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Stability and synchronization of fractional-order complex-valued neural networks with time delay: LMI approach

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Abstract. In this paper, we investigate the problem of stability and synchronization of fractional-order complex-valued neural networks with time delay. By using Lyapunov–Krasovskii functional approach, some linear matrix inequality (LMI) conditions are proposed to ensure that the equilibrium point of the addressed neural networks is globally Mittag–Leffler stable. Moreover, some sufficient conditions for projective synchronization of considered fractional-order complex-valued neural networks are derived in terms of LMIs. Finally, two numerical examples are given to demonstrate the effectiveness of our theoretical results.

1 Introduction

Artificial neural networks (NNs) are generally remembered as one of the simplified model of neural processing in the human brain. The dynamics of NNs have aroused enormous interest of many researchers owing to their broad application prospects in many fields such as associative memory, image processing, optimization problems and pattern recognition [1–4]. Therefore, in the past decades, the analysis of dynamical characteristics of NNs has received much attention and interest from many researchers [4,5]. Recently, extended dissipative analysis was proposed by adjusting the weight matrices in a new performance index in [6].

Fractional calculus is one of the emerging issues of mathematics in recent years, which is the generalization of integer-order differentiation and integration. Some practical applications are modeled more accurately by using fractional-order derivatives than classical integer-order derivatives. Therefore, it is necessary in various fields of science and engineering, such as physics, signal and image processing, pattern recognition, biophysics, aerodynamics, economics biology, viscoelasticity, electron-analytical chemistry [4,5,7,8]. As we know that the concept of memory is one of the important property of NNs. When compared with the integer-order differential equations, the fractional-order differential equations have infinite memory. The incorporation of

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memory terms into NNs models by using fractional calculus which is an important improvement in the existing literature and it is necessary to analysis the dynamical properties of fractional-order NNs. In recent decades, many interesting results about stability analysis of fractional-order NNs which are reported in [9–13]. Some sufficient conditions were presented for finite-time stability of network with Caputo fractional-derivative in [14]. In [12], the authors investigated the global asymptotic stability of equilibrium point of the fractional-order NNs by using contraction mapping principle and inequality scaling skills. Global Mittag-leffler stability and synchronization problem for fractional-order NNs with time-varying delays was studied in [4,15,16] by using fractional-order Lyapunov method and Mittag-leffler function.

Complex-valued neural networks (CVNNs) are extension of the real-valued neural networks (RVNNs) and its state vectors, output, activation functions, connection weights are defined in the complex plane. CVNNs have specific successful applications different from RVNNs such as physical systems, image analysis optoelectronic, signal processing, pattern recognition, ultrasonic, light, network communications and so on [7,17–19]. In fact, analyzing the frequency domain, each signal is characterized by magnitude and phase [1,20]. The magnitude and phase information's are properly treat by using the complex domain which is the most characterized advantage of CVNNs. Thus, the analysis of dynamical characteristics of CVNNs has gained increasing interest and more attention by many researchers, scientists in past few years [1,12,21–25]. For example, in [26] the authors proposed several sufficient conditions to ensure the existence of equilibrium point and global stability of fractional-order complex-valued BAM NNs based on Lyapunov method. In [25], authors investigated dual function projective synchronization of fractional-order complex-valued chaotic systems.

Time delay phenomenon were first discussed in biological models and theoretical research about the delay related systems [10] such as fluid-mechanical transmissions, network control systems are very attractive attention of many researchers. Meanwhile, the existence of time delay systems, which may degrades the performance of the systems, such as oscillations, deterioration of system performance, divergence and even instability of control systems. Therefore, the study of dynamical characteristics of time delay systems has both theoretical and practical importance. Recently, several remarkable results about time-varying delay systems have been investigated in [13,19,27–29]. For example, in [30], the stability analysis on time-delayed NNs has also been investigated using Writinger based integral inequality. Very recently, in [28], the authors have derived some delay-dependent stability criteria in the form of linear matrix inequality which guarantee the asymptotic stability of the addressed time-varying delay systems. By constructing Lyapunov functional candidate which consisted of two quadratic functions with a special structural matrix, some sufficient conditions were obtained to checking the stability of considered time-varying delayed systems in [29].

Nowadays, the dynamical analysis of fractional-order CVNNs has received interesting attention and some results have been reported [4,5,11,12,25,31–35]. Global asymptotic stability of impulsive fractional-order CVNNs and stability analysis of fractional-order CVNNs with both leakage and discrete delay was pointed out in [12,35]. The idea of synchronization of fractional-order CVNNs systems has a lot of attention from the scientists for verity of research areas such as ecological systems, secure communication and system identification [36]. In addition, many results of synchronization have been proposed such as exponential, finite-time synchronization and global projective synchronization in [37–39].

Motivated by the above discussions, the purpose of this paper is to examine the problem of stability and synchronization of fractional-order CVNNs with time delays. Some sufficient LMI conditions have been proposed for checking the global asymptotic stability and projective synchronization of fractional-order CVNNs with time delays. First, LMI stability condition for fractional-order CVNNs are derived by using fractional-order Lyapunov's Direct Method [40]. Then, some numerical simulations are presented to verify our theoretical results.

The rest of this paper is organized as follows. In Section 2, we give some basic definitions of fractional calculus involving fractional-order Lyapunov direct method, Mittag-Leffler stability and Caputo fractional-derivative. The global stability and projective synchronization conditions of fractional-order CVNNs with time delay were proposed in Section 3. In Section 4, two numerical examples are given to show the effectiveness of our main results. Finally, the conclusion of the manuscript is given in Section 5.

2 Preliminaries

Definition 1 ([41])**.** The Riemann–Liouville fractional integral of order $\alpha > 0$ for a continuous function $h(y) : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$I^\alpha h(y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y-\tau)^{(\alpha-1)} h(\tau) d\tau,$$

where I^α denotes the Riemann–Liouville fractional integral of order α and $\Gamma(.)$ is the Gamma function.

Definition 2 ([42])**.** For a function $u(t) : (0, \infty) \to \mathbb{R}$, the Riemann–Liouville fractional derivative of order $\alpha > 0, n = [\alpha] + 1$ ($[\alpha]$ denotes the integer part of the real number α) is defined as

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} u(\tau) d\tau.$$

Definition 3 ([43])**.** The Caputo fractional-derivative of order $\alpha > 0$ for a function $g \in \mathcal{C}^{n+1}([0, \infty), \mathbb{R})$ the set of all $n+1$ order continuous differentiable functions on $[0, \infty)$ is defined as follows

$${}_0^C D_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{(n-\alpha-1)} g^{(n)}(\tau) d\tau,$$

where n is the first integer greater than α, that is $n - 1 < \alpha < n$.

Especially, when $\alpha \in (0, 1)$

$${}_0^C D_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} g'(\tau) d\tau.$$

The Gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad (Re(z) > 0),$$

where the $Re(z)$ is the real part of z.

Let us consider the fractional-order CVNNs with time delay described as follows:

$$ {}_{t_0}D_t^{\alpha} z_i(t) = -c_i z_i(t) + \sum_{j=1}^{n} a_{ij} f_j(z_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(z_j(t-\tau)) + I_i, \tag{1} $$

where $i = 1, 2, 3, \ldots, n, z_i(t) \in \mathbb{C}$ is the complex state of the ith neuron, $c_i > 0$ is the self-feedback connection weights, $(a_{ij})_{n\times n} \in \mathbb{C}^{n\times n}$ and $(b_{ij})_{n\times n} \in \mathbb{C}^{n\times n}$ are the connection weight matrices without and with time delay respectively. $I_i \in \mathbb{C}$ is the external input vector, τ is the constant time delay and $f_i(z_i(t)), f_i(z_i(t-\tau))$ are complex-valued nonlinear activation functions for without and with time delay respectively.

In order to obtain the LMI conditions for global stability of fractional-order CVNNs (1), we need the following Lemmas and some Assumptions to prove our main results.

Lemma 1 (Fractional-order Lyapunov Direct Method [16])**.** *The equilibrium point $\bar{x} = 0$ is Mittag–Leffler stable if there exist positive constants l_1, l_2, l_3, a, b and a continuously differentiable function $V(t, x(t))$ satisfying*

$$ l_1 \|x(t)\|^a \le V(t, x(t)) \le l_2 \|x(t)\|^{ab}, \tag{2} $$

$$ {}_{t_0}D_t^{\alpha} V(t, x(t)) \le -l_3 \|x(t)\|^{ab}, \tag{3} $$

where $t \ge 0, \alpha \in (0,1)$, and $V(t, x(t)) : [0,\infty) \times D \to \mathbb{R}$ satisfies locally Lipschitz condition on x; $D \subset \mathbb{R}^n$ is a domain containing the origin. If (2) and (3) holds globally on $\mathbb{R}^n$, then $\bar{x} = 0$ is globally Mittag–Leffler stable.

Remark 1. Mittag-Leffler stability implies asymptotic stability for fractional-order CVNNs (1), i.e., $\lim\limits_{t\to+\infty} ||z(t)|| = 0$.

Lemma 2 ([16])**.** *Let $x(t) \in \mathbb{R}^{2n}$ be continuous and derivable, it implies, for any positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$*

$$ \frac{1}{2}\ {}_{t_0}D_t^q x^T(t) P x(t) \le x^T(t) P\ {}_{t_0}D_t^q x(t), \quad \forall\ q \in (0,1). $$

Assumption 1. $g(x)$ is continuous and satisfies Lipschitz condition on $\mathbb{R}^{2n}$ and there exists a Lipschitz constant $L_g > 0$, such that

$$ \|g(y) - g(x)\|_2 \le L_g \|(y - x)\|_2, \quad x, y \in \mathbb{R}^{2n}. $$

Assumption 2. There exists a unique solution of a system (1), if the activation function $f(x, y)$ satisfies locally Lipschitz condition on x, y.

Assumption 3. Every activation function f_i is continuous and satisfies Lipschitz condition on $\mathbb{R}$ with Lipschitz constants l_i, q_i ,

$$ |f_i(y_i) - f_i(x_i)| \le l_i |y_i - x_i|, $$
$$ |g_i(y_i) - g_i(x_i)| \le q_i |y_i - x_i|, $$

for all $x_i, y_i \in \mathbb{R}$ and $i = 1, 2, 3, \ldots, n$. In addition, the compact form for any diagonal matrix $L > 0$

$$ ||f(y) - f(x)||_2 \le L||(y - x)||_2. $$

Assumption 4. The complex-valued nonlinear activation functions $f_i(z_i(t))$ and $f_i(z_i(t-\tau))$ are separated by its real and imaginary parts and it can be expressed as follows

$$f_i(z_i(t)) = f_i^R(z_i(t)) + \mathbf{i} f_i^I(z_i(t))$$

and

$$f_i(z_i(t-\tau)) = f_i^R(z_i(t-\tau)) + \mathbf{i} f_i^I(z_i(t-\tau)),$$

where $f_i^R(z_i(t)), f_i^I(z_i(t)) : \mathbb{R} \to \mathbb{R},\ i = 1, 2, 3, \ldots, n.$

Then, there exist positive numbers $\nu_1, \nu_2 \in \mathbb{R}$ and $\nu_1 \neq \nu_2$,

$$\vartheta_i^{R-} \leq \frac{f_i^R(\nu_1) - f_i^R(\nu_2)}{\nu_1 - \nu_2} \leq \vartheta_i^{R+},$$
$$\vartheta_i^{I-} \leq \frac{f_i^I(\nu_1) - f_i^I(\nu_2)}{\nu_1 - \nu_2} \leq \vartheta_i^{I+}, \quad \text{for } i = 1, 2, 3, \ldots, n,$$

where $\vartheta_i^{R-}, \vartheta_i^{R+}, \vartheta_i^{I-}, \vartheta_i^{I+}$ are some real constants and may be positive, zero or negative.

3 Main results

For representation convenience, the following notations are introduced:

$$\begin{aligned}
F_1 &= \text{diag}\{\hat{F}_1, \tilde{F}_1\},\\
\hat{F}_1 &= \text{diag}\{\vartheta_1^{R-}\vartheta_1^{R+}, \vartheta_2^{R-}\vartheta_2^{R+}, \ldots, \vartheta_n^{R-}\vartheta_n^{R+}\},\\
\tilde{F}_1 &= \text{diag}\{\vartheta_1^{I-}\vartheta_1^{I+}, \vartheta_2^{I-}\vartheta_2^{I+}, \ldots, \vartheta_n^{I-}\vartheta_n^{I+}\},\\
F_2 &= \text{diag}\{\hat{F}_2, \tilde{F}_2\},\\
\hat{F}_2 &= \text{diag}\left\{\tfrac{\vartheta_1^{R-}+\vartheta_1^{R+}}{2}, \tfrac{\vartheta_2^{R-}+\vartheta_2^{R+}}{2}, \ldots, \tfrac{\vartheta_n^{R-}+\vartheta_n^{R+}}{2}\right\},\\
\tilde{F}_2 &= \text{diag}\left\{\tfrac{\vartheta_1^{I-}+\vartheta_1^{I+}}{2}, \tfrac{\vartheta_2^{I-}+\vartheta_2^{I+}}{2}, \ldots, \tfrac{\vartheta_n^{I-}+\vartheta_n^{I+}}{2}\right\}.
\end{aligned}$$

3.1 Stability of fractional-order CVNNs

Let us consider the equivalent vector form of (1) is defined as follows

$$_{t_0}D_t^{\alpha} Z(t) = -CZ(t) + Af(Z(t)) + Bf(Z(t-\tau)) + I, \tag{4}$$

where $Z(t) = x(t) + \mathbf{i}y(t) \in \mathbb{C}$, with $x(t) = \text{Re}(Z(t))$, $y(t) = \text{Im}(Z(t))$. $\mathbf{i}$ denotes the imaginary unit, $\mathbf{i} = \sqrt{-1}$. $C = \text{diag}\{c_1, c_2, \ldots, c_n\}$ $A = (a_{ij})_{n\times n} \in \mathbb{C}^{n\times n}, B = (b_{ij})_{n\times n} \in \mathbb{C}^{n\times n}, f(Z(t)) = (f_1(z_1(t)), f_2(z_2(t)), \ldots, f_n(z_n(t)))^T, f(Z(t-\tau)) = (f_1(z_1(t-\tau)), f_2(z_2(t-\tau)), \ldots, f_n(z_n(t-\tau)))^T$ are the complex-valued nonlinear activation functions without and with time delays. $f_i(\cdot) \in \mathbb{C}$ denotes the activation function for ith neuron. $I = (I_1, I_2, \ldots, I_n)^T$ denotes the external input.

Equation (4) can be separated into its real and imaginary parts, we have

$$\begin{cases} _{t_0}D_t^{\alpha} x(t) = -Cx(t) + A^R f^R(x,y) - A^I f^I(x,y) + B^R g^R(x_\tau, y_\tau) - B^I g^I(x_\tau, y_\tau) + I^R,\\ _{t_0}D_t^{\alpha} y(t) = -Cy(t) + A^I f^R(x,y) + A^R f^I(x,y) + B^I g^R(x_\tau, y_\tau) + B^R g^I(x_\tau, y_\tau) + I^I, \end{cases} \tag{5}$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T$, $y(t) = (y_1(t), \ldots, y_n(t))^T$, $C = \mathrm{diag}\{c_1, \ldots, c_n\}$. $A^R, B^R, A^I, B^I \in \mathbb{R}^{n\times n}$ are real and imaginary parts of the connection weight matrices of A and B respectively. $f^R(x(t), y(t)), f^I(x(t), y(t)), f^R(x(t-\tau), y(t-\tau)), f^I(x(t-\tau), y(t-\tau)) : \mathbb{R}^2 \to \mathbb{R},\ i = 1, 2, \ldots, n$, are the real and imaginary activation function of $f(Z(t))$ and $f(Z(t-\tau))$ respectively. I^R, I^I are the real and imaginary parts of external input I.

Then, the system of equation (5) can be written in the compact form as,

$$ {}_{t_0}D_t^{\alpha} X(t) = -CX(t) + \bar{A}\bar{f}(X(t)) + \bar{B}\bar{f}(X(t-\tau)) + \bar{I}. \tag{6}$$

where,
$X(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T, C = \mathrm{diag}[c_1,\ c_2,\ c_3,\ \ldots\ c_{2n}],$
$\bar{A} = \begin{bmatrix} A^R & -A^I \\ A^I & A^R \end{bmatrix}, \bar{B} = \begin{bmatrix} B^R & -B^I \\ B^I & B^R \end{bmatrix}, \bar{I} = \begin{bmatrix} I^R \\ I^I \end{bmatrix},$
$\bar{f}(X(t)) = (f_1^R(x,y), f_2^R(x,y), f_3^R(x,y), \ldots, f_n^R(x,y),$
$f_1^I(x,y), f_2^I(x,y), f_3^I(x,y), \ldots, f_n^I(x,y))^T,$
$\bar{f}(X(t-\tau)) = (f_1^R(x_\tau,y_\tau), f_2^R(x_\tau,y_\tau), f_3^R(x_\tau,y_\tau), \ldots, f_n^R(x_\tau,y_\tau),$
$f_1^I(x_\tau,y_\tau), f_2^I(x_\tau,y_\tau), f_3^I(x_\tau,y_\tau), \ldots, f_n^I(x_\tau,y_\tau))^T.$

Assumption 5. There exists a constant θ, such that $0 \le \theta < 1$ for

$$(\bar{A} + \bar{B})^T (C^{-1})^T C^{-1} (\bar{A} + \bar{B}) \le \theta (L^{-1})^2.$$

Theorem 1. *The system (6) has a unique equilibrium point if Assumption 1–5 holds.*

Proof. We show that the existence and uniqueness of the equilibrium point of equation (6). Let us define a mapping $E(w) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$,

$$E(w) = C^{-1}\bar{A}\bar{f}(w) + C^{-1}\bar{B}\bar{f}(w) + C^{-1}\bar{I}. \tag{7}$$

where $w = (w_1, w_2, w_3, \ldots, w_{2n})^T$.

Then we have to take two vectors $\phi, \varphi \in \mathbb{R}^{2n}$

$$\begin{aligned} \|E(\phi) - E(\varphi)\|_2^2 &= \|(C^{-1}\bar{A} + C^{-1}\bar{B})(\bar{f}(\phi) - \bar{f}(\varphi))\|_2^2, \\ &= (\bar{f}(\phi) - \bar{f}(\varphi))^T (C^{-1}(\bar{A}+\bar{B}))^T (C^{-1}(\bar{A}+\bar{B}))(\bar{f}(\phi) - \bar{f}(\varphi)), \\ &= (\bar{f}(\phi) - \bar{f}(\varphi))^T (\bar{A}+\bar{B})^T (C^{-1})^T C^{-1} (\bar{A}+\bar{B})(\bar{f}(\phi) - \bar{f}(\varphi)), \\ &\le \sum_{i=1}^{n} \theta l_i^{-2} (\bar{f}_i(\phi) - \bar{f}_i(\varphi))^2, \\ &= \theta \|\phi - \varphi\|_2^2. \end{aligned} \tag{8}$$

From the inequality (8), $\|E(\phi) - E(\varphi)\|_2 \le \sqrt{\theta}\|\phi - \varphi\|_2$ which shows that the mapping is the contraction mapping on $\mathbb{R}^{2n}$. Thus, there exists a unique fixed point $w^* \in \mathbb{R}^{2n}$ such that $E(w^*) = w^*$, we have

$$w^* = C^{-1}\bar{A}\bar{f}(w^*) + C^{-1}\bar{B}\bar{f}(w^*) + C^{-1}\bar{I}. \tag{9}$$

It follows that,

$$-Cw^* + \bar{A}\bar{f}(w^*) + \bar{B}\bar{f}(w^*) + \bar{I} = 0,$$

where the w^* is the unique zero solution of system (6).

Theorem 2. *Under Assumption 1–5, the unique equilibrium point of system (6) is globally Mittag–Leffler stable, if there exists an $2n \times 2n$ matrices $P > 0$ and $Q > 0$ satisfying*

$$\Xi = \begin{bmatrix} -PC - C^TP + Q - F_1\Lambda_1 & 0 & P\bar{A} + F_2\Lambda_1 & P\bar{B} \\ * & -F_1\Lambda_2 - Q & 0 & F_2\Lambda_2 \\ * & * & -\Lambda_1 & 0 \\ * & * & * & -\Lambda_2 \end{bmatrix} < 0. \tag{10}$$

Proof. From Theorem 1, the system (6) has a unique equilibrium point $X^* = 0$. By the change of variables for the system (6) as $Y(t) = X(t) - X^*$ then,

$$\begin{aligned} {}_{t_0}D_t^\alpha Y(t) &= -C(Y(t) + X^*) + \bar{A}\bar{f}(Y(t) + X^*) + \bar{B}\bar{f}(Y(t-\tau) + X^*) + \bar{I}, \\ &= -CY(t) + \bar{A}[\bar{f}(Y(t) + X^*) - \bar{f}(X^*)] + \bar{B}[\bar{f}(Y(t-\tau) + X^*) - \bar{f}(X^*)] \\ &\quad +(-CX^* + \bar{A}\bar{f}(X^*) + \bar{B}\bar{f}(X^*) + \bar{I}). \end{aligned} \tag{11}$$

Since by (9), X^* is the unique equilibrium of the system (6). Thus,

$$-CX^* + \bar{A}\bar{f}(X^*) + \bar{B}\bar{f}(X^*) + \bar{I} = 0. \tag{12}$$

Then, the system (11) can be converted as follows,

$${}_{t_0}D_t^\alpha Y(t) = -CY(t) + \bar{A}\tilde{f}(Y(t)) + \bar{B}\tilde{f}(Y(t-\tau)) + \bar{I}, \tag{13}$$

where $\tilde{f}(Y(t)) = f((Y(t)+X^*) - f(X^*))$, $\tilde{f}(Y(t-\tau)) = f((Y(t-\tau)+X^*) - f(X^*))$. Let $\tilde{f}(Y(t)), \tilde{f}(Y(t-\tau))$ satisfies the Lipschitz condition on $\mathbb{R}^{2n}$ such that, for all $Y, X \in \mathbb{R}^{2n}$,

$$\|\tilde{f}(Y) - \tilde{f}(X)\|_2 \leq L\|(Y - X)\|_2. \tag{14}$$

Now, consider the following Lyapunov–Krasovskii functional,

$$V(t, Y(t)) = Y^T(t)PY(t) + {}_{t_0}D_t^{1-\alpha} \int_{1-\tau}^t Y^T(r)QY(r)dr. \tag{15}$$

Calculating the caputo fractional-derivative α of $V(t, Y(t))$ along the solution of system (13), we have

$$\begin{aligned} {}_{t_0}D_t^\alpha V(t) &= {}_{t_0}D_t^\alpha(Y^T(t)PY(t)) + {}_{t_0}D_t \int_{t-\tau}^t Y^T(r)QY(r)dr, \\ &\leq 2Y^T(t)P_{t_0}D_t^\alpha Y(t) + Y^T(t)QY(t) - Y^T(t-\tau)QY(t-\tau), \\ &= -2CY^T(t)PY(t) + 2Y^T(t)P\bar{A}\tilde{f}(Y(t)) + 2Y^T(t)P\bar{B}\tilde{f}(Y(t-\tau)) \\ &\quad +Y^T(t)QY(t) - Y^T(t-\tau)QY(t-\tau), \\ &= -Y^T(t)PCY(t) - Y^T(t)C^TPY(t) + 2Y^T(t)P\bar{A}\tilde{f}(Y(t)) \\ &\quad +2Y^T(t)P\bar{B}\tilde{f}(Y(t-\tau)) + Y^T(t)QY(t) - Y^T(t-\tau)QY(t-\tau). \end{aligned} \tag{16}$$

For any positive diagonal matrices Λ_1, Λ_2, from Assumption 4, we can write the inequality as follows:

$$\begin{bmatrix} Y^T(t) \\ \tilde{f}^T(Y(t)) \end{bmatrix}^T \begin{bmatrix} F_1\Lambda_1 & -F_2\Lambda_1 \\ * & \Lambda_1 \end{bmatrix} \begin{bmatrix} Y(t) \\ \tilde{f}(Y(t)) \end{bmatrix} \leq 0, \tag{17}$$

$$\begin{bmatrix} Y^T(t-\tau) \\ \tilde{f}^T(Y(t-\tau)) \end{bmatrix}^T \begin{bmatrix} F_1\Lambda_2 & -F_2\Lambda_2 \\ * & \Lambda_2 \end{bmatrix} \begin{bmatrix} Y(t-\tau) \\ \tilde{f}(Y(t-\tau)) \end{bmatrix} \leq 0. \tag{18}$$

From (16) to (18), we obtain

$${}_{t_0}D_t^\alpha V(t) \leq \zeta^T(t)\ \Xi\ \zeta(t) < 0, \tag{19}$$

where $\zeta(t) = \begin{bmatrix} Y(t) & Y(t-\tau) & \tilde{f}(Y(t)) & \tilde{f}(Y(t-\tau)) \end{bmatrix}^T$ and Ξ is defined in (10). Hence, as a consequence of Lemma 1, we can conclude that the fractional-order CVNNs system (6) is global Mittag–Leffler stable. The proof is completed.

3.2 Synchronization of fractional-order CVNNs

In this subsection, we derive the conditions for checking projective synchronization of considered fractional-order CVNNs with time delay by using drive-response approach.

Let us consider the drive system and the corresponding response system as follows:

$$D_t^\alpha u(t) = -Cu(t) + Af(u(t)) + Bf(u(t-\tau)) + I, \tag{20}$$

$$D_t^\alpha v(t) = -Cv(t) + Af(v(t)) + Bf(v(t-\tau)) + I + \bar{v}(t), \tag{21}$$

where $u(t) = x(t) + \mathbf{i}y(t)$,$v(t) = c(t) + \mathbf{i}d(t) \in \mathbb{C}^n$ are the state vector of (20) and (21) respectively. $A = (a_{ij})_{n\times n} \in \mathbb{C}^{n\times n}, B = (b_{ij})_{n\times n} \in \mathbb{C}^{n\times n}, I,\ f(u(t)), f(u(t-\tau)), f(v(t)), f(v(t-\tau)) \in \mathbb{C}^n$ are defined as the same as those in system (4).

$\bar{v}(t) = (\bar{v}_1(t), \bar{v}_2(t), \dots, \bar{v}_n(t))^T \in \mathbb{C}^n$ is the control law. Now, separating the real and imaginary parts of (20) and (21) and the corresponding vector form as follows:

$$D_t^\alpha X(t) = -CX(t) + \bar{A}\bar{f}(X(t)) + \bar{B}\bar{f}(X(t-\tau)) + \bar{I}, \tag{22}$$

$$D_t^\alpha Y(t) = -CY(t) + \bar{A}\bar{f}(Y(t)) + \bar{B}\bar{f}(Y(t-\tau)) + \bar{I} + U(t), \tag{23}$$

where

$$\begin{aligned}
&X(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T, Y(t) = \begin{bmatrix} c(t) & d(t) \end{bmatrix},^T \\
&x(t) = [x_1(t), x_2(t), x_3(t), \dots, x_n(t)]^T, y(t) = [y_1(t), y_2(t), y_3(t), \dots, y_n(t)]^T, \\
&c(t) = [c_1(t), c_2(t), c_3(t), \dots, c_n(t)]^T, d(t) = [d_1(t), d_2(t), d_3(t), \dots, d_n(t)]^T, \\
&C = \mathrm{diag}[c_1,\ c_2,\ c_3,\ \dots\ c_{2n}], \\
&\bar{A} = \begin{bmatrix} A^R & -A^I \\ A^I & A^R \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B^R & -B^I \\ B^I & B^R \end{bmatrix}, \quad \bar{I} = \begin{bmatrix} I^R \\ I^I \end{bmatrix}, \quad U(t) = \begin{bmatrix} \bar{v}^R(t) \\ \bar{v}^I(t) \end{bmatrix}, \\
&\bar{f}(X(t)) = (f_1^R(x,y), f_2^R(x,y), \dots, f_n^R(x,y), f_1^I(x,y), f_2^I(x,y), \dots, f_n^I(x,y))^T, \\
&\bar{f}(X(t-\tau)) = (f_1^R(x_\tau,y_\tau), f_2^R(x_\tau,y_\tau), \dots, f_n^R(x_\tau,y_\tau), f_1^I(x_\tau,y_\tau), f_2^I(x_\tau,y_\tau), \\
&\qquad \dots, f_n^I(x_\tau,y_\tau))^T, \\
&\bar{f}(Y(t)) = (f_1^R(c,d), f_2^R(c,d), \dots, f_n^R(c,d), f_1^I(c,d), f_2^I(c,d), \dots, f_n^I(c,d))^T, \\
&\bar{f}(Y(t-\tau)) = (f_1^R(c_\tau,d_\tau), f_2^R(c_\tau,d_\tau), \dots, f_n^R(c_\tau,d_\tau), f_1^I(c_\tau,d_\tau), f_2^I(c_\tau,d_\tau), \\
&\qquad \dots, f_n^I(c_\tau,d_\tau))^T.
\end{aligned}$$

Definition 4. The drive (20) and response systems (21) are said to be globally asymptotically projective synchronized, if there exists a $\Lambda \in \mathbb{R}^{2n\times 2n}$, such that for any initial values $X(t_0), Y(t_0) \in \mathbb{R}^{2n}$, the error vector $E(t) = Y(t) - \Lambda X(t)$ converges to zero,

$$\lim_{t\to+\infty} ||E(t)|| = \lim_{t\to+\infty} ||Y(t) - \Lambda X(t)|| = 0. \tag{24}$$

Remark 2. There are some special cases of projective synchronization under projection scaling matrix Λ. If there exists a $\Lambda = \mathrm{diag}\{\Lambda_1, \Lambda_2, \Lambda_3, \ldots, \Lambda_{2n}\}$ such that $\lim_{t\to+\infty} ||Y(t) - \Lambda X(t)|| = 0$, then we call the system (20) and (21) are achieve modified projective synchronization.

Remark 3. When, $\Lambda_1 = \Lambda_2 = \Lambda_3 = \cdots = \Lambda_{2n} = 1$, then we call the systems (20) and (21) are achieve complete synchronization, $\Lambda_1 = \Lambda_2 = \Lambda_3 = \cdots = \Lambda_{2n} = -1$, it is called anti synchronization and $\Lambda_1 = \Lambda_2 = \Lambda_3 = \cdots = \Lambda_{2n}$, then systems (20) and (21) are said to realize projective synchronization.

Remark 4. If the error system $E(t)$ is asymptotically stable, which implies the modified projective synchronization between the systems (20) and (21) is realized, i.e., $\lim_{t\to+\infty} ||E(t)|| = \lim_{t\to+\infty} ||Y(t) - \Lambda X(t)|| = 0$.

Now, we can design the projective synchronization control in the response system (23). Consider the drive system (22) with projective coefficient matrix Λ and response system (23) as

$$\begin{aligned}\Lambda D_t^\alpha X(t) &= -\Lambda C X(t) + \Lambda \bar{A}\bar{f}(X(t)) + \Lambda \bar{B}\bar{f}(X(t-\tau)) + \Lambda \bar{I},\\ D_t^\alpha Y(t) &= -CY(t) + \bar{A}\bar{f}(Y(t)) + \bar{B}\bar{f}(Y(t-\tau)) + \bar{I} + U(t).\end{aligned}$$

Since, the error system is represented as $E(t) = Y(t) - \Lambda X(t)$, one can obtain,

$$\begin{aligned}D_t^\alpha Y(t) - \Lambda D_t^\alpha X(t) &= -CY(t) + \bar{A}\bar{f}(Y(t)) + \bar{B}\bar{f}(Y(t-\tau)) + \bar{I} + U(t) + \Lambda C X(t)\\ &\quad -\Lambda \bar{A}\bar{f}(X(t)) - \Lambda \bar{B}\bar{f}(X(t-\tau)) - \Lambda \bar{I},\\ &= -C\Lambda X(t) + \bar{A}\bar{f}(\Lambda X(t)) + \bar{B}\bar{f}(\Lambda X(t-\tau)) + \bar{I} + U(t)\\ &\quad -\Lambda C X(t) + \Lambda \bar{A}\bar{f}(X(t)) + \Lambda \bar{B}\bar{f}(X(t-\tau)) + \Lambda \bar{I}.\end{aligned}$$

Then, the control input $U(t)$ in response system (23) as the following form

$$\begin{cases} U(t) = j_1(t) + j_2(t),\\ j_1(t) = KE(t),\\ j_2(t) = (C\Lambda - \Lambda C)X(t) - \bar{A}\bar{f}(\Lambda X(t)) + \Lambda \bar{A}\bar{f}(X(t)) - \bar{B}\bar{f}(\Lambda X(t-\tau))\\ \qquad\quad +\Lambda \bar{B}\bar{f}(X(t-\tau)) + \Lambda \bar{I} - \bar{I},\end{cases} \tag{25}$$

where K is the control gain and Λ is called the projective coefficient matrix.

The control (25) is a hybrid control, $j_1(t)$ is an adaptive feedback control, and $j_2(t)$ is an open loop control. For the given drive-response system (22) and (23) we find the error system $E(t)$ is described as follows

$$\begin{aligned}
D_t^{\alpha}E(t) &= D_t^{\alpha}Y(t) - \Lambda D_t^{\alpha}X(t),\\
&= -CY(t) + \bar{A}\bar{f}(Y(t)) + \bar{B}\bar{f}(Y(t-\tau)) + \bar{I} + U(t) + \Lambda CX(t)\\
&\quad -\Lambda\bar{A}\bar{f}(X(t)) - \Lambda\bar{B}\bar{f}(X(t-\tau)) - \Lambda\bar{I},\\
&= -CY(t) + \bar{A}\bar{f}(Y(t)) + \bar{B}\bar{f}(Y(t-\tau)) + \bar{I} + KE(t) + (C\Lambda - \Lambda C)X(t)\\
&\quad -\bar{A}\bar{f}(\Lambda X(t)) + \Lambda\bar{A}\bar{f}(X(t)) - \bar{B}\bar{f}(\Lambda X(t-\tau)) + \Lambda\bar{B}\bar{f}(X(t-\tau)) + \Lambda\bar{I} - \bar{I}\\
&\quad +\Lambda CX(t) - \Lambda\bar{A}\bar{f}(X(t)) - \Lambda\bar{B}\bar{f}(X(t-\tau)) - \Lambda\bar{I},\\
&= -C(Y(t) - \Lambda X(t)) + \bar{A}(\bar{f}(Y(t)) - \bar{f}(\Lambda X(t)))\\
&\quad +\bar{B}(\bar{f}(Y(t-\tau)) - \bar{f}(\Lambda X(t-\tau))) + KE(t),\\
D_t^{\alpha}E(t) &= (K-C)E(t) + \bar{A}\tilde{f}(E(t)) + \bar{B}\tilde{f}(E(t-\tau)),
\end{aligned}$$

where $\tilde{f}(E(t)) = \bar{f}(Y(t)) - \bar{f}(\Lambda X(t)), \tilde{f}(E(t-\tau)) = \bar{f}(Y(t-\tau)) - \bar{f}(\Lambda X(t-\tau))$.

Theorem 3. *For the drive-response systems (20) and (21) can realize the globally asymptotically projective synchronization under control law (25). If the Assumption 1–5 holds then the inequality satisfies,*

$$\xi = \begin{bmatrix} P(K-C) + (K-C)^T P + Q - F_1\Lambda_1 & 0 & P\bar{A} + F_2\Lambda_1 & P\bar{B} \\ * & -Q - F_1\Lambda_2 & 0 & F_2\Lambda_2 \\ * & * & -\Lambda_1 & 0 \\ * & * & * & -\Lambda_2 \end{bmatrix} < 0. \tag{26}$$

Proof. According to drive-response systems (22) and (23) and control law (25) we get the error system as,

$$D_t^{\alpha}E(t) = (K-C)E(t) + \bar{A}\tilde{f}(E(t)) + \bar{B}\tilde{f}(E(t-\tau)). \tag{27}$$

Consider the Lyapunov–Krasovskii functional,

$$V(t, E(t)) = E^T(t)PE(t) +_{t_0} D_t^{1-\alpha}\int_{1-\tau}^{t} E^T(r)QE(r)dr. \tag{28}$$

Calculating the fractional-derivative α of the $V(t, E(t))$ along the solution of system (27),

$$\begin{aligned}
{}_{t_0}D_t^{\alpha}V(t) &= {}_{t_0}D_t^{\alpha}(E^T(t)PX(t)) +_{t_0} D_t\int_{t-\tau}^{t} E^T(r)QE(r)dr,\\
&\le 2E^T(t)P_{t_0}D_t^{\alpha}E(t) + E^T(t)QE(t) - E^T(t-\tau)QE(t-\tau),\\
&= -2CE^T(t)P(K-C)E(t) + 2E^T(t)P\bar{A}\tilde{f}(E(t)) + 2E^T(t)P\bar{B}\tilde{f}(\bar{E}(t-\tau)),\\
&\quad +E^T(t)QE(t) - E^T(t-\tau)QE(t-\tau)\\
&= E^T(t)P(K-C)E(t) + E^T(t)(K-C)^T PE(t) + 2E^T(t)P\bar{A}\tilde{f}(E(t))\\
&\quad +2E^T(t)P\bar{B}\tilde{f}(E(t-\tau)) + E^T(t)QE(t) - E^T(t-\tau)QE(t-\tau).
\end{aligned} \tag{29}$$

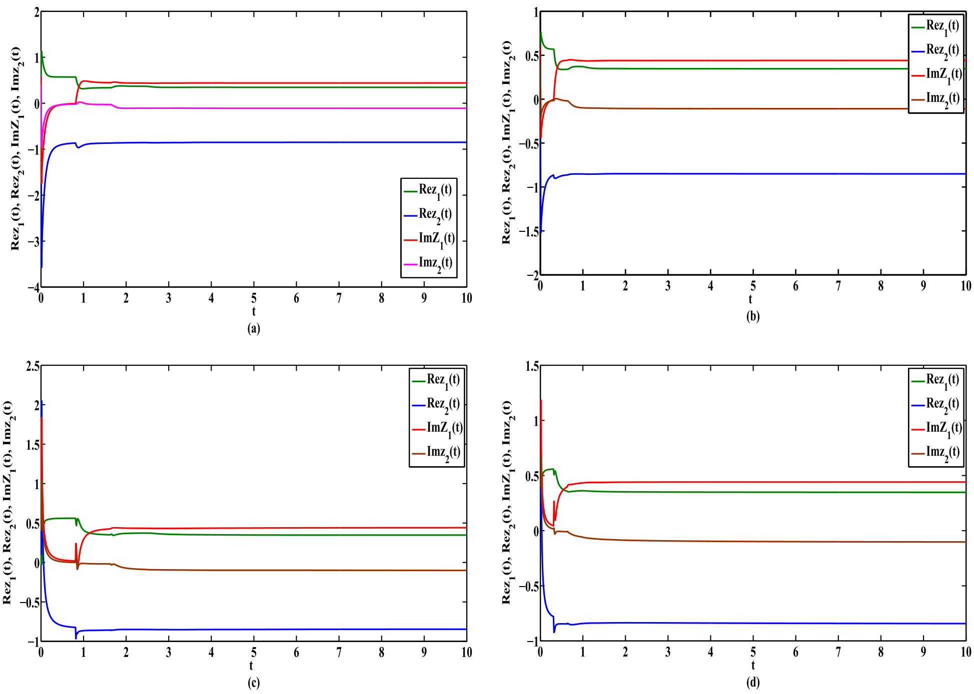

Fig. 1. The real and imaginary parts of the state trajectories of the system (33) is converges approximately to the unique equilibrium point $\bar{z} \approx (\bar{z}_1 = 0.3461 + \mathbf{i}\ 0.4405, \bar{z}_2 = -0.8505 - \mathbf{i}\ 0.1100)^T$ with initial condition $z(t_0) = (z_1(t_0) = 0.5 + \mathbf{i}\ 0.6, z_2(t_0) = 0.1 + \mathbf{i}\ 0.4)^T$ and $I = (4 + \mathbf{i}\ 2, -8 + \mathbf{i})^T$ for $\alpha = 0.9, \tau = 0.8$ in (a), $\alpha = 0.9, \tau = 0.3$ in (b) and $\alpha = 0.6, \tau = 0.8$ in (c), $\alpha = 0.6, \tau = 0.3$ in (d).

For positive diagonal matrices Λ_1, Λ_2, from Assumption 4, one can obtain the inequalities as follows,

$$\begin{bmatrix} E^T(t) \\ \tilde{f}^T(E(t)) \end{bmatrix}^T \begin{bmatrix} F_1\Lambda_1 & -F_2\Lambda_1 \\ * & \Lambda_1 \end{bmatrix} \begin{bmatrix} E(t) \\ \tilde{f}(E(t)) \end{bmatrix} \leq 0, \tag{30}$$

$$\begin{bmatrix} E^T(t-\tau) \\ \tilde{f}^T(E(t-\tau)) \end{bmatrix}^T \begin{bmatrix} F_1\Lambda_2 & -F_2\Lambda_2 \\ * & \Lambda_2 \end{bmatrix} \begin{bmatrix} E(t-\tau) \\ \tilde{f}(E(t-\tau)) \end{bmatrix} \leq 0. \tag{31}$$

From (29) to (31), we obtain

$$_{t_0}D_t^{\alpha} V(t) \leq \zeta^T(t)\ \xi\ \zeta(t) < 0, \tag{32}$$

where $\zeta(t) = \begin{bmatrix} E(t) & E(t-\tau) & \bar{f}(E(t)) & \bar{f}(E(t-\tau)) \end{bmatrix}^T$ and ξ is defined on (26).

Hence, the error system (27) is globally Mittag–Leffler stable. Then, the drive and response systems (20) and (21) are said to be globally asymptotically generalized projective synchronized. The proof is completed.

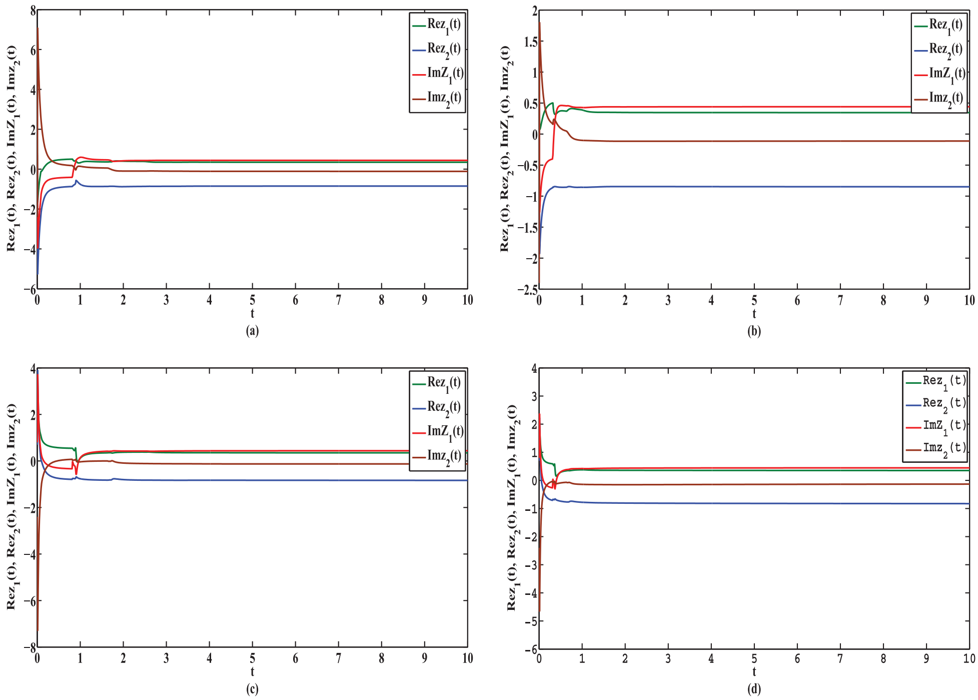

Fig. 2. The real and imaginary parts of the state trajectories of the system (33) is converges approximately to the unique equilibrium point $\bar{z} \approx (\bar{z}_1 = 0.3461 + \mathbf{i}\ 0.4405, \bar{z}_2 = -0.8505 - \mathbf{i}\ 0.1100)^T$ with initial condition $z(t_0) = (z_1(t_0) = 1.5 + \mathbf{i}\ 0.8, z_2(t_0) = 0.7 - \mathbf{i}\ 2.4)^T$ and $I = (4 + \mathbf{i}\ 2, -8 + \mathbf{i})^T$ for $\alpha = 0.9, \tau = 0.8$ in (a), $\alpha = 0.9, \tau = 0.3$ in (b) and $\alpha = 0.6, \tau = 0.8$ in (c), $\alpha = 0.6, \tau = 0.3$ in (d).

4 Numerical examples

In this section, two illustrative examples are proposed to demonstrate the effectiveness of our theoretical results.

Example 1. Consider the two dimensional fractional-order CVNNs with time delays as follows.

$$\begin{cases} {}_0D_t^{\alpha} z_1(t) = -11.4 z_1(t) - (0.3 + \mathbf{i}\ 2)\tanh(z_1(t)) + (0.2 - \mathbf{i}\ 0.6)\tanh(z_2(t)) \\ \qquad\qquad -(0.5 + \mathbf{i})\tanh(z_1(t - \tau) + (0.65 - \mathbf{i}\ 6)\tanh(z_2(t - \tau)) + (4 + \mathbf{i}\ 2), \\ {}_0D_t^{\alpha} z_2(t) = -11.4 z_2(t) - (0.8 + \mathbf{i}\ 0.1)\tanh(z_1(t)) + (-0.5 + \mathbf{i}\ 1.3)\tanh(z_2(t)) \\ \qquad\qquad +(-2.1 + \mathbf{i}\ 0.5)\tanh(z_1(t - \tau) + (1.2 + \mathbf{i}\ 0.4)\tanh(z_2(t - \tau)) + (-8 + \mathbf{i}). \end{cases} \tag{33}$$

Then by the Assumptions 3 and 5 the inequality $(\bar{A} + \bar{B})^T (C^{-1})^T C^{-1} (\bar{A} + \bar{B}) \leq \theta (L^{-1})^2$ holds for the chosen parameters

$$(\bar{A} + \bar{B})^T (C^{-1})^T C^{-1} (\bar{A} + \bar{B}) \approx \begin{bmatrix} -0.0630 & 0.0062 & 0.1868 & 0.0287 \\ -0.0062 & -0.0171 & -0.0287 & 0.1682 \\ -0.1868 & -0.0287 & -0.0630 & 0.0062 \\ 0.0287 & -0.1682 & -0.0062 & -0.0171 \end{bmatrix}.$$

By Theorems 1 and 2, the system (33) has unique equilibrium point $\bar{z} \approx (\bar{z}_1 = 0.3461 + \mathbf{i}\ 0.4405, \bar{z}_2 = -0.8505 - \mathbf{i}\ 0.1100)^T$, and $\bar{z}$ is globally Mittag-Leffler stable for different initial conditions, fractional-order α with time delay. Figures 1 and 2 show

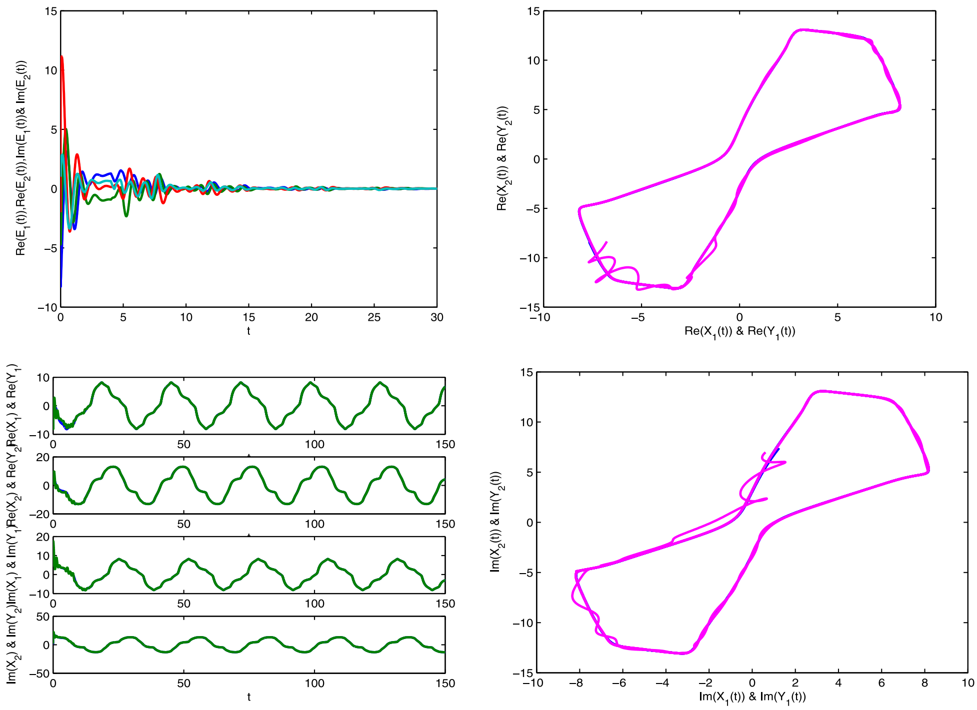

Fig. 3. (a) Evaluation of error state between (34) and (35) and evaluation of drive-response system with $\Lambda = \mathrm{diag}(1,1,1,1)$ (see left banner). (b) Chaotic behavior of drive-response system with $\Lambda = \mathrm{diag}(1,1,1,1)$ (see right banner).

that the equilibrium solution of the system (33) converges to the unique equilibrium point $\bar{z} \approx (\bar{z}_1 = 0.3461 + \mathbf{i}\,0.4405, \bar{z}_2 = -0.8505 - \mathbf{i}\,0.1100)^T$.

Example 2. Consider the two dimensional drive and response fractional-order CVNNs with time delays as follows. The drive system is represented as

$$\begin{aligned} D_t^{0.98}u_1(t) &= -u_1(t) + (1.2 + \mathbf{i}\,0.7)\tanh(u_1(t)) + (2.6 + \mathbf{i}\,1.5)\tanh(u_2(t)) \\ &\quad + (1.3 + \mathbf{i}\,0.4)\tanh(u_1(t-4)) + (-1.5 + \mathbf{i}\,0.8)\tanh(u_2(t-4)), \\ D_t^{0.98}u_2(t) &= -u_2(t) + (6.2 - \mathbf{i}\,1.3)\tanh(u_1(t)) + (-0.3 + \mathbf{i}\,0.5)\tanh(u_2(t)) \\ &\quad + (0.3 + \mathbf{i}\,2.5)\tanh(u_1(t-4)) + (3.2 + \mathbf{i}\,1.7)\tanh(u_2(t-4). \end{aligned} \tag{34}$$

The response system is represented as

$$\begin{aligned} D_t^{0.98}v_1(t) &= -v_1(t) + (1.2 + \mathbf{i}\,0.7)\tanh(v_1(t)) + (2.6 + \mathbf{i}\,1.5)\tanh(v_2(t)) \\ &\quad + (1.3 + \mathbf{i}\,0.4)\tanh(v_1(t-4)) + (-1.5 + \mathbf{i}\,0.8)\tanh(v_2(t-4)) + \bar{v}_1(t), \\ D_t^{0.98}v_2(t) &= -v_2(t) + (6.2 - \mathbf{i}\,1.3)\tanh(v_1(t)) + (-1.3 + \mathbf{i}\,0.5)\tanh(v_2(t)) \\ &\quad + (0.3 + \mathbf{i}\,2.5)\tanh(v_1(t-4)) + (3.2 + \mathbf{i}\,1.7)\tanh(v_2(t-4)) + \bar{v}_2(t). \end{aligned} \tag{35}$$

We have to derive the error system as $E(t) = Y(t) - \Lambda X(t)$, with the projective coefficient matrix Λ and the initial conditions of the drive system and response system are taken as $u_1(t_0) = 1.3 + \mathbf{i}\,0.6, u_2(t_0) = 6.2 + \mathbf{i}\,0.7$ and $v_1(t_0) = 2.3 + \mathbf{i}\,0.4, v_2(t_0) = 6.2 + \mathbf{i}\,0.7$ respectively. From (25) the control can be deduced as

$$U(t) = KE(t) + \Lambda\bar{A}\bar{f}(X(t)) - \bar{A}\bar{f}(\Lambda X(t)) + \Lambda\bar{B}\bar{f}(X(t-\tau)) - \bar{B}\bar{f}(\Lambda X(t-\tau)). \tag{36}$$

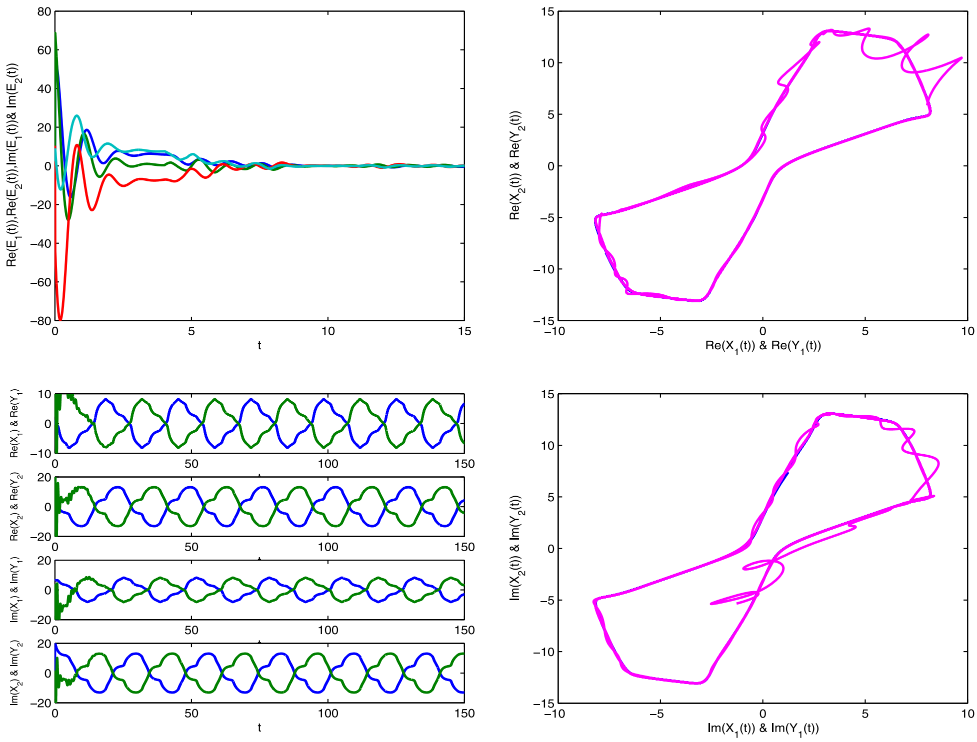

Fig. 4. (a) Evaluation of error state between (34) and (35) and evaluation of drive-response system with $\Lambda = \mathrm{diag}(-1,-1,-1,-1)$ (see left banner). (b) Chaotic behavior of drive-response system with $\Lambda = \mathrm{diag}(-1,-1,-1,-1)$ (see right banner).

It is a real valued system to be designed from the real valued drive-response systems (22) and (23). $K^{2n\times 2n}$ is the coefficient matrix of the adaptive feedback control.

By Theorem 3 and Assumption 3 the systems (34) and (35) are globally asymptotically generalized projective synchronized under the control law (36) for any initial conditions $u_0, v_0 \in \mathbb{C}^n$ when the error vector $E(t)$ converges to zero. Under chosen projective coefficient matrix Λ some special synchronization classes are discussed.

Case (i): In this case, we discus about the complete synchronization of (34) and (35), i.e., Λ is identity. We select the projective coefficient matrix Λ and the coefficient matrix K of the adaptive control is defined as

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{37}$$

$$K = \begin{bmatrix} 0.1454 & 1.4595 & 1.9563 & 2.6119 \\ -1.0327 & 0.0158 & 2.3522 & 3.4510 \\ -2.7535 & -4.4164 & -0.9371 & 0.4532 \\ -1.4580 & -2.1111 & -0.5341 & 0.1251 \end{bmatrix}. \tag{38}$$

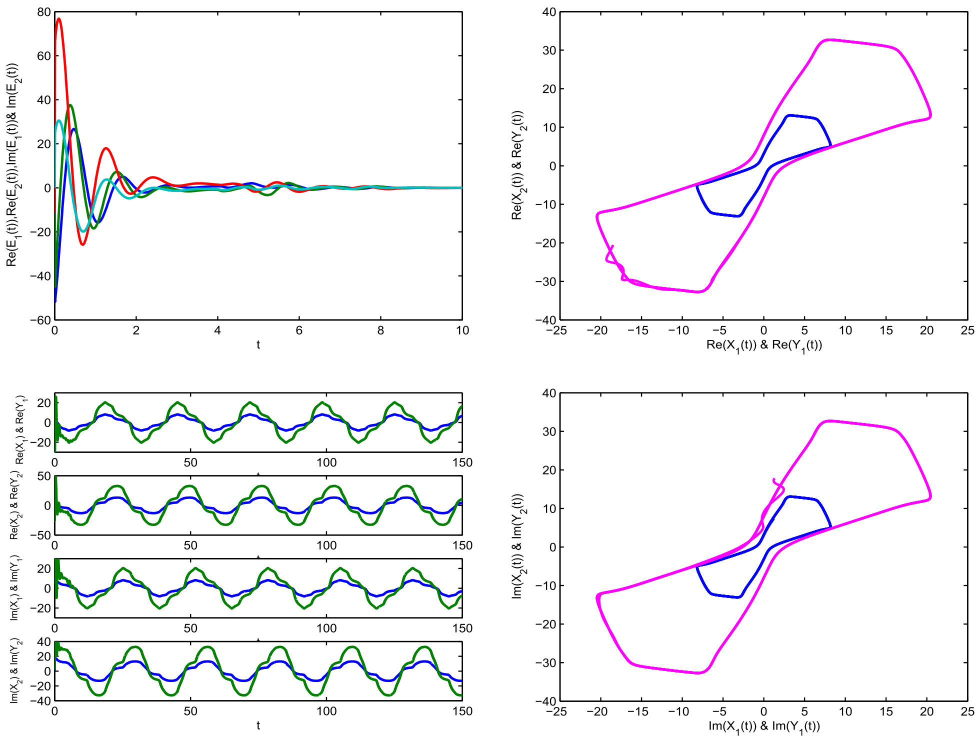

Fig. 5. (a) Evaluation of error state between (34) and (35) and evaluation of drive-response system with $\Lambda = \text{diag}(2.5, 2.5, 2.5, 2.5)$ (see left banner). (b) Chaotic behavior of drive-response system with $\Lambda = \text{diag}(2.5, 2.5, 2.5, 2.5)$ (see right banner).

Figure 3 shows that the state variables of drive system and response system synchronize to the corresponding scaling matrix (37). Also, the time response of the synchronization errors of drive-response systems is stable.

Case (ii): For Λ is chosen as (39) and the controller gain matrix K is defined as (38)

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{39}$$

Figure 4 shows that the state variables of drive system and response system synchronize to the corresponding scaling matrix (39). Also, the time response of the synchronization errors of drive-response systems is stable.

Case (iii): For Λ is chosen as (40) and the controller gain matrix K is defined as (38)

$$\Lambda = \begin{bmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 2.5 \end{bmatrix}. \tag{40}$$

Figure 5 shows that the state variables of drive system and response system synchronize to the corresponding scaling matrix (40). Also, the time response of the synchronization errors of drive-response systems is stable.

5 Conclusion

In this paper, the authors extensively studied the problem of stability and synchronization of fractional-order CVNNs with time delay. Some sufficient conditions have obtained to checking the global stability of considered fractional-order CVNNs with time delay by constructing appropriate Lyapunov–Krasovskii functional approach. In addition, some sufficient criteria were developed in terms of LMI to guarantee the projective synchronization for delayed fractional-order CVNNs with projective scaling matrix. Finally, two numerical examples were provided to demonstrate the effectiveness of our theoretical results.

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