these equilibrium points in their integer order counterpart are stable [223]. Inspired by this result, the following question is raised: Can a fractional order chaotic system be found such that the structure of its integer order counterpart is a non-chaotic one, i.e. cannot generate chaotic oscillations for all involved parameters? Finding a reasonable response for this question invites future research works. It is worth noting that if the answer of the question raised in Section 6.1 is positive, the found fractional order chaotic system with lower than three pseudo-state variables will be a sample system justifying the positiveness of the answer of the above question. Nevertheless, the answer of the above question can be positive, whereas the answer of the question of Section 6.1 is negative.

6.5 The paradox of absence of periodic orbits and route to chaos in fractional order systems

It has been known that chaotic attractors in integer order chaotic systems can be well approximated by their unstable periodic orbits (UPOs) [224–226]. In fact, study on such periodic orbits has increased our knowledge about the occurrence of chaos in integer order chaotic systems [227–229]. But, as a surprising fact, it has been proved that fractional order systems in the form (2) cannot have periodic orbits [230], [231]. It seems that we face a paradox. Chaos occurs in fractional order chaotic systems, whereas no UPO exits in such systems. To address this paradox, more analytical tools for analysis of the complex behaviors in fractional order systems need to be developed.

7 Conclusions

This review paper dealt with researches on fractional order dynamical systems generating chaotic oscillations. From a historical point of view, the pioneering works on the subject of the existence of chaos in fractional order systems were reviewed. Then, a survey on the recent research works done on this subject was presented. Also, applications of fractional order systems with chaotic behaviors were summarized. Moreover, some relevant open problems were described, which can be considered in the future research works on fractional order chaotic systems.

References

1. B. Ross, Hist. Math. **4**, 75 (1977)
2. B. Ross, Math. Mag. **50**, 115 (1977)
3. D. Cafagna, IEEE Ind. Electron. Mag. **1**, 35 (2007)
4. M.D. Ortigueira, IEEE Circuits Syst. Mag. **8**, 19 (2008)
5. A.S. Elwakil, IEEE Circuits Syst. Mag. **10**, 40 (2010)
6. L. Chen, J. Zhang, J. Zhao, W. Cao, H. Wang, J. Zhang, Comput. Phys. Commun. **245**, 106842 (2019)
7. T. Jin, Y. Sun, Y. Zhu, Physica A **534**, 122357 (2019)
8. R. Brociek, D. Słota, M. Król, G. Matula, W. Kwaśny, Int. J. Heat Mass Transfer **143**, 118440 (2019)
9. M. Amabili, P. Balasubramanian, I. Breslavsky, J. Mech. Behav. Biomed. Mater. **99**, 186 (2019)
10. R.K. Miller, T.C. Walker, *Fractals and Chaos: Exploiting Real-World Applications* (SEAI Technical Publications, 1991)
11. G.L. Baker, J.P. Gollub, *Chaotic Dynamics: An Introduction*, 2nd edn. (Cambridge University Press, 2012)
12. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999)
13. K. Diethelm, *The Analysis of Fractional Differential Equations* (Springer, Berlin, 2010)
14. M. Tavakoli-Kakhki, M. Haeri, M.S. Tavazoei, IEEE Trans. Circuits Syst. I: Regul. Pap. **58**, 1099 (2011)
15. T.T. Hartley, C.F. Lorenzo, H.K. Qammer, IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. **42**, 485 (1995)
16. H.K. Khalil, *Nonlinear Systems*, 3rd edn. (Pearson, 2001)
17. T.T. Hartley, F. Mossayebi, J. Circuits Syst. Comput. **3**, 173 (1993)
18. P. Arena, R. Caponetto, L. Fortuna, D. Porto, Int. J. Bifurc. Chaos **8**, 1527 (1998)
19. S. Nimmo, A.K. Evans, Chaos Solitons Fractals **10**, 1111 (1999)
20. J.C. Sprott, Phys. Rev. E **50**, 647 (1994)
21. I. Grigorenko, E. Grigorenko, Phys. Rev. Lett. **91**, 034101 (2003)
22. I. Grigorenko, E. Grigorenko, Phys. Rev. Lett. **96**, 199902 (2006)
23. C. Li, G. Chen, Physica A **341**, 55 (2004)
24. C. Li, G. Chen, Chaos Solitons Fractals **22**, 549 (2004)
25. J.G. Lu, Phys. Lett. A **354**, 305 (2006)
26. J.G. Lu, Chaos Solitons Fractals **26**, 1125 (2005)
27. J.G. Lu, Chin. Phys. **14**, 1517 (2005)
28. J.G. Lu, Chin. Phys. **15**, 301 (2006)
29. G.M. Zaslavsky, A.A. Stanislavsky, M. Edelman, Chaos **16**, 013102 (2006)
30. M. Seredynska, A. Hanyga, Int. J. Bifurc. Chaos **14**, 1291 (2004)
31. F.B.M. Duarte, J.A.T. Machado, Nonlinear Dyn. **22**, 315 (2002)
32. W.M. Ahmad, A.M. Harb, Chaos Solitons Fractals **18**, 693 (2003)
33. I. Petras, Acta Montanistica Slovaca **11**, 273 (2006)
34. W.M. Ahmad, J.C. Sprott, Chaos Solitons Fractals **16**, 339 (2003)
35. M.S. Tavazoei, M. Haeri, Phys. Lett. A **367**, 102 (2007)
36. M.S. Tavazoei, M. Haeri, Physica D **237**, 2628 (2008)

37. M.S. Tavazoei, IEEE Trans. Circuits Syst. II: Express Briefs **56**, 519 (2009)
38. M.S. Tavazoei, IEEE Trans. Circuits Syst. I: Regul. Pap. **62**, 329 (2015)
39. M.S. Tavazoei, M. Haeri, IET Signal Proc. **1**, 171 (2007)
40. M.S. Tavazoei, M. Haeri, S. Bolouki, M. Siami, SIAM J. Numer. Anal. **47**, 321 (2008)
41. M.S. Tavazoei, M. Haeri, Automatica **46**, 94 (2010)
42. X. Lin, B. Liao, C. Zeng, Int. J. Bifurc. Chaos **27**, 1750207 (2017)
43. J. Čermák, L. Nechvátal, Int. J. Bifurc. Chaos **28**, 1850098 (2018)
44. Z. Odibat, N. Corson, M.A. Aziz-Alaoui, A. Alsaedi, Int. J. Bifurc. Chaos **27**, 1750161 (2017)
45. E.F.D. Goufo, Int. J. Bifurc. Chaos **28**, 1850125 (2018)
46. M. Borah, B.K. Roy, Chaos Solitons Fractals **102**, 372 (2017)
47. E. Bonyah, Chaos Solitons Fractals **116**, 316 (2018)
48. E.F.D. Goufo, Chaos Solitons Fractals **104**, 443 (2017)
49. L. Chen, W. Pan, R. Wu, J.A. Tenreiro Machado, A.M. Lopes, Chaos **26**, 084303 (2016)
50. J. Palanivel, K. Suresh, S. Sabarathinam, K. Thamilmaran, Chaos Solitons Fractals **95**, 33 (2017)
51. M.P. Kennedy, IEEE Trans. Circuits Syst. I: Fundam. Theor. Appl. **40**, 657 (1993)
52. D.B. Strukov, G.S. Snider, D.R. Stewart, R.S. Williams, Nature **453**, 80 (2008)
53. I. Petras, IEEE Trans. Circuits Syst. II: Express Briefs **57**, 975 (2010)
54. D. Cafagna, G. Grassi, Nonlinear Dyn. **70**, 1185 (2012)
55. J. Ruan, K. Sun, J. Mou, S. He, L. Zhang, Eur. Phys. J. Plus **133**, 3 (2018)
56. L. Teng, H.H.C. Iu, X. Wang, X. Wang, Nonlinear Dyn. **77**, 231 (2014)
57. W. Hu, D. Ding, Y. Zhang, N. Wang, D. Liang, Optik **130**, 189 (2017)
58. Y. Fan, X. Huang, Z. Wang, Y. Li, Nonlinear Dyn. **93**, 611 (2018)
59. K. Rajagopal, A. Karthikeyan, A. Srinivasan, Nonlinear Dyn. **91**, 1491 (2018)
60. K. Rajagopal, S.T. Kingni, A.J.M. Khalaf, Y. Shekofteh, F. Nazarimehr, Eur. Phys. J. Special Topics **228**, 2035 (2019)
61. A.G. Soriano–Sánchez, C. Posadas–Castillo, M.A. Platas–Garza, A. Arellano–Delgado, Appl. Math. Comput. **332**, 250 (2018)
62. D.K. Shah, R.B. Chaurasiya, V.A. Vyawahare, K. Pichhode, M.D. Patil, Int. J. Electron. Commun. **78**, 245 (2017)
63. X. Zhang, Z. Li, D. Chang, Int. J. Electron. Commun. **82**, 435 (2017)
64. L.F. Ávalos-Ruiz, C.J. Zúñiga-Aguilar, J.F. Gómez-Aguilar, R.F. Escobar-Jiménez, H.M. Romero-Ugalde, Chaos Solitons Fractals **115**, 177 (2018)
65. D. Liu, Y. Xu, J. Li, J. Sound Vib. **399**, 182 (2017)
66. P.R. Nwagoum Tuwa, C.H. Miwadinou, A.V. Monwanou, J.B. Chabi Orou, P. Woafo, Mech. Res. Commun. **97**, 8 (2019)
67. L.M. Anague Tabejieu, B.R. Nana Nbendjo, P. Woafo, Chaos Solitons Fractals **93**, 39 (2016)
68. Y. Long, B. Xu, D. Chen, W. Ye, Appl. Math. Model. **58**, 128 (2018)
69. H. Ye, Y. Ding, Math. Prob. Eng. **2009**, 378614 (2009)
70. I. N'Doye, H. Voos, M. Darouach, Proc. Eur. Control Conf. **2014**, 171 (2014)
71. B. Bodo, A. Mvogo, S. Morfu, Chaos Solitons Fractals **102**, 426 (2017)
72. M. Das, A. Maiti, G.P. Samanta, Ecol. Genet. Genomics **7-8**, 33 (2018)
73. C. Wen, J. Yang, Chaos Solitons Fractals **128**, 242 (2019)
74. G. Zhang, P. Qian, Z. Su, Chaos Solitons Fractals **128**, 219 (2019)
75. H. Wang, C. Weng, Z. Song, J. Cai, Chaos Solitons Fractals **131**, 109462 (2020)
76. C. Huang, L. Cai, J. Cao, Chaos Solitons Fractals **113**, 326 (2018)
77. V.T. Pham, S.T. Kingni, C. Volos, S. Jafari, T. Kapitaniak, Int. J. Electron. Commun. **78**, 220 (2017)
78. S. Shao, M. Chen, IEEE/CAA J. Autom. Sinica **6**, 1000 (2019)
79. V.T. Pham, A. Ouannas, C. Volos, T. Kapitaniak, Int. J. Electron. Commun. **86**, 69 (2018)
80. J. Mishra, Chaos Solitons Fractals **116**, 43 (2018)
81. E.F.D. Goufo, Chaos Solitons Fractals **119**, 24 (2019)

82. S. Fang, Z. Li, X. Zhang, Y. Li, Braz. J. Phys. **49**, 846 (2019)
83. K. Rajagopal, A. Karthikeyan, A.K. Srinivasan, Nonlinear Dyn. **87**, 2281 (2017)
84. D. Cafagna, G. Grassi, Chin. Phys. B **24**, 080502 (2015)
85. J.M. Munoz-Pacheco, E. Zambrano-Serrano, C. Volos, S. Jafari, J. Kengne, K. Rajagopal, Entropy **20**, 564 (2018)
86. J.M. Munoz-Pacheco, Eur. Phys. J. Special Topics **228**, 2185 (2019)
87. S.T. Kingni, S. Jafari, H. Simo, P. Woafo, Eur. Phys. J. Plus **129**, 76 (2014)
88. Z. Wei, A. Akgul, U.E. Kocamaz, I. Moroz, W. Zhang, Chaos Solitons Fractals **111**, 157 (2018)
89. C. Volos, V.T. Pham, E. Zambrano-Serrano, J.M. Munoz-Pacheco, S. Vaidyanathan, E. Tlelo-Cuautle, Advances in memristors, memristive devices and systems, in *Studies in Computational Intelligence*, edited by S. Vaidyanathan, C. Volos (Springer, 2017), Vol. 701, pp. 207–235
90. S.T. Kingni, V.T. Pham, S. Jafari, P. Woafo, Chaos Solitons Fractals **99**, 209 (2017)
91. S.T. Kingni, V.T. Pham, S. Jafari, G.R. Kol, P. Woafo, Circuits Syst. Signal Process. **35**, 1933 (2016)
92. M.F. Danca, Nonlinear Dyn. **89**, 577 (2017)
93. C. Goodrich, A.C. Peterson, *Discrete Fractional Calculus* (Springer, 2015)
94. G.C. Wu, D. Baleanu, Nonlinear Dyn. **75**, 283 (2014)
95. Y.D. Ji, L. Lai, S.C. Zhong, L. Zhang, Commun. Nonlinear Sci. Numer. Simul. **57**, 352 (2018)
96. G.C. Wu, D. Baleanu, S.D. Zeng, Phys. Lett. A **378**, 484 (2014)
97. A. Ouannas, A.A. Khennaoui, Z. Odibat, V.T. Pham, G. Grassi, Chaos Solitons Fractals **123**, 108 (2019)
98. L. Huang, L. Wang, D. Shi, IEEE/CAA J. Autom. Sinica, https://doi.org/10.1109/JAS.2016.7510148
99. M.K. Shukla B.B. Sharma, Int. J. Electron. Commun. **78**, 265 (2017)
100. A.A. Khennaoui, A. Ouannas, S. Bendoukha, G. Grassi, R. Lozi, V.T. Pham, Chaos Solitons Fractals **119**, 150 (2019)
101. A. Ouannas, A.A. Khennaoui, G. Grassi, S. Bendoukha, J. Comput. Appl. Math. **358**, 293 (2019)
102. A.A. Khennaoui, A. Ouannas, S. Bendoukha, X. Wang, V.T. Pham, Entropy **20**, 530 (2018)
103. Y. Peng, K. Sun, D. Peng, W. Ai, Physica A **525**, 96 (2019)
104. S. He, K. Sun, Y. Peng, Phys. Lett. A **383**, 2267 (2019)
105. Z. Liu, T. Xia, Appl. Comput. Inf. **14**, 177 (2018)
106. A. Ouannas, A.A. Khennaoui, S. Bendoukha, G. Grassi, Int. J. Bifurc. Chaos **29**, 1950078 (2019)
107. R. Almeida, D. Tavares, D.F.M. Torres, *The Variable-order Fractional Calculus of Variations,* Springer Briefs in Applied Sciences and Technology (Springer, 2019)
108. S. He, S. Banerjee, K. Sun, Chaos Solitons Fractals **115**, 14 (2018)
109. C.J. Zuñiga-Aguilar, J.F. Gómez-Aguilar, R.F. Escobar-Jiménez, H.M. Romero-Ugalde, Eur. Phys. J. Plus **133**, 13 (2018)
110. L. Zhang, Ch. Yu, T. Liu, Nonlinear Dyn. **86**, 1967 (2016)
111. L.F. Ávalos-Ruiz, C.J. Zúñiga-Aguilar, J.F. Gómez-Aguilar, R.F. Escobar-Jiménez, H.M. Romero-Ugalde, Chaos Solitons Fractals **115**, 083103 (2018)
112. G.C. Wu, Z.G. Deng, D. Baleanu, D.Q. Zeng, Chaos **29**, 083103 (2019)
113. G.C. Wu, D. Baleanu, H.P. Xie, S.D. Zeng, Int. J. Bifurc. Chaos **26**, 1650013 (2016)
114. Z. Jiao, Y.Q. Chen, I. Podlubny, *Distributed-order Dynamic Systems: Stability, Simulation, Applications and Perspectives*, Springer Briefs in Electrical and Computer Engineering/Springer Briefs in Control, Automation and Robotics (Springer, 2012)
115. H. Taghavian, M.S. Tavazoei, Int. J. Syst. Sci. **49**, 523 (2018)
116. T.T. Hartley, C.F. Lorenzo, Int. J. Bifurc. Chaos **22**, 1250253 (2012)
117. C. Li, Z. Gong, D. Qian, Y.Q. Chen, Chaos **20**, 013127 (2010)
118. R. Caponetto, S. Fazzino, Commun. Nonlinear Sci. Numer. Simul. **18**, 22 (2013)
119. M.F. Danca, Nonlinear Dyn. **81**, 227 (2015)

120. A. Wolf, J.B. Swift, H.L. Swinney, J.A. Vastano, Physica D **16**, 285 (1985)
121. M.F. Danca, N. Kuznetsov, Int. J. Bifurc. Chaos **28**, 1850067 (2018)
122. K. Diethelm, N. Ford, A. Freed, Nonlinear Dyn. **29**, 3 (2002)
123. B.R. Andrievskii, A.L. Fradkov, Autom. Remote Control **64**, 673 (2003)
124. B.R. Andrievskii, A.L. Fradkov, Autom. Remote Control **65**, 505 (2004)
125. M.S. Tavazoei, J. Comput. Nonlinear Dyn. **6**, 031012 (2011)
126. M.S. Tavazoei, M. Haeri, Eur. J. Control **14**, 247 (2008)
127. M.S. Tavazoei, M. Haeri, J. Dyn. Syst. Meas. Control **132**, 021008 (2010)
128. B. Aguiar, T. González, M. Bernal, IEEE Trans. Autom. Control **61**, 2796 (2016)
129. R. Luo, H. Su, Complexity **2018**, 2796 (2018)
130. S. Zhang, H. Liu, S. Li, Adv. Difference Equ. **2018**, 412 (2018)
131. Y. Zheng, Y. Nian, D. Wang, Phys. Lett. A **375**, 125 (2010)
132. K. Khettab, S. Ladaci, Y. Bensafia, IEEE/CAA J. Autom. Sinica **6**, 125 (2019)
133. L. Shaohua, L. Shaobo, F. Tajaddodianfar, H. Jianjun, J. Franklin Inst. **355**, 6435 (2018)
134. R. Li, W. Li, Optik **126**, 2965 (2015)
135. Y. Zheng, Z. Ji, Chaos Solitons Fractals **87**, 307 (2016)
136. Y. Zheng, Optik **126**, 5645 (2015)
137. Y. Wu, H. Lv, AIP Adv. **6**, 085121 (2016)
138. G. Li, C. Sun, Adv. Difference Equ. **2019**, 148 (2019)
139. B. Bourouba, S. Ladaci, Algorithms **11**, 101 (2018)
140. J.S.H. Tsai, T.H. Chien, S.M. Guo, Y.P. Chang, L.S. Shieh, IEEE Trans. Circuits Syst. I: Regul. Pap. **54**, 632 (2007)
141. K. Su, C. Li, Optik **126**, 2671 (2015)
142. A. Gjurchinovski, T. Sandev, V. Urumov, J. Phys. A: Math. Theor. **43**, 445102 (2010)
143. M.S. Tavazoei, M. Haeri, Phys. Lett. A **372**, 798 (2008)
144. M.S. Tavazoei, M. Haeri, S. Jafari, S. Bolouki, M. Siami, IEEE Trans. Ind. Electron. **55**, 4094 (2008)
145. M.S. Tavazoei, M. Haeri, S. Jafari, Proc. Inst. Mech. Eng. Part I: J. Syst. Control Eng. **222**, 175 (2008)
146. E. Zambrano-Serrano, J.M. Muñoz-Pacheco, E. Campos-Cantón, Int. J. Electron. Commun. **79**, 43 (2017)
147. M.S. Tavazoei, M. Haeri, Nonlinear Anal. Ser. B: Real World Appl. **11**, 332 (2010)
148. W. Xingyuan, Q. Xue, Math. Prob. Eng. **2012**, 601309 (2012)
149. I. N'Doye, K.N. Salama, T.M. Laleg-Kirati, IEEE/CAA J. Autom. Sinica **6**, 268 (2019)
150. M. Bettayeb, U.M. Al-Saggaf, S. Djennoune, IET Control Theory Appl. **11**, 3171 (2017)
151. M.S. Tavazoei, M. Haeri, Physica A **387**, 57 (2008)
152. Z. Alam, L. Yuan, Q. Yang, IEEE/CAA J. Autom. Sinica **3**, 157 (2016)
153. X.L. Gong, X.H. Liu, X. Xiong, Physica A **522**, 33 (2019)
154. L. Chen, Y. Chai, R. Wu, Chaos **21**, 043107 (2011)
155. A.K. Singh, V.K. Yadav, S. Das, Optik **133**, 98 (2017)
156. M. Chen, S.Y. Shao, P. Shi, Y. Shi, IEEE Trans. Circuits Syst. II: Express Briefs **64**, 417 (2017)
157. P. Jafari, M. Teshnehlab, M. Tavakoli-Kakhki, IET Control Theory Appl. **12**, 183 (2018)
158. F. Meléndez-Vázquez, R. Martínez-Guerra, IET Control Theory Appl. **12**, 1755 (2018)
159. D. Chen, R. Zhang, J.C. Sprott, H. Chen, X. Ma, Chaos **22**, 023130 (2012)
160. X.Q. Zhang, J. Xiao, Q. Zhang, Optik **130**, 1139 (2017)
161. Q. Zhang, J. Xiao, X.Q. Zhang, D. Cao, Optik **141**, 90 (2017)
162. Y. Li, H. Lv, D. Jiao, AIP Adv. **7**, 035106 (2017)
163. P. Zhoua, R. Ding, J. Franklin Inst. **348**, 2839 (2011)
164. C.D. Cruz-Ancona, R. Martínez-Guerra, J. Franklin Inst. **354**, 3054 (2017)
165. P. Zhou, W. Zhu, Nonlinear Anal.: Real World Appl. **12**, 811 (2011)
166. W. Zhang, J. Cao, R. Wu, F.E. Alsaadi, A. Alsaedi, J. Franklin Inst. **356**, 1522 (2019)

167. D.A. Yousri, A.M. Abdel, A.A. Said, A.S. Elwakil, B. Maundy, A.G. Radwan, Nonlinear Dyn. **95**, 2491 (2019)
168. F. Liu, X. Li, Z. Liu, Y. Tang, Syst. Sci. Control Eng. **5**, 42 (2016)
169. L. Yuan, Q. Yang, C. Zeng, Nonlinear Dyn. **73**, 2491 (2013)
170. Y. Huang, F. Guo, Y. Li, Y. Liu, PLoS One **10**, e0114910 (2015)
171. E. Cuevas, J. Gálvez, O. Avalos, Comput. Sistemas **21**, 369 (2017)
172. Y. Al-Assaf, R. El-Khazali, W. Ahmad, Chaos Solitons Fractals **22**, 897 (2004)
173. Y. Tang, X. Zhang, C. Hua, L. Li, Y. Yang, Phys. Lett. A **376**, 457 (2012)
174. J. Lin, Z.G. Wang, Nonlinear Dyn. **90**, 1243 (2017)
175. R.G. Li, H.N. Wu, Nonlinear Dyn. **95**, 1221 (2019)
176. J. Liu, Z. Wang, M. Shu, F. Zhang, S. Leng, X. Sun, Complexity **2019**, 7242791 (2019)
177. I. N'Doye, H. Voos, M. Darouach, IEEE Journal on Emerg. Sel. Top. Circuits Syst. **3**, 442 (2013)
178. R.G. Li, H.N. Wu, Nonlinear Dyn. **92**, 935 (2018)
179. P. Muthukumar, P. Balasubramaniam, K. Ratnavelu, Chaos **24**, 033105 (2014)
180. P. Mani, R. Rajan, L. Shanmugam, Y.H. Joo, Inf. Sci. **491**, 74 (2019)
181. Y.R. Bai, D. Baleanuc, G.C. Wua, Optik **168**, 553 (2018)
182. J. Hou, R. Xi, P. Liu, T. Liu, IEEE/CAA J. Autom. Sinica **4**, 381 (2017)
183. J.J. Montesinos-García, R. Martinez-Guerra, IET Image Process. **12**, 1913 (2018)
184. A.J.A. El-Maksoud, A.A.A. El-Kader, B.G. Hassan, N.G. Rihan, M.F. Tolba, L.A. Said, A.G. Radwan, M.F. Abu- Elyazeed, Microelectron. J. **90**, 323 (2019)
185. L.J. Sheu, Nonlinear Dyn. **65**, 103 (2011)
186. Y.G. Yang, B.W. Guan, J. Li, D. Li, Y.H. Zhou, W.M. Shi, Opt. Laser Technol. **119**, 105661 (2019)
187. M. Lahdir, H. Hamiche, S. Kassim, M. Tahanout, K. Kemih, S.A. Addouche, Opt. Laser Technol. **109**, 534 (2019)
188. F. Tao, W. Qian, Measurement **134**, 866 (2019)
189. J.L. Chen, C.H. Huang, Y.C. Du, C.H. Lin, IET Image Proc. **8**, 354 (2014)
190. J.C. Sprot, Phys. Lett. A **266**, 19 (2000)
191. K.M. Owolabi, A. Atangana, Chaos **29**, 023111 (2019)
192. A. Atangana, J.F. Gómez-Aguilar, Chaos Solitons Fractals **14**, 516 (2018)
193. E.F. Doungmo Goufo, M. Mbehou, M.M. Kamga Pene, Chaos Solitons Fractals **115**, 170 (2018)
194. P.R. Nwagoum Tuwa, C.H. Miwadinou, A.V. Monwanou, J.B. Chabi Orou, P. Woafo, Mech. Res. Commun. **97**, 8 (2019)
195. S. Faraji, M.S. Tavazoei, Cent. Eur. J. Phys. **11**, 836 (2013)
196. H.T. Yau, S.Y. Wu, C.L. Chen, Y.C. Li, IEEE Trans. Ind. Electron. **63**, 3824 (2016)
197. H.K. Chen, C.I. Lee, Chaos Solitons Fractals **21**, 957 (2004)
198. H.T. Yau, C.C. Wang, C.T. Hsieh, S.Y. Wu, IET Gener. Transm. Distrib. **9**, 2775 (2015)
199. M.H. Wang, H.T. Yau, Energies **7**, 6340 (2014)
200. C.K. Chen, Y.C. Li, Int. J. Adv. Manuf. Technol. **100**, 907 (2019)
201. J.C. Sprott, Phys. Lett. A **366**, 397 (2007)
202. D.P. Wang, J.B. Yu, J. Electro. Sci. Technol. Chin. **6**, 289 (2008)
203. M. Busłwicz, A. Makarewicz, Pomiary Automatyka Robotyka **2**, 321 (2013)
204. R. Almeida, M. Guzowska, T. Odzijewicz, Open Math. **14**, 1122 (2016)
205. S. Westerlund, Phys. Scr. **43**, 174 (1991)
206. A.S. Deshpande, V. Daftardar-Gejji, Chaos Solitons Fractals **102**, 119 (2017)
207. M.S. Tavazoei, M. Haeri, Nonlinear Dyn. **54**, 213 (2008)
208. D. Peng, K. Sun, S. He, A.O.A. Alamodi, Chaos Solitons Fractals **119**, 163 (2019)
209. M.S. Tavazoei, J. Comput. Nonlinear Dyn. **9**, 021011 (2014)
210. C.P. Silva, IEEE Trans. Circuits Syst. I: Fundam. Theory App. **40**, 675 (1993)
211. T.Y. Li, J.A. Yorke, Am. Math. Mon. **82**, 985 (1975)
212. E. Akin, E. Glanser, W. Huang, S. Shao, X. Ye, Ergodic Theory Dyn. Syst. **30**, 1277 (2010)
213. T. Mitra, J. Econ. Theory **96**, 133 (2001)

214. W. Tucker, C.R. Acad. Sci., Ser. I Math. **328**, 1197 (1999)
215. I. Stewart, Nature **406**, 948 (2000)
216. T. Zhou, Y. Tang, G. Chen, Int. J. Bifurc. Chaos **14**, 3167 (2004)
217. S. He, K. Sun, Y. Peng, Phys. Lett. A **383**, 2267 (2019)
218. M.S. Tavazoei, M. Haeri, Nonlinear Dyn. **57**, 363 (2009)
219. Z.M. Wu, J.G. Lu, J.Y. Xie, Chin. Phys. **15**, 1201 (2006)
220. H. Zhu, Discrete Dyn. Nat. Soc. **2013**, 593856 (2013)
221. S. Kou and K. Han, IEEE Trans. Autom. Control **20**, 291 (1975)
222. S. Engelberg, IEEE Trans. Autom. Control **47**, 1887 (2002)
223. V. Badri, M.S. Tavazoei, IEEE Trans. Circuits Syst. Express Briefs, http://doi.org/10.1109/TCSII.2019.2922771
224. P. So, Scholarpedia **2**, 1353 (2007)
225. Y. Saiki, Nonlinear Processes Geophys. **14**, 615 (2007)
226. E. Ott, *Chaos in Dynamical Systems*, 2nd edn. (Cambridge University Press, 1993)
227. R. Barrio, A. Dena, W. Tucker, Comput. Phys. Commun. **194**, 76 (2015)
228. B.M. Boghosian, A. Brown, J. Lätt, H. Tang, L.M. Fazendeiro, P.V. Coveney, Philos. Trans. R. Soc. London, Ser. A **369**, 2345 (2011)
229. X. Yu, Y. Xia, Int. J. Bifurc. Chaos **10**, 1987 (2000)
230. M.S. Tavazoei, M. Haeri, Automatica **45**, 1886 (2009)
231. M.S. Tavazoei, Automatica **46**, 945 (2010)