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Bifurcations, chaos and synchronization of a predator–prey system with Allee effect and seasonally forcing in prey's growth rate

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Abstract. In this paper, a predator–prey model with Allee effect and seasonally forcing in the prey's growth rate is introduced and analysed. Nonlinear analysis, using equilibrium points computed via Symbolic Math software tools, bifurcation diagrams, phase diagrams, Lyapunov exponents and Kaplan–Yorke dimension, proves that such effects lead to undesirable biological dynamics including chaos behaviours. In order to establish a certain balance in the ecosystem and avoid chaotic dynamics, the biological system is controlled in order to follow an unforced reference model via a synchronization approach. Simulation results prove the efficiency of the proposed approach to allow the predator–prey system regaining natural dynamics.

1 Introduction

Nowadays, protection of biological and ecological systems are considered as one of the most exciting and potentially ground breaking research topics. In this framework, several research works have been developed to study predator–prey relationships considered as the basic interaction model in nature [1,2]. The most basic predator–prey model is the Lotka–Volterra model [3]. Study of predator–prey models can help to understand and control the undesirable dynamics of nature. In this context, many complex models integrating two or more interacting species have been proposed for which the behaviour of species is affected by socio-environmental factors such as time delay [4], impulsive effect [5], Allee effect [6] and seasonal effect [7,8]. On the other hand, the most crucial elements of the model in which their effects are integrated, are the predator growth rate [9], the functional response of the predation [10] and the prey's growth rate. To the best of our knowlage incorporating a seasonal effect in a predator–prey system suffering of Allee effect is not investigated yet. It is shown that the last extra factor, which is a positive relationship between any component of fitness of species and either numbers or densities of conspecifics [11], influence the system dynamics by occurring more complexities such as multi-stability, pseudo-periodicity or routes to chaos [7,12]. Recently multistability and coexisting attractors have been a hot topic [20–22].

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Control of predator–prey systems is a hot topic in biology. Synchronization can be considered as one of the most popular control approaches specially for prey–predator systems [13]. Several chaos synchronization approaches have been proposed [14]. There are many kinds of synchronization such as instance phase synchronization (PS) [15], generalized synchronization (GS) [16], and complete synchronization (CS) [17].

The main purpose of this paper is to analyse the influence of a periodic force in the prey's growth rate of predator–prey system suffering from Allee effect. A complete synchronization approach of the seasonally forced system with the basic model is then investigated to stabilize the chaotic system towards its natural dynamics. This paper is organised as follow. Section 2 presents the forced predator–prey model. In Section 3, a stability and bifurcation analysis of the forced system is investigated. Then, a chaos synchronization of forced system is achieved in Section 4 and finally the paper is concluded in Section 5.

2 The seasonally forced predator–prey model

Consider the Lotka–Volterra predator–prey model given by [3]

$$\begin{cases} \frac{dX_1}{dT} = RX_1(X_1 - L)(K - X_1) - AX_1X_2 \\ \frac{dX_2}{dT} = (EX_1 - M)X_2 \end{cases}, \tag{1}$$

where X_1 and X_2 are the size of the prey population and the predator population, respectively. They should be positive. R is the prey's growth rate in the absence of the predator and K is the carrying capacity. L is the Allee effect threshold which should not exceed the interval [–1, 1]. E represents the predator's conversion efficiency and M is the mortality rate of the predator depending on the predator's efficiency containing the interval [0, 1].

By introducing the following non-dimensional variables and parameters

$$t = \frac{TRK^2}{k^2},\ x_1 = \frac{kX_1}{K},\ x_2 = \frac{k^2X_2A}{RK^2},\ l = \frac{kL}{K},\ e = \frac{E}{RK} \text{ and } m = \frac{M}{EK},$$

and for $k = e = 1$, the dimensionless model related to system (1) is given by [18],

$$\begin{cases} \frac{dx_1}{dt} = x_1(x_1 - l)(1 - x_1) - x_1x_2 \\ \frac{dx_2}{dt} = (x_1 - m)x_2 \end{cases}. \tag{2}$$

Considering the two case studies of the Allee effect constraint which belonging to the domain [–1, 1]

- Strong Allee effect when $l \in [0\ 1]$,
- Weak Allee effect when $l \in [-1\ 0]$,

and for a periodic growth rate described by

$$R = R_0 + R_1(1 - \cos(\theta t))/(X_1 - L)(K - X_1) \tag{3}$$

and using the same non-dimensional variables and parameters, system (1) can be written as

$$\begin{cases} \frac{dx_1}{dt} = x_1(x_1 - l)(1 - x_1) - x_1 x_2 + \lambda x_1 (1 - \cos(\theta t)) \\ \\ \frac{dx_2}{dt} = (x_1 - m)x_2 \end{cases} \tag{4}$$

where $\lambda = \frac{R_1}{R_0}$ and $\theta = \frac{\omega}{R_0} \in [0,\ 2\pi]$ are the amplitude and frequency of the forcing term, respectively.

3 Dynamics of seasonally forced BB-model

Proposition 1. For weak Allee effect, $\forall l \in [-1, 0]$, system (2) admits 4 equilibriums

$$\begin{array}{llll} E_1 & : & x_1 = 0 & ; \quad x_2 = 0 \\ E_2 & : & x_1 = l & ; \quad x_2 = 0 \\ E_3 & : & x_1 = 1 & ; \quad x_2 = 0 \\ E_4 & : & x_1 = m & ; \quad x_2 = (m - l)(1 - m), \end{array}$$

where

$$m \neq 0. \tag{5}$$

The stability analysis is investigated by computing the Jacobian matrix of system (2) at each equilibrium E_i ($i = 1,\ 2,\ 3,\ 4$) for the two case studies of Allee effect

$$J_i = \begin{pmatrix} -3x_1^2 + 2(l+1)x_1 - (l + x_2) & -x_1 \\ \\ x_2 & (x_1 - m) \end{pmatrix}. \tag{6}$$

Using the Jacobian matrix at each equilibrium point, eigenvalues of each equilibrium point are as follows,

$$\begin{aligned} &E_1 : \lambda_1 = -l, \lambda_2 = -m \\ &E_2 : \lambda_1 = l - m, \lambda_2 = -l^2 + l \\ &E_3 : \lambda_1 = l - 1, \lambda_2 = 1 - m \\ &E_4 : \lambda_{1,2} = \frac{(1+l)m - 2m^2 \pm \sqrt{m(l^2 m - 4lm^2 - 2lm + 4l + 4m^3 - 3m)}}{2}. \end{aligned} \tag{7}$$

So type of equilibrium points can be discussed as follows where $m \in [0, 1]$:

- For weak Allee effect, equilibrium E_1 of system (2) is a saddle point. E_2 is a stable node in this condition and E_3 is saddle point. Eigenvalues of E_4 are nonlinear function of l and m, and it is not easy to determine the type of this equilibrium point.

- For strong Allee effect, system (2) admits, at E_1, a stable node. While E_2 can be an unstable or a saddle point and E_3 is a saddle point.

Now, we focus on the forced system (4). In order to identify the stability regions of the forced system, bifurcation diagrams of system (4) for both Allee effects types are presented in Figures 1 and 3, respectively. Figure 1 shows the periodic system (4) dynamic with strong Allee effect. It is varied between the periodic and the chaotic behaviour with respect to the value of the seasonality frequency as follow,

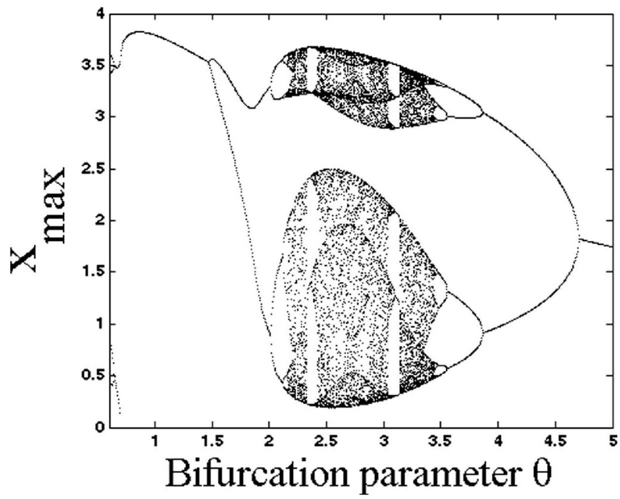

Fig. 1. Bifurcation diagram of the seasonally forced predator–prey system with strong Allee effect in parameters $m = 0.75$, $l = 0.5$ and $\lambda = 5$.

- When $\theta \in [0, 2.18]$, system (4) admits a periodic/pseudo-periodic behaviour
- When $\theta \in [2.19, 3.44]$, system (4) admits a chaotic behaviour
- When $\theta \in [3.45, 6]$, system (4) admits a periodic/pseudo-periodic behaviour.

In addition, chaotic and pseudo-periodic regions are identified by attractors which correspond to the interval limits. Then, they are presented in Figures 2 and 4 for both case studies, respectively. Figure 3 shows the dynamic behaviour of the forced system (4) for the weak Allee effect case study. It is varied between periodic and chaotic behaviour with respect to the value of the seasonality frequency as follow,

- When $\theta \in [0, 2.09]$, system (4) admits a periodic/pseudo-periodic behaviour
- When $\theta \in [2.1, 3.06]$, system (4) admits a chaotic behaviour
- When $\theta \in [3.07, 3.6]$, system (4) admits a periodic/pseudo-periodic behaviour.
- When $\theta \in [3.61, 3.77]$, system (4) admits a chaotic behaviour
- When $\theta \in [3.78, 6]$, system (4) admits a periodic/pseudo-periodic behaviour.

In order to investigate chaos dynamics of predator–prey system, we use the changing variable $x_3 = \theta t$ to calculate Lyapunov spectrum. Lyapunov exponent (LE) and Lyapunov dimension calculus are realized by choosing $\theta = 2.5$. We compute LEs by using the MATLAB package Matds and presented them in Figure 5. For strong and weak Allee effect, LEs are equal to (0.079863, 0, –2.257) and (0.17916, 0, –1.2078), respectively. Also, a relationship exists between LEs and the Lyapunov dimension which is called the Kaplan–Yorke dimension. For the n-dimensional system, suppose that the LEs $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and let j denote the largest integer such that $\sum_{i=1}^{j} \alpha_i \geq 0$. The system admits a strange attractor when D_L is fractal and contained in the interval $[j,n]$. The corresponding Kaplan–Yorke dimension is written, as follow [19]

$$D_L \begin{cases} 0 & \text{if no such } j \text{ exists} \\ j + \frac{1}{|\alpha_{j+1}|} \sum_{i=1}^{j} (\alpha_i) & \text{if} \quad j < n \\ n & \text{if} \quad j = n \end{cases} . \tag{8}$$

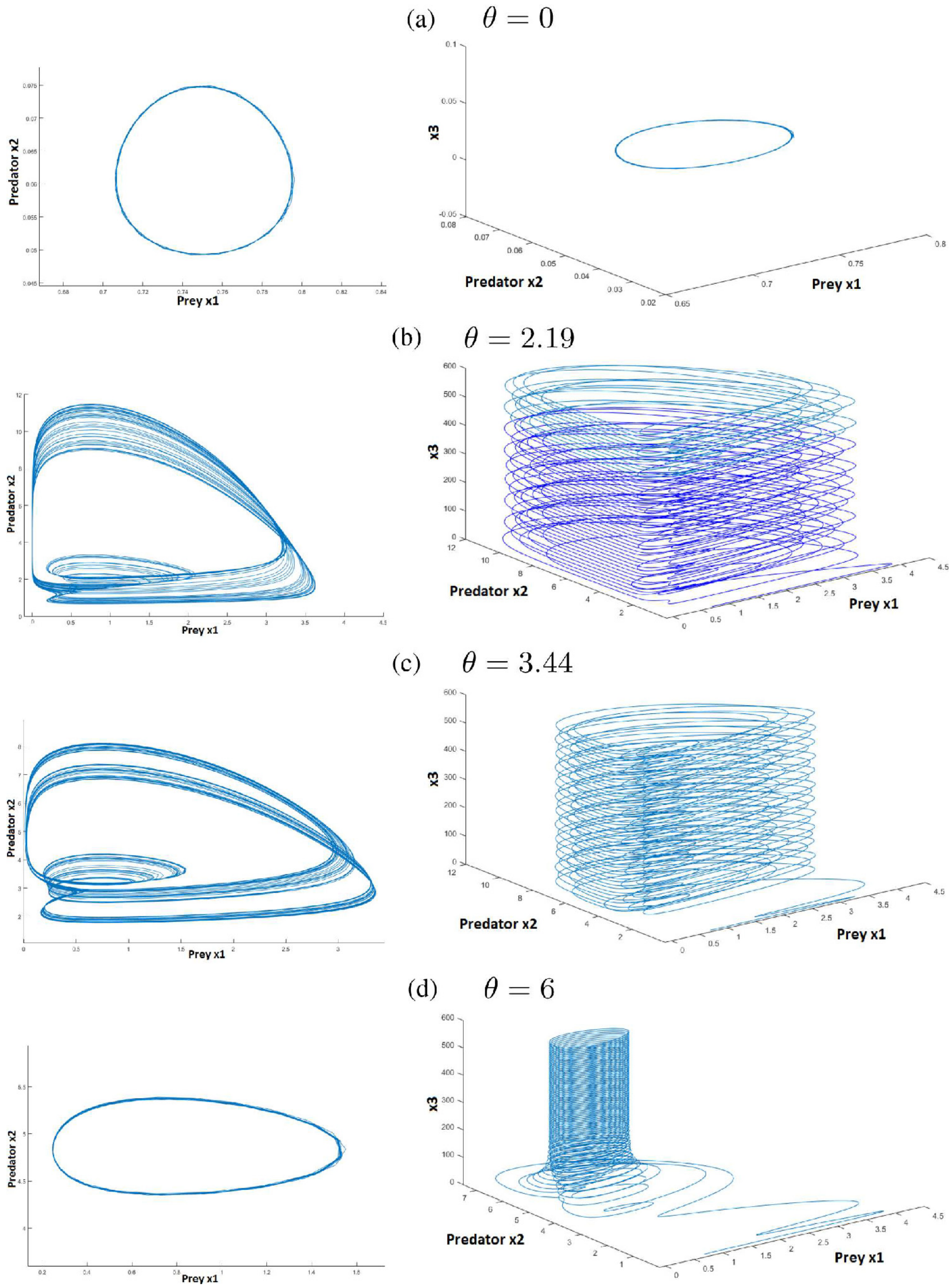

Fig. 2. Phase trajectories of the seasonally forced predator–prey system with strong Allee effect.

Thus, system (4) admits obviously a chaotic attractor since the corresponding Lyapunov dimension is fractal and equal to 2.0353 and 2.1483 for the strong and weak Allee effect, respectively.

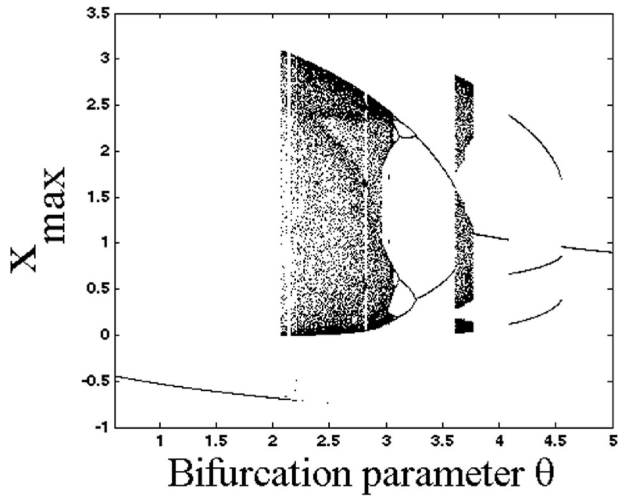

Fig. 3. Bifurcation diagram of the seasonally forced predator–prey system with weak Allee effect in parameters $m = 0.4$, $l = -0.2$ and $\lambda = 5$.

4 Chaos synchronization

4.1 Problem position

Let consider the basic predator–prey model described by equation (2) where $x(t) = [x_1(t),x_2(t)]^T$ is the state vector, m and l are constant parameters. As discussed in the previous section, the seasonality forcing of the prey's growth rate is a chaos generator creating an imbalance in the ecosystem. Therefore, a feedback control is necessary to establish a balance of the nature system. In order to reach such an objective, we propose a synchronisation approach. The basic model, described by system (2), is considered as the drive system and the chaotic model is considered as the driven system. This latter is described by

$$\begin{cases} \frac{dy_1}{dt} = y_1(y_1 - l)(1 - y_1) - y_1 y_2 + \lambda y_1(1 - \cos(\theta t)) + U_1 \\ \frac{dy_2}{dt} = (y_1 - m)y_2 + U_2 \end{cases}, \tag{9}$$

where $y = [y_1(t), y_2(t)]^T$ is the state vector of the driven system, λ and θ are the seasonal amplitude and frequency, respectively. The objective of the synchronization problem is to design a control law $u(t) = [u_1(t), u_1(t)]^T$ such that

$$\lim_{t \to +\infty} \|e(t)\| = 0, \tag{10}$$

where $e(t) = [e_1(t), e_2(t)]^T$ is the error synchronization vector defined by

$$\begin{cases} e_1(t) = y_1 - x_1 \\ e_2(t) = y_2 - x_2 \end{cases}. \tag{11}$$

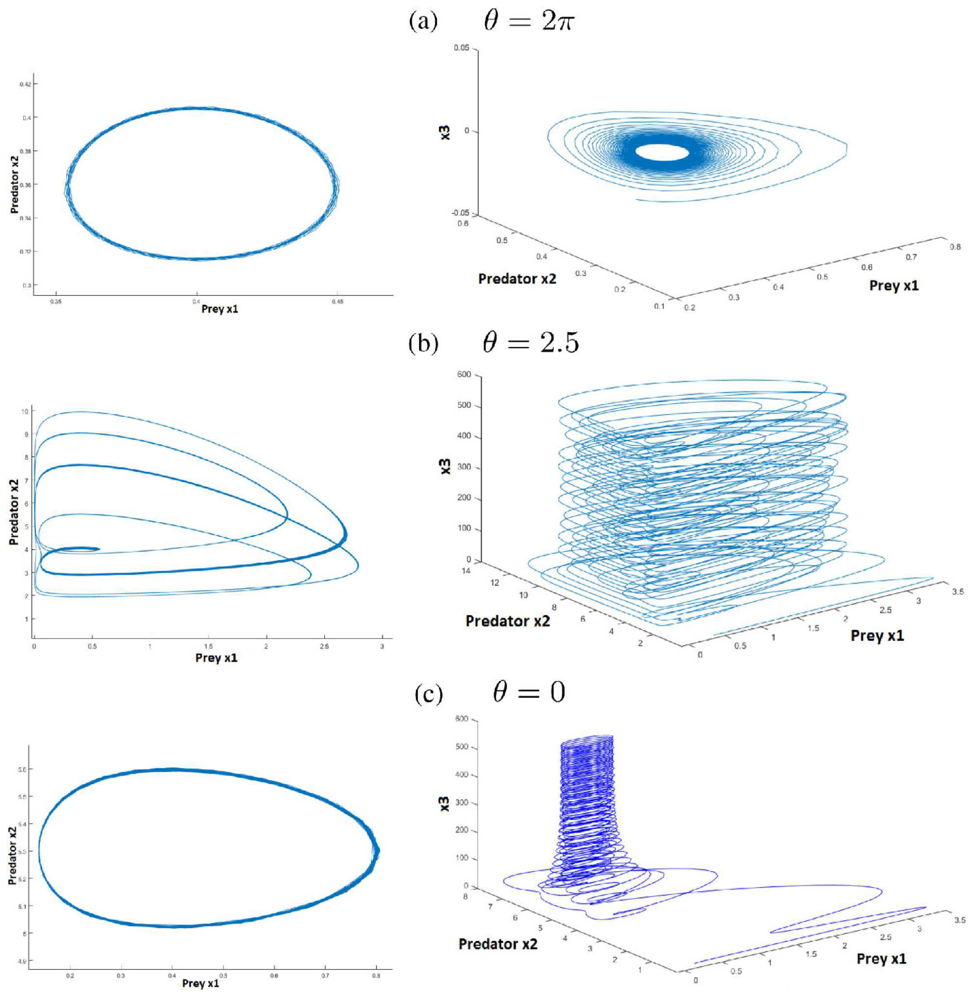

Fig. 4. Phase trajectories of the seasonally forced predator–prey system with weak Allee effect.

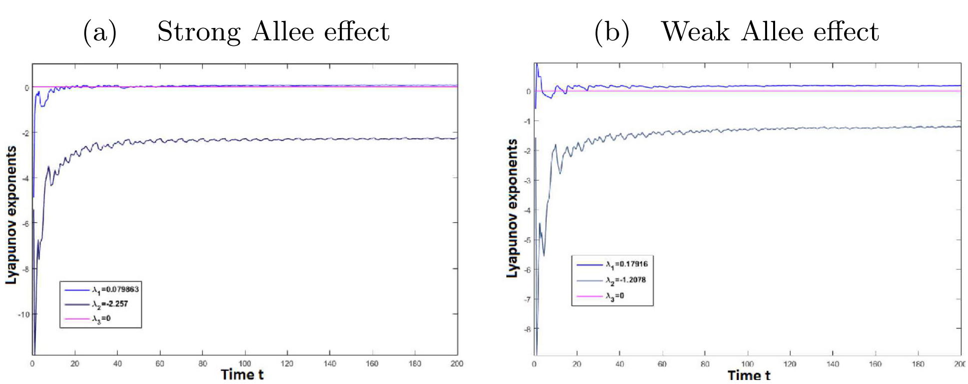

Fig. 5. Lyapunov exponents for the two cases of the Allee effect.

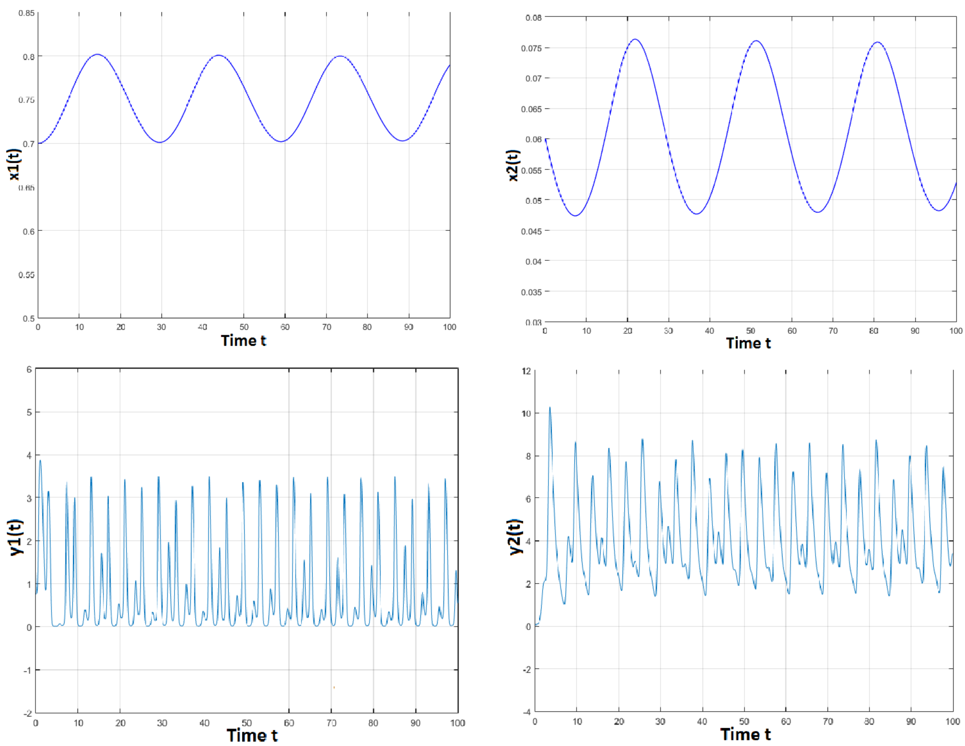

Fig. 6. Unsynchronized predator–prey systems with strong Allee effect.

(a) Prey (b) Predator

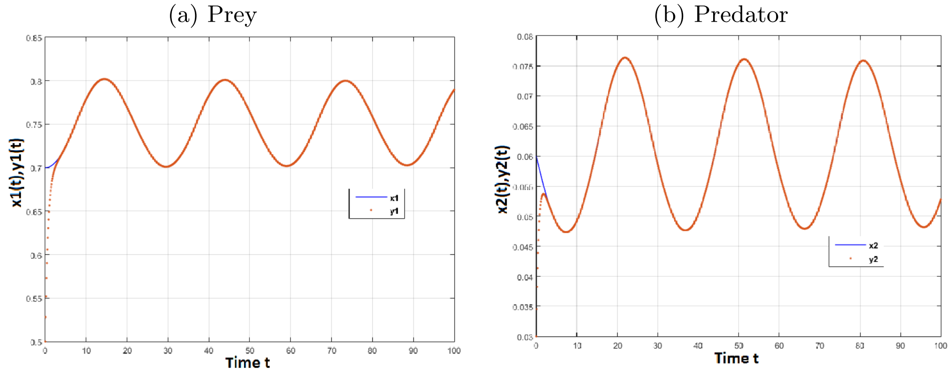

Fig. 7. Chaos synchronization of the predator–prey systems with strong Allee effect.

4.2 Nonlinear feedback control design

Theorem 1. *For the control laws*

$$\begin{cases} U_1 = (y_1^3 - x_1^3) - (l+1)(y_1 + x_1)e_1 - x_1x_2 + y_1y_2 + \lambda y_1(\cos(\theta t) - 1) - k_1e_1 \\ U_2 = x_1x_2 - y_1y_2 - k_2e_2 \end{cases} \tag{12}$$

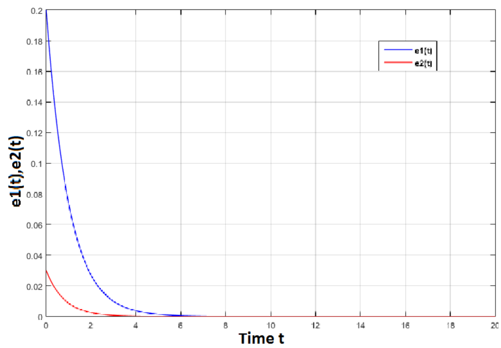

Fig. 8. Synchronization errors: strong Allee effect.

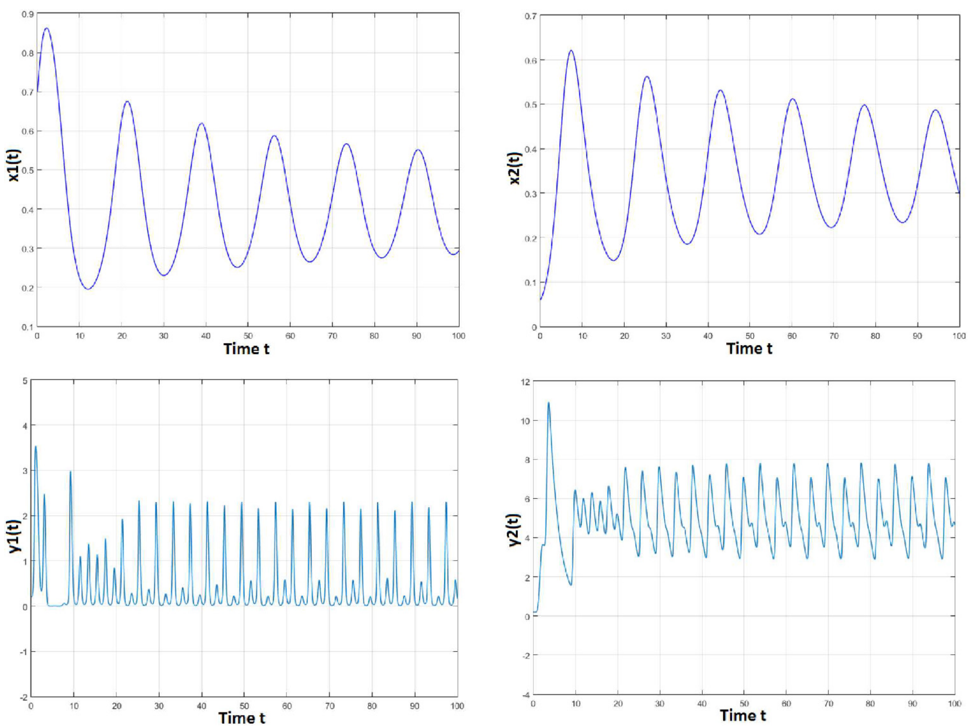

Fig. 9. Unsynchronized predator–prey systems with weak Allee effect.

the synchronization errors (11) between the drive system (2) and the driven system (9) are asymptotically stable if the following conditions are satisfied

$$\begin{cases} k_1 > -l \\ k_2 > -m \end{cases}. \tag{13}$$

Proof. Let subtract the dynamics of the drive system (2) from the dynamics of the driven system (9)

$$\begin{cases} \dot{e_1} = \dot{y_1} - \dot{x_1} \\ \dot{e_2} = \dot{y_2} - \dot{x_2} \end{cases}. \tag{14}$$

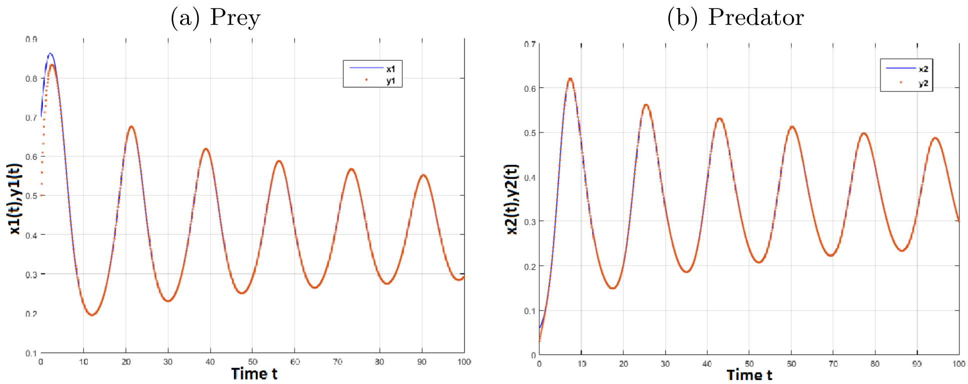

Fig. 10. Chaos synchronization of the predator–prey systems with weak Allee effect.

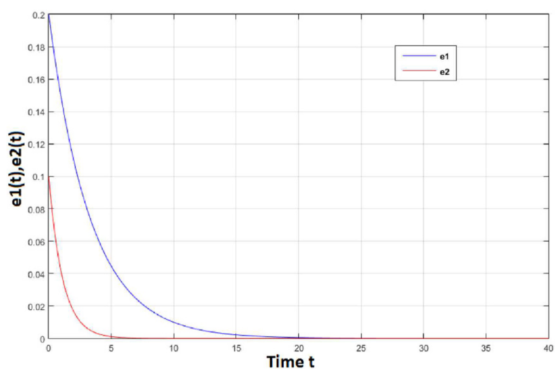

Fig. 11. Synchronization errors: weak Allee effect.

Then, using the proposed control laws (12) in (14), the synchronization error can be written as

$$\dot{e} = \begin{pmatrix} -(l + k_1) & 0 \\ 0 & -(m + k_2) \end{pmatrix} e. \tag{15}$$

The synchronization errors (15) are asymptotically stable if conditions (13) are fulfilled.

4.3 Simulation results

For simulation results of the strong Allee effect case, initial conditions and parameters are chosen as $(x_1(0), x_2(0), y_1(0), y_2(0)) = (0.7, 0.06, 0.5, 0.03)$, $m = 0.75$, $l = 0.5$ and λ is fixed at 5. For weak Allee effect case, they are chosen as $(x_1(0), x_2(0), y_1(0), y_2(0)) = (0.4, 0.3, 0.2, 0.2)$, $m = 0.4$, $l = -0.2$ and $\lambda = 5$. For $U_1 = U_2 = 0$, the temporal evolution of system (2) and system (9) with strong and weak Allee effect are shown in Figures 6 and 9, respectively. However, for U_1 and U_2 following the laws (12), Figures 7, 8, 10 and 11 prove that chaos synchronization is well achieved for both case studies where the control gains (k_1, k_2) and θ are chosen as (0.5,0.5) and 2.5, respectively.

5 Conclusion

In this paper, the significant impact of seasons on predator–prey system via a periodically forced prey's growth rate has been investigated and controlled. The obtained bifurcation diagrams have proved that the populations incorporated a seasonal forcing on the intrinsic growth rate have chaotic dynamic where the frequency of seasonality was $\theta \in [2.19, 3.44]$ in the strong case. However, in the weak case study, chaotic dynamic occurs where $\theta \in [2.1, 3.06]$ and $\theta \in [3.61, 3.77]$. This study permits to determine the set of seasonality frequency which can destabilize the ecosystem and exhibit the chaos dynamic and control it. Thus, in order to establish a balance of the nature system, a synchronization of the seasonal forced system with the basic predator–prey system using nonlinear feedback control was proposed. Furthermore, numerical simulations have illustrated the effectiveness of the proposed synchronization.

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