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Occupational arbitrage equilibrium as an entropy maximizing solution

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Abstract. A standard critique of attempts to apply entropy-maximizing perspectives to income distribution phenomena in economics is that they do not have appropriate characterizations of individuals making choices, which is at the core of economic modeling. This paper presents a possible bridge between these two seemingly separate universes of discourse. With a specific illustration we show that a conventional model of choice between occupations by individuals can lead to an economic equilibrium which can also be characterized as an outcome which maximizes the entropy of the distribution of individuals across occupations (and hence across incomes). This occupational choice interpretation can provide economic and institutional basis to what has, up to now, often been characterized as somewhat mechanical translation of methods from one discipline to another, without substantive content. The illustrations provided in the paper are a first step in exploring the possible linkages between occupational choice and maximum entropy approaches in modelling income distribution outcomes.

1 Occupational arbitrage equilibrium in economics

Consider the following model of returns to economic occupations or activities. There are n occupations, indexed by $i = 1, 2, \dots, n$. There are N individuals indexed by $j = 1, 2, \dots, N$. Let the set of individuals engaged in activity i be denoted A_i with the number of such individuals denoted N_i and the fraction $x_i = N_i/N$. Denote the vector $\bar{x} = (x_1, \dots, x_n)$. Clearly,

$$\sum_{i=1}^{n} x_i = 1. \tag{1}$$

Let the return to an individual j from engaging in occupation i be denoted h_{ij} and let this depend on three factors. First is a factor which is specific to occupation i, denoted v_i. Second is the distribution of population across all occupations, in other words the set $A = [A_i]_{i=1}^{n}$ where A_i is the set of all individuals in occupation i. Third is a vector of characteristics specific to individual j, δ_j. Thus

$$h_{ij} = h\left(v_i, A, \delta_j\right). \tag{2}$$

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An allocation A^* of the N individuals across the n activities, and the associated returns h^*_{ij}, is an equilibrium if and only if

$$h^*_{ij} \geq h^*_{kj} \forall j \quad \text{and} \quad \forall\, k \neq i. \tag{3}$$

In other words, no single individual could get a strictly higher return by moving to an alternative occupation or activity.

While the above statement is useful to position the problem at a certain level of generality, it is clear that we will have to specify the $h(.)$ function further to get tractable insights and conclusions. At this level of generality, even the existence of equilibrium is not guaranteed, depending as it does on the specific properties of the function h. But helpful specialization is possible if we assume that (i) all individuals are identical, i.e., $\delta_j = \delta$ for all j – since this is a constant parameter now, we do not mention it explicitly; (ii) h depends not on the detailed allocation of individuals to occupations, but only on the number of them in each occupation. Normalizing by total population size we have that (2) becomes

$$h_{ij} = h_i = h\left(v_i, \bar{x}\right). \tag{4}$$

The return to any individual in activity i is given as a function of v_i, a characteristic specific to occupation i, and the distribution of population across the occupations as given by the vector $\bar{x} = (x_1, \ldots, x_n)$. Now existence can be built on the base of a set of conditions on the function h, in particular that h_i be decreasing in x_i and increasing in $x_k, k \neq i$. With this we have an equilibrium allocation of population across the activities $\bar{x}^*$ such that

$$h^*_i = h\left(v_i, \bar{x}^*\right) = h^* \ \forall i. \tag{5}$$

Solving the equations (5) together with the constraint (1) gives us the equilibrium distribution of population across occupations, $x^*_i, i = 1, 2, \ldots, n$.

The general formulation in (3) and the specialization in (5) capture between them a large range of phenomena that have been modeled in the economics literature. We restrict ourselves here to an illustration with three cases.

Case 1. Adam Smith and smuggling

First and foremost, the general idea that high returns in certain activities pull people into those activities, and such a reallocation tends to reduce returns in those activities, is presented by Adam Smith [21] in the founding text of modern economics, *An Inquiry into the Nature and Causes of the Wealth of Nations*, generally referred to, of course, as *The Wealth of Nations.* The opening paragraph of Book 1, Chapter 10, provides a classic statement of the forces in play in occupational choice and equilibrium:

> "The whole of the advantages and disadvantages of the different employments of labor and stock must, in the same neighborhood, be either perfectly equal or continually tending to equality. If in the same neighborhood, there was any employment evidently either more or less advantageous than the rest, so many people would crowd into it in the one case, and so many would desert it in the other, that its advantages would soon return to the level of other employments. This at least would be the case in a society where things were left to follow their natural course, where there was perfect liberty, and where every man was perfectly free both to choose what occupation he thought proper, and to change it as often as he thought proper. Every man's interest would prompt him to seek the advantageous, and to shun the disadvantageous employment."

Smith goes on to give many examples of the process and the equilibrium, sometimes in colorful terms. Here is his account of returns to a particular risky occupation:

> "Bankruptcies are most frequent in the most hazardous trades. The most hazardous of all trades, that of a smuggler, though when the adventure succeeds it is likewise the most profitable, is the infallible road to bankruptcy. The presumptuous hope of success seems to act here as upon all other occasions, and to entice so many adventurers into those hazardous trades, that their competition reduces their profit below what is sufficient to compensate the risk. To compensate it completely, the common returns ought, over and above the ordinary profits of stock, not only to make up for all occasional losses, but to afford a surplus profit to the adventurers of the same nature with the profit of insurers."

Case 2. Entrepreneurship and safety

To move from this general discussion to specifics requires further detailing of the economics of returns of different activities and how individuals respond to them. Consider then a second example from the economics literature, in the arena of choice between risky occupations. Friedman [7] followed the lead of Smith as follows, looking at Robinson Crusoe on his island:

> "At any moment, Robinson Crusoe has many courses of action open to him – that is, different ways of using his time and the resources on the island. He can cultivate the arable land intensively or extensively, make one or another kind of capital goods to assist in cultivation, hunt or fish or do both, and so on in infinite variety ... the actual result of the course of action adopted depends not only on what Robinson Crusoe does but also on such chance events as the weather, the number of fish in the neighborhood when he happens to fish, the quality of the seed he plants, the state of his health, and so on." (p. 279)

With this framework, Friedman [7] goes on to predict income distribution in society based on risk characteristics of activities and attitudes towards risk of individuals.

However, Kanbur [12] argues that Friedman [7] neglects a key insight of Adam Smith that individual returns to an activity depend on the number of individuals in that activity. As more individuals crowd into an attractive looking occupation, returns to that activity will fall, and will continue to do so till the advantage is removed – in other words, the equilibrium, is as set out in (5). Kanbur [12] goes on to the specific occupational arbitrage equilibrium as follows:

> "... suppose that there are n activities obeying probability densities $f(y; x_1, x_2, \ldots, x_n)$ where y is the return and f_i the density in the ith activity, shown to depend on x_i, the proportion of population engaged in the various activities. Let the utility function of the identical agents be $U(y)$. Then equilibrium is defined by a set of x_i such that $EU(y) = \int U(y) f_i(y; x_1^*, \ldots, x_n^*) dy = \int U(y) f_j(y; x_1^*, \ldots, x_n^*) dy$, for all i, j, where E is the expectation operator." (p. 772)

The core specification thus makes an identification of h_i in (4) with $\int U(y) f_i(y; x_1, \ldots, x_n)\, dy$, and the application of (5) becomes

$$\int U(y) f_i(y; x_1^*, \ldots, x_n^*) dy = EU^*(y) \quad \text{for all} \quad i. \tag{6}$$

After further specification of the dependence of the densities f_i on the x_i based on economic production functions, and the shape of the utility function $U(y)$, Kanbur

[12] derives explicit expressions for income distribution and inequality as a function of attitudes to risk. Let there be two occupations, one of them safe wage labor which receives certain wage, and the other uncertain entrepreneurship which hires labor but in which each individual's productivity is not known till after the occupation is entered. Kanbur [12] assumes that output is given by

$$q = \theta l^{\alpha} \tag{7}$$

where l is labor hired by entrepreneur of productivity α. If the wage is w, entrepreneurial profit is given by

$$y = \theta l^{\alpha} - wl. \tag{8}$$

Profit maximizing labor demand, and maximized profit are given by

$$l = \left(\frac{\theta\alpha}{w}\right)^{\frac{1}{1-\alpha}} \tag{9}$$

$$y = \left[(1-\alpha)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}\right]\theta^{\frac{1}{1-\alpha}}. \tag{10}$$

Assume that (i) labor demand is determined after entry to occupation and after knowledge of productivity, (ii) the density of θ is lognormal,

$$\theta \sim \Lambda\left(\mu, \sigma^2\right) \tag{11}$$

and (iii) the utility function displays constant relative risk aversion R:

$$U(y) = \frac{y^{1-R}}{1-R} \quad \text{if} \quad R \neq 1 \quad \text{and} \quad \log R \quad \text{if} \quad R = 1. \tag{12}$$

Kanbur [12] then solves for the full occupational choice equilibrium which leaves individuals indifferent between expected utility in the two occupations, and the wage adjusts to clear the labor market, with those who choose to be entrepreneurs hiring those who choose to be workers. In this society, as attitudes to risk become less risk averse (lower R), more people crowd into the risky occupation in Smithian fashion, depressing returns there and raising the safe wages till equality of expected utility is restored. Further, the overall income distribution is a mixture of the lognormal distribution of profits in entrepreneurship and the safe wage, the weights being the equilibrium population distribution. Kanbur [12] then carries out an analysis of how overall inequality changes when individuals become less risk averse.

Case 3. Rural-urban migration

As a third and final example, consider a classic paper in development economics by Harris and Todaro [9]. The paper sets out to account for rural-urban migration, or the allocation of population across rural and urban sectors, in developing countries:

> "The basic model which we shall employ can be described as a two-sector internal trade model with unemployment. The two sectors are the permanent urban and the rural ... The crucial assumption to be made in our model is that rural-urban migration will continue so long as the expected urban real income at the margin exceeds real agricultural product – i.e., prospective rural migrants behave as maximizers of expected utility ... (p. 127) ... an equilibrium condition, is derived from the hypothesis that migration to the urban area is a positive function of the urban-rural expected wage differential ... clearly then, migration will cease only when the expected income differential is zero ..." (p. 129)

The model is further specified to capture the Smithian postulate that as population crowds into an activity, returns from that activity to any individual will fall. In this case, the model is that as the urban population increases unemployment in the urban area increases, thereby reducing the attractiveness of that activity (or, in this case, that location) since the prospects of high income in the urban area have to be balanced against the possibility of unemployment. The Harris–Todaro [9] equilibrium is then an allocation of population across the rural and urban sectors which equalizes expected income (h_i) in the two locations ($i = 1$ being rural, say, and $i = 2$ being urban or the city).

A specialized version of the Harris–Todaro model, developed first in Todaro [23] suffices to illustrate the basic equilibrating processes. We follow the exposition in Christiaensen et al. [3]. There is a single, certain, rural income denoted w_r. In the city, however, there are two incomes, a high "formal employment" income w_c and a low "informal employment" income w_r:

$$w_c > w_r > w_{oc}. \tag{13}$$

Total population is $\bar{N}$, and the populations of the two locations are denoted N_r, and N_c. Formal and informal employment in the city are respectively E_c and I_c,

$$N_c = E_c + I_c. \tag{14}$$

The probability of getting a formal sector job is then specified as $E_c/(E_c + I_c)$. The basic literature then goes on to assume identical individuals who choose location on the basis of expected income (i.e., they are risk neutral). Thus the equilibrium condition analogous to (5) and (6) is given by:

$$w_r = [E_c/\left(E_c + I_c\right)]\, w_c + [1 - E_c/\left(E_c + I_c\right)]\, w_{oc}. \tag{15}$$

Equation (15) together with total population solves the distribution of population across the two locations r and c, and also within c between formal and informal employment:

$$N_c = \left[\frac{w_c - w_{oc}}{w_r - w_{oc}}\right] E_c;\ I_c = \left[\frac{w_c - w_r}{w_r - w_{oc}}\right] E_c;\ \bar{N} = N_r + N_c. \tag{16}$$

This equilibrium has been extended and elaborated, but it has been the mainstay of many branches of development economics (for a recent use of the basic framework, see [2]). The central point, however, is that the full income distribution is determined through occupational choice equilibrium as a special case of (5).

2 Towards entropy maximization equivalence

Consider now a fourth case, where $h_i = h\left(v_i, \bar{x}\right)$ from (4) is specified, following Venkatasubramanian et al. [25], as follows:

$$h_i = v_i - \gamma \ln x_i. \tag{17}$$

The choice of $\ln x_i$ for the *congestion* or the *competition* term, i.e., for the reduction in an agents utility due to "overcrowding" from others in the occupational category i, is a particularly interesting one (we discuss this in more detail in the next section). As Venkatasubramanian [26] explains, this choice leads to the direct connection between entropy maximization in statistical mechanics and potential maximization in potential game theory. That is, the statistical equilibrium of statistical mechanics and the arbitrage equilibrium of game theory are shown to be equivalent under this choice.

Now if we apply the equilibrium condition (5) together with condition (1)–(17), we get the equilibrium distribution of populations across occupations as:

$$x_i^* = \frac{e^{\frac{v_i}{\gamma}}}{\sum_i e^{\frac{v_i}{\gamma}}}. \tag{18}$$

This is, of course, the Gibbs equation which is the solution to the following maximum entropy problem:

$$\begin{aligned} &\mathrm{Max}_{x_i} - \sum_i x_i \ln x_i \\ &\text{subject to} \\ &\qquad \sum_i x_i v_i = V \\ &\qquad \sum_i x_i = 1. \end{aligned} \tag{19}$$

Further, from (18), if

$$v_i = \alpha \ln S_i - \beta (\ln S_i)^2 \tag{20}$$

then it can be shown that x_i^* has a lognormal density [25,26]. Note that in this case the constraint $\sum\limits_i x_i v_i = V$ is not applicable, as discussed by Venkatasubramanian [26].

On the other hand, if

$$v_i = \bar{S} - S_i \tag{21}$$

then x_i^* has an exponential density [1]. Different specifications of the functional relationship between v_i and S_i lead to different densities for the equilibrium density x_i^* as a function of S_i. For example, the Gamma, Beta, Weibull, Fisk and many others can be generated by appropriate specification of the functional relationship (see Tab. 1 of [15]).

3 Entropy maximization, arbitrage equilibrium, and income distribution

There are thus several steps needed to convert the general statement of occupation arbitrage equilibrium to an equivalence with entropy maximizing distribution of population across occupations. In other words, we need to move sequentially to greater and greater degrees of specialization, from (3) to (4) to (5) to (17). And then beyond that to a specification like (20) or (21) to arrive at a specific income distribution density. Along the way we encounter substantive traditions in economics, from Smith to Friedman to Harris–Todaro, each of which addresses an economic issue through the lens of occupational arbitrage equilibrium, but none of which necessarily falls into the specialization depicted in (17) and then (20), or (21).

Of these steps, the crucial one is the particular choice of $\ln x_i$ in (17) for the congestion term. As Venkatasubramanian ([26], pp. 61, 62, 120, 121, 246–248) discusses, without this choice one does not get the connection to entropy maximization. This term leads directly to entropy in statistical mechanics and information theory. So, as Venkatasubramanian [26] points out, all the econophysics models in the literature on income distributions have been making this assumption *implicitly* without realizing it. One can miss this point easily if one stays entirely in the statistical mechanics space. However, if one invokes the occupational choice equilibrium perspective, one is forced to reckon with this issue. Here equilibrium is reached when the payoffs or utilities of all choices are exactly the same. In other words, equilibrium is reached when the opportunity for arbitrage, i.e., the ability to increase one's payoff or utility by simply switching to another option at low or no cost, disappears. This arbitrage equilibrium (demanded explicitly by the game theoretic formulation in [25]) makes one recognize the importance of the special choice of $\ln x_i$ for the connection to entropy maximization happen, as we will see below.

We know how the specific forms of h_i were justified in the occupational choice stories leading to (6) and (16). Venkatasubramanian [26] presents the following justification for the particular form of h_i as a function of x_i and salary S_i as shown in (17) and (20). It will be seen that this involves certain departures from a full-blown neoclassical model of optimizing individuals.

Venkatasubramanian [26] argues as follows. The utility h_i of a job can be thought of, at the most fundamental level, as the ability to pay bills now so that one can make a living and the hope that the current job will lead to a better future. One hopes that the present job will lead to a better one in time, acquired based on the experience from the current job, and to a series of better jobs in the future, hence to a better life. This opportunity for a better future, with the expectation of *upward mobility*, is valuable to most, if not all, of us. This is the utility of having a *fair shot* at better future prospects. We are, of course, prepared to put in the requisite effort, make the appropriate contribution, to earn such a life. This effort includes not only the effort we would put into our present jobs but also the effort (and time and money) we invested in the past to acquire the requisite education, skills, and experience. Thus, the utility derived from a job is made up of two components: the *immediate* benefit of making a living (i.e., "present" utility) and the prospect of a better *future* life (i.e., "future" utility).

Hence, the overall utility h_i from a job can be seen as determined by three dominant elements: (1) utility u_i from salary S_i, (2) disutility v_i from effort, and (3) utility w_i from a fair opportunity for a better future. Thus, the overall utility of a job for an agent is given by

$$h_i = u_i - v_i + w_i. \tag{22}$$

Before modeling these elements further, Venkatasubramanian [26] invokes the last four decades of research in behavioral economics, that real-life people making complex real-life decisions are of bounded rationality and are subject to a variety of cognitive traits, attitudes, and preferences [4,10,11,13,18–20,22,24]. They are also limited in their abilities to process complex information, to compute various quantities, to evaluate different choices, and to arrive at decisions. Empirical studies have shown that people suffer from a variety of limitations such as overestimating probabilities of good outcomes (such as winning a lottery), underestimating probabilities of bad outcomes (such as the chances of suffering from a disease), experiencing difficulties in computing expected utilities, and so on.

This real-life behavior is very different from that of the mythical *Homo Economicus*, the perfectly rational, unbiased, super-human of unlimited capacity to process, compute, evaluate, and decide on complex choices as pictured in neoclassical economics. However, as one might intuitively expect, behavioral studies have shown that while people are incapable of such Herculean feats, they generally succeed by using heuristics and approximate models to cope with the complexity in making quick decisions in real-life [8]. Ballpark estimates, heuristics, and approximate models are important tools in the problem-solving arsenal of ordinary people in all kinds of situations. For example, people use ballpark figures and reasoning with approximate models in negotiations for the price of a home or a car, or for the salary of a new job offer, and so on, to assess cost-benefit tradeoffs.

To accomplish this, people often quickly focus on a few key quantities, ignoring a number of other factors, make estimates, and then reason with an approximate mental model of the reality using these key quantities. They do this knowing the potential pitfalls, because they also realize this is the only pragmatic option available for them. They realize that a more thorough, detailed, and accurate analysis is virtually impossible for most real-life situations because the costs – in terms of money, time, and effort – are too prohibitive. Sometimes, it simply cannot be done, no matter how much time, money, or effort is spent, owing to problem complexity. Driven by such practical considerations, people practice rational inattention [20] and ignore a variety of less important factors. Furthermore, for the few key elements they have focused on, they develop approximate mental models and heuristics to help with the decision-making, evaluating various pros and cons to perform a cost-benefit analysis to determine the optimal or an acceptable decision under these conditions.

With this background, Venkatasubramanian [26] develops specifications for u_i, v_i, and w_i. The goal is to develop models that may reflect the approximations used by people in real-life situations. For the utility derived from salary, we employ the commonly used logarithmic utility function

$$u_i = \alpha \ln S_i \tag{23}$$

where α is a positive parameter. Again, as noted, we are not claiming that people are necessarily computing logarithms, but people do have an intuitive feel for diminishing marginal utility of resources. They know that the value of something diminishes and saturates as they have more and more of it. Such diminishing marginal utility is a standard assumption in economics.

This kind of approximate model people use is captured by a logarithmic function.

As for the second element, u and v are combined to compute $u_{\text{net}} = au - bv$ (a and b are positive constant parameters), which is the net utility (i.e., net benefit or gain) derived from a job after accounting for its cost. Typically, net utility will increase as u increases (e.g., because of salary increase). However, generally, after a point, the cost has increased so much – due to personal sacrifices such as working overtime, missing quality time with family, giving up on hobbies, job stress resulting in poor mental and physical health, etc. – that u_{net} begins to decrease after reaching a maximum.

The simplest model of this commonly occurring inverted-U profile is a quadratic function, as in

$$u_{\text{net}} = au - bu^2. \tag{24}$$

Since $u_i \sim \ln S_i$, we get $v_i = u_i^2 \sim (\ln S_i)^2$.

Therefore,

$$v_i = \beta (\ln S_i)^2 \tag{25}$$

where β is a positive parameter.

Again, the motivation here is to develop the simplest possible model that captures the essence of the underlying phenomenon. For the third element, Venkatasubramanian [26] considers the following scenario. A group of freshly minted law school graduates have just been hired by a prestigious New York law firm as associates. They have been told that one of them will be promoted as partner in eight years depending on their performance. Let us say that the partnership is worth \$$Q$ millions over the career of a partner. So any associate's chance of winning the coveted partnership goes as $1/N_i$, where N_i is the number of associates in her peer group i, her *local competition*. Therefore, her expected value for the award is Q/N_i, and the utility derived from it goes as $\ln(Q/N_i)$ because of diminishing marginal utility. While the reward Q lies in the future, were she to win the partnership, the cost is paid here and now in the form of working hard to compete with her peers.

Hence we have:

$$w_i = -\gamma \ln N_i. \tag{26}$$

Again, as noted, the *Homo Economicus* way of properly modelling this element would require invoking discounted future utility over many time periods, using a complicated discounting factor, and performing complicated computations. Instead, we use approximate models to estimate the potential benefits and costs. Equation (26) is proposed in that spirit – a model that captures the cost of competition.

Combining all three, we have

$$h_i(S_i, N_i) = \alpha \ln S_i - \beta(\ln S_i)^2 - \gamma \ln N_i \tag{27}$$

where α, β, $\gamma > 0$. This equation is essentially the same as the combination of (17) and (20), if we replace N_i by x_i $(= \frac{N_i}{N})$.

In general, α, β, and γ, which model the relative importance an agent assigns to these three elements, can vary from agent to agent. But for a 1-class society, the simplest case, all agents have the same preferences and hence we can treat these as constant parameters.

4 Potential game theory and arbitrage equilibrium

Venkatasubramanian et al. [25] show the connection between the arbitrage equilibrium and entropy maximization directly through potential game theory. In potential games [14,16], there exists a single scalar-valued global function, called potential, that captures the necessary information about the utilities or payoffs. The gradient of the potential is the payoff or the utility.

Therefore, for the utility h_i in (27), we have

$$h_i\left(\mathbf{x}\right) \equiv \frac{\partial \phi\left(\mathbf{x}\right)}{\partial x_i} \tag{28}$$

where $\phi(\mathbf{x})$ is the potential. This naturally leads to

$$\phi\left(\mathbf{x}\right) = \sum_{i=1}^{n} \int h_i\left(\mathbf{x}\right) dx_i. \tag{29}$$

When one performs the integration using the information in (27), one obtains the following:

$$\phi\left(\mathbf{x}\right) = \phi_u + \phi_v + \phi_w + \text{constant} \tag{30}$$

where

$$\begin{aligned} \phi_u &= \alpha \sum_{i=1}^{n} x_i \ln S_i \\ \phi_v &= -\beta \sum_{i=1}^{n} x_i \left(\ln S_i\right)^2 \\ \phi_w &= \frac{\gamma}{N} \ln \frac{N!}{\prod_{i=1}^{n} \left(N x_i\right)!}. \end{aligned} \tag{31}$$

We have used Stirling's approximation to ϕ_w obtain. As Venkatasubramanian et al. [25] observe is essentially the same as entropy in statistical mechanics except for the missing Boltzmann factor k.

In potential games, arbitrage equilibrium is reached when the potential is maximized. We can show that $\phi(\mathbf{x})$ the potential is strictly concave, as given by:

$$\frac{\partial^2 \phi(\mathbf{x})}{\partial x_i^2} = -\frac{\gamma}{x_i} < 0. \tag{32}$$

Therefore, a unique Nash equilibrium exists for this game exists when $\phi(\mathbf{x})$ is maximized [16].

Formulating this optimization using the method of Lagrange multipliers, subject to the constraint $\sum_{i=1}^{n} x_i = 1$, given in (1), we get

$$L = \phi + \lambda \left(1 - \sum_{i=1}^{n} x_i\right). \tag{33}$$

Solving $\partial L / \partial x_i = 0$, we obtain

$$x_i = \frac{1}{S_i D} \exp\left\{-\frac{\left[\ln S_i - (\alpha + \gamma)/2\beta\right]^2}{\gamma/\beta}\right\} \tag{34}$$

where $D = N \exp\left[h^*/\gamma - (\alpha+\gamma)^2/4\beta\gamma\right]$ and h^* is given by (5).

This, of course, is the lognormal distribution of salaries S_i.

So by maximizing the game theoretic potential $\phi(\mathbf{x})$, subject to the constraint in (1), we see that from (31) that what we have essentially accomplished is to maximize entropy subject to the constraints implied by the terms ϕ_u and ϕ_v, and of course by (1). Thus, the arbitrage equilibrium demanded by the potential game theoretic framework, achieved by maximizing potential, is equivalent to the statistical equilibrium accomplished by maximizing entropy, subject to the appropriate constraints. However, for this to happen, as noted, the crucial requirement is the choice of $\ln x_i$ (or equivalently $\ln N_i$) for the congestion term.

5 Discussion and conclusion

An important aspect of the typical econophysics approaches to income distribution phenomena in economics is that they do not have appropriate modeling of individuals making choices, which is, of course, central to conventional economic models. This paper shows how these two seemingly separate perspectives could be connected by invoking a perspective of equilibrium through arbitrage in occupational choice.

This paper highlights a mathematical dualism in the determination of individual incomes via the elimination of arbitrage opportunities: that between occupational choice equilibrium for individuals with a certain type of preferences and the maximization of entropy functionals under constraints on the distribution of individuals across income levels or occupations. We show that a particular model of choice between occupations by individuals leads to an economic equilibrium which can also be characterized as an outcome which maximizes the entropy of the distribution of individuals across occupations (and hence across incomes) under a specific choice of the congestion term in the utility function. This choice leads to entropy maximization in statistical mechanics. We also link this outcome to the equilibrium structure of potential games. The arbitrage equilibrium of payoffs in potential games, which is achieved by maximizing potential, is equivalent to the statistical equilibrium accomplished by maximizing entropy, subject to the appropriate constraints. In arbitrage equilibrium the payoffs of all occupational choices are equal for identical people.

In statistical equilibrium, the chemical potentials of all options are equal for molecules. These turn out to be equivalent when the congestion term is $\ln x_i$, as postulated in the specific model in Venkatasubramanian [26].

The equivalence established in this paper between occupational choice equilibrium, maximum entropy equilibrium and the equilibrium of potential games is of course for a specific model with specific functional forms. We fully recognize this limitation. At the very least it shows what precisely is required as a necessary (and perhaps sufficient) condition to postulate equivalence between the different approaches. It is a first step in the further exploration of the relationship between these similar outcomes from very different perspectives. If the exploration is successful, the power of maximum entropy methods and potential games can be brought to bear on the analysis of income distribution. Further, the occupational choice interpretation can provide economic and institutional substance, which is largely missing in the typical econophysics models. The equivalence can also provide the basis for economic comparative static analysis, where key economic parameters can be altered to gauge their impact on income inequality.

Despite the value added we have claimed for bringing together these perspectives to enrich econophysics on the one hand and income distribution analysis on the other, we are aware of the conceptual, methodological and modeling challenges involved. We are grateful to the anonymous referees of this paper for highlighting these challenges in constructive fashion, and we use some of their own words in what follows to outline the issues that will need further and deeper and dialogue across economics and statistical physics.

Coming to the specific and key role of the specification of the congestion effect as a separable term in the logarithm of the fraction of population in an occupation, more thought clearly needs to be given to the underlying economics which leads to this formulation. Venkatasubramanian [26] interprets and motivates as this as the probability of big success, winning the ultimate prize, in an occupation. But Smithian congestion could arise in a number of different ways. The typical economic thinking (including Smith's) is that equalization across occupations (adjusted for differences in risk, etc.) works through differences in pay or rates of return. In (17) together with (20), salary appears entirely disconnected from labor supply in occupation i. In Kanbur [12], there is a wage that ensures market clearing in the labor market, but it is not obvious how one gets from that to something like (17), where the congestion penalty is a separate term from remuneration.

More generally, the issue arises as to how specific economic models of occupational choice as laid out in Section 1 could lead to a separable logarithmic congestion term – what specifications of production and preferences could be consistent with a characterization which would be consistent with entropy maximization. The demonstration exercises in this paper, while interesting and illuminating, raise further questions of generalizability.

Turning to broader methodological issues, it can be argued that the paper effectively casts entropy as something phenomenological, reflecting homogeneous subjective preferences, and not as a simple combinatorial heuristic quantifying the support across the economic system's micro-level configurations enjoyed by different macroscopic distributions of income or occupations. It is of course possible to say large-N systems have a "preference" for higher-entropy states, but that is simply a metaphorical recognition that those states are more common across all possible states the system may occupy. As such, we statistically expect systems to evolve toward higher entropy states. But this is different from the paper's effective contention that maximum-entropy states are expressions of a specific pattern of subjective preferences governing the pursuit of arbitrage opportunities in labor markets. These methodological differences are open for further discussion and debate.

Adam Smith spoke of a society where "every man was perfectly free both to choose what occupation he thought proper, and to change it as often as he thought proper" as the basis of his occupational choice equilibrium. Recent developments in behavioral economics have widened considerably our notions of individual choice beyond the standard neoclassical model of what is the "proper" basis of choice. While we have focused on the linkages between the choice based and entropy maximization outcomes, information theory and statistical mechanics could enable the pursuit of a nuanced understanding between the micro- and macro-level functioning of systems. This opens the possibility to move beyond the conventional understanding in economics that "scientific rigor" requires casting macroscopic patterns or results in terms of strongly specified models of homogeneous individual behavior based on quantities and suppositions that may be difficult to observe or investigate scientifically.

This is not to deny the value of contributions that do indeed seek to characterize macroscopic outcomes in terms of detailed descriptions of individual characteristics. It is simply to point to the broader array of possibilities that can be pursued with these techniques we are taking from information theory and statistical mechanics. Here recent work by Scharfenaker and Foley [17] on the Quantal-Response Statistical Equilibrium models is interesting. They may offer a way to draw on Kanbur's earlier work (engaging with Friedman's contributions on occupational choice) to produce new information-theoretic models of the outcomes of labor-market arbitrage. Specifically, if hypotheses concerning the elimination of arbitrage opportunities can be framed in terms of moment constraints (say, expected values of returns across families of occupations are the same), it may be possible to postulate interesting maximum-entropy models to express some of the possible macroscopic consequences of the pursuit of labor-market arbitrage.

Here entropy would be used in the conventional, combinatorial manner, as a way to characterize the most likely distributions of income under certain types of no-arbitrage conditions or participation constraints.

Finally, we recognize the long history of attempts to incorporate entropy perspectives into economic modeling. Specifically, Foley [5] already makes the claim that because arbitrage opportunities are systematically eliminated in competitive markets, they are entropy maximizing. Like the present paper, Foley [5] relaxes the mainstay behavioral assumptions typical in economics, though in very limited and specific manner: agents accept any utility improving exchange rather than only the unique utility maximizing exchange. This makes an interesting case for maximum entropy approaches for understanding the distribution of market outcomes. Foley and Smith [6] also build a bridge between classical thermodynamics and general equilibrium modeling in economics, covering entropy analogues and when they break down. Our contribution can be seen to be in this longer tradition, but with a focus on deriving implications for income distribution as well as on bridging statistical mechanics and potential game theory.

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