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Positive solutions for a coupled system of semipositone fractional differential equations with the integral boundary conditions

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Abstract. In this paper, we investigate the existence of positive solutions for a system of nonlinear fractional differential equations with sign-changing nonlinearities. The result obtained in this paper essentially improves and extends some well-known results. An example demonstrates the main results.

1 Introduction

In this paper, we consider the system of nonlinear fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha_1}u(t) + \lambda_1 f_1(t,u(t),v(t)) = 0, & \in (0,1), \\ D_{0^+}^{\alpha_2}v(t) + \lambda_2 f_2(t,u(t),v(t)) = 0, & t \in (0,1), \end{cases} \tag{1}$$

with the integral boundary conditions

$$\begin{cases} u(0) = h_1\left(\int_0^1 (\beta_1 u(s) + \gamma_1 v(s))d\phi_1(s)\right), & u^{'}(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = g_1\left(\int_0^1 (\delta_1 u(s) + \xi_1 v(s))d\theta_1(s)\right), & \\ v(0) = h_2\left(\int_0^1 (\beta_2 u(s) + \gamma_2 v(s))d\phi_2(s)\right), & v^{'}(0) = \cdots = v^{(m-2)}(0) = 0, \\ v(1) = g_2\left(\int_0^1 (\delta_2 u(s) + \xi_2 v(s))d\theta_2(s)\right), & \end{cases} \tag{2}$$

where $n-1 < \alpha_1 \le n$, $m-1 < \alpha_2 \le m$, for $n, m \ge 3$, λ_1 and λ_2 are positive parameters, $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \xi_1, \xi_2$ are nonnegative real numbers. $D_{0^+}^{\alpha_1}$ and $D_{0^+}^{\alpha_2}$ are the standard Niemann-Knoxville fractional derivatives, $\int_0^1(\beta_1 u(s) + \gamma_1 v(s))d\phi_1(s)$, $\int_0^1(\delta_1 u(s) + \xi_1 v(s))d\theta_1(s)$, $\int_0^1(\beta_2 u(s) + \gamma_2 v(s))d\phi_2(s)$ and $\int_0^1(\delta_2 u(s) + \xi_2 v(s))d\theta_2(s)$ are Niemann-Stilettoes integrals, we give the following assumptions:

(H_0) h_1, h_2, g_1, g_2: $[0,+\infty) \to [0,+\infty)$ are continuous and nondecreasing;

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(H_1) ϕ_1, ϕ_2, θ_1, θ_2: $[0,1] \to (-\infty, +\infty)$ are increasing nonconstant functions;

(H_2) there exist $M_i > 0$ such that $f_i \in C([0,1] \times [0,+\infty) \times [0,+\infty), [-M_i, +\infty))$, for $i = 1, 2$.

Driven by the wide range of the applications, the FLAPS have been studied by more and more researchers. However, to the best of our knowledge, few papers can be found in the literature for the FLAPS with sign changing nonlinearity, most papers are dealing with the existence of positive solutions when the nonlinear term f is nonnegative, see [1,3,11–14,16,17,19,20]. In fact, (H_2) implies that f is not necessarily nonnegative, monotone, superlunar and sublimer. And also this assumption implies that FLAPS (1) and (2) is semiquinone.

Let $h_i(t) = 0$, $g_i(t) = t$, for $i = 1, 2$, $\delta_1 = \xi_2 = 1$, $\xi_1 = \delta_2 = 0$, the boundary conditions (2) becomes the following boundary conditions:

$$\begin{cases} u(0) = 0, u^{'}(0) = \cdots = u^{(n-2)}(0) = 0, u(1) = \int_0^1 u(s)d\theta_1(s), \\ v(0) = 0, v^{'}(0) = \cdots = v^{(m-2)}(0) = 0, v(1) = \int_0^1 v(s)d\theta_2(s). \end{cases} \tag{3}$$

The existence of positive solutions for the system (1) and (3) has been investigated in [4–6,15].

If $h_i(t) = 0$, $g_i(t) = t$, for $i = 1, 2$, $\xi_1 = \delta_2 = 1$, $\delta_1 = \xi_2 = 0$, the boundary conditions (2) becomes the following boundary conditions:

$$\begin{cases} u(0) = 0, u^{'}(0) = \cdots = u^{(n-2)}(0) = 0, u(1) = \int_0^1 v(s)d\theta_1(s), \\ v(0) = 0, v^{'}(0) = \cdots = v^{(m-2)}(0) = 0, v(1) = \int_0^1 u(s)d\theta_2(s). \end{cases} \tag{4}$$

The existence of positive solutions for the system (1) and (4) has been studied by several authors, see for example [7–10].

By applying the well-known property of the Niemann-Stilettoes (see Lemma 1 in [2]):

$\int_0^1 x(t)d\varphi(t) = x(\eta)(\varphi(1) - \varphi(0)) = \zeta x(\eta)$, for $\eta \in (0,1)$, the boundary conditions (4) becomes the following there-point boundary conditions:

$$\begin{cases} u(0) = 0, u^{'}(0) = \cdots = u^{(n-2)}(0) = 0, u(1) = \zeta_1 v(\eta_1), \\ v(0) = 0, v^{'}(0) = \cdots = v^{(m-2)}(0) = 0, v(1) = \zeta_2 u(\eta_2). \end{cases} \tag{5}$$

In [18], the authors studied the existence and multiplicity of positive solutions for system (1) and (5) with $\alpha_1 = \alpha_2$, $\lambda_1 = \lambda_2$, $\eta_1, \eta_2 \in (0,1)$, $0 < \zeta_1\eta_1\zeta_2\eta_2 < 1$.

It is to be observed that there is no literature to discuss the other types of works on the FLAPS (1) with the conditions (2) except the conditions (3)–(5) which are the special cases of (2). In the other words, the FLAPS (1) and (2) we studied in this paper are more extensive. It is noticed that we could obtain the corresponding results if we give the similar assumptions and use the same method in [4–10,15,18]. However, in this paper, the assumption (H_2) is different from the corresponding assumptions in [4–10,15,18]. This is more controllable and convenient for practical applications. The purpose of this paper is to establish the existence of positive solutions of the FLAPS (1)–(2) by using Guo-Krasnosel'skii fixed point theorem in cones. The associated Green's function for the above problem is given at first, and some useful properties of the Green's function are also obtained. As applications, an example is presented to illustrate the main results.

2 Preliminaries and lemmas

For the convenience of the reader, we recall some basic concepts on fractional calculus. For more details, we refer to [4,11].

Definition 2.1 ([11]). The fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to R$ is given by

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0,$$

provided the right side is pointiest defined on $(0, +\infty)$.

Definition 2.2 ([11]). The Niemann-Knoxville fractional derivative of order $\alpha > 0$ of a function: $f : [0, 1] \to R$ is given by

$$(D_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side is pointiest defined on $(0, +\infty)$.

Lemma 2.3 ([4]). *Let $\alpha > 0$, n be the smallest integer greater than or equal to $\alpha \in (n-1, n]$ and $y \in L^1(0,1)$. The solutions of the fractional equation $D_{0+}^{\alpha} u(t) + y(t) = 0$, $0 < t < 1$,*

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad \textit{for some } t \in (0,1),$$

where c_i $(i = 1, 2, \ldots, n)$ are arbitrary real constants.

Lemma 2.4. *Let $w \in L^1[0,1]$, then the FLAPS*

$$\begin{cases} D_{0+}^{\alpha_1} u(t) + w(t) = 0, \quad t \in (0,1), \\ u(0) = h_1 \left(\int_0^1 (\beta_1 u(s) + \gamma_1 v(s)) d\phi_1(s) \right), u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = g_1 \left(\int_0^1 (\delta_1 u(s) + \xi_1 v(s)) d\theta_1(s) \right), \end{cases} \tag{6}$$

has a unique solution

$$u(t) = \int_0^1 G_1(t,s) w(s) ds + t^{\alpha_1 - 1} g_1 \left(\int_0^1 (\delta_1 u(s) + \xi_1 v(s)) d\theta_1(s) \right) \\ + (t^{\alpha_1 - n} - t^{\alpha_1 - 1}) h_1 \left(\int_0^1 (\beta_1 u(s) + \gamma_1 v(s)) d\phi_1(s) \right),$$

where

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha_1-1}(1-s)^{\alpha_1-1} - (t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha_1-1}(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, & 0 \le t \le s \le 1. \end{cases} \tag{7}$$

Proof. From Lemma 2.3, we have

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} w(s) ds + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2} + \cdots + c_n t^{\alpha_1-n}.$$

By $u^{'}(0) = \cdots = u^{(n-2)}(0) = 0$, we get that $c_i = 0$, for $i = 2, 3, \ldots, n-1$, so,

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} w(s)ds + c_1 t^{\alpha_1 - 1} + c_n t^{\alpha_1 - n}.$$

By $u(0) = h_1\left(\int_0^1 (\beta_1 u(s) + \gamma_1 v(s))d\phi_1(s)\right)$ and $u(1) = g_1\left(\int_0^1 (\delta_1 u(s) + \xi_1 v(s))d\theta_1(s)\right)$, we have

$$\begin{aligned} c_1 &= \int_0^1 \frac{(1-s)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} w(s)ds + g_1\left(\int_0^1 (\delta_1 u(s) + \xi_1 v(s))d\theta_1(s)\right) \\ &\quad - h_1\left(\int_0^1 (\beta_1 u(s) + \gamma_1 v(s))d\phi_1(s)\right), \\ c_n &= h_1\left(\int_0^1 (\beta_1 u(s) + \gamma_1 v(s))d\phi_1(s)\right). \end{aligned}$$

Hence,

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} w(s)ds + \int_0^1 \frac{(1-s)^{\alpha_1 - 1} t^{\alpha_1 - 1}}{\Gamma(\alpha_1)} w(s)ds \\ &\quad + t^{\alpha_1 - 1} g_1\left(\int_0^1 (\delta_1 u(s) + \xi_1 v(s))d\theta_1(s)\right) \\ &\quad + (t^{\alpha_1 - n} - t^{\alpha_1 - 1}) h_1\left(\int_0^1 (\beta_1 u(s) + \gamma_1 v(s))d\phi_1(s)\right) \\ &= \int_0^1 G_1(t,s) w(s)ds + + t^{\alpha_1 - 1} g_1\left(\int_0^1 (\delta_1 u(s) + \xi_1 v(s))d\theta_1(s)\right) \\ &\quad + (t^{\alpha_1 - n} - t^{\alpha_1 - 1}) h_1\left(\int_0^1 (\beta_1 u(s) + \gamma_1 v(s))d\phi_1(s)\right). \end{aligned}$$

Lemma 2.5 ([15])**.** *The Green function G_1 defined by (7) satisfies the following properties:*

(i) $G_1 : [0,1] \times [0,1] \to [0, +\infty)$ is a continuous function, $G_1(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$ and

$G_1(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$;

(ii) $G_1(t,s) \leq \varphi_1(s)$, for all $(t,s) \in [0,1] \times [0,1]$, where $\varphi_1(s) = \frac{s(1-s)^{\alpha_1 - 1}}{\Gamma(\alpha_1 - 1)}$;

(iii) $G_1(t,s) \geq k_1(t)\varphi_1(s)$, for all $(t,s) \in [0,1] \times [0,1]$, where

$$k_1(t) = \min\left\{\frac{(1-t)t^{\alpha_1 - 2}}{\alpha_1 - 1}, \frac{t^{\alpha_1 - 1}}{\alpha_1 - 1}\right\} = \begin{cases} \frac{t^{\alpha_1 - 1}}{\alpha_1 - 1}, & 0 \leq t \leq \frac{1}{2}, \\ \frac{(1-t)t^{\alpha_1 - 2}}{\alpha_1 - 1}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Remark 2.6. We can also formulate similar results as Lemmas 2.4 and 2.5 for the FBVPs

$$\begin{cases} D_{0+}^{\alpha_2} v(t) + w(t) = 0, \quad t \in (0,1), \\ v(0) = h_2\left(\int_0^1 (\beta_2 u(s) + \gamma_2 v(s)) d\phi_2(s)\right), \quad v^{'}(0) = \cdots = v^{(m-2)}(0) = 0, \\ v(1) = g_2\left(\int_0^1 (\delta_2 u(s) + \xi_2 v(s)) d\theta_2(s)\right). \end{cases} \tag{8}$$

We denote by G_2, φ_2, k_2 the corresponding functions for the FBVPs (8) defined in a similar manner as G_1, φ_1, k_1, respectively.

The main tool we will use is the following Guo-Krasnosel'skii fixed point theorem in a cone.

Lemma 2.7. *Let E be a Banach space, $K \subseteq E$ be a cone, and Ω_1, Ω_2 be two bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that either*

(i) $\|Tx\| \le \|x\|$, $x \in K \cap \partial\Omega_1$ and $\|Tx\| \ge \|x\|$, $x \in K \cap \partial\Omega_2$ or

(ii) $\|Tx\| \ge \|x\|$, $u \in K \cap \partial\Omega_1$ and $\|Tx\| \le \|x\|$, $x \in K \cap \partial\Omega_2$,

holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 The main results and proofs

We consider the system of nonlinear fractional boundary value problems:

$$\begin{cases} D_{0+}^{\alpha_1} x(t) + \lambda_1(f_1(t, [x(t) - u_0(t)]^*, [y(t) - v_0(t)]^*) + M_1) = 0, \quad t \in (0,1), \\ D_{0+}^{\alpha_2} y(t) + \lambda_2(f_2(t, [x(t) - u_0(t)]^*, [y(t) - v_0(t)]^*) + M_2) = 0, \quad t \in (0,1), \end{cases} \tag{9}$$

with the boundary conditions

$$\begin{cases} x(0) = h_1\left(\int_0^1 (\beta_1 [x(t) - u_0(t)]^* + \gamma_1 [y(t) - v_0(t)]^*) d\phi_1(s)\right), x^{'}(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = g_1\left(\int_0^1 (\delta_1 [x(t) - u_0(t)]^* + \xi_1 [y(t) - v_0(t)]^*) d\theta_1(s)\right), \\ y(0) = h_2\left(\int_0^1 (\beta_2 [x(t) - u_0(t)]^* + \gamma_2 [y(t) - v_0(t)]^*) d\phi_2(s)\right), y^{'}(0) = \cdots = y^{(m-2)}(0) = 0, \\ y(1) = g_2\left(\int_0^1 (\delta_2 [x(t) - u_0(t)]^* + \xi_2 [y(t) - v_0(t)]^*) d\theta_2(s)\right), \end{cases} \tag{10}$$

where $[z(t)]^* = \begin{cases} z(t), & z(t) \ge 0, \\ 0, & z(t) < 0, \end{cases}$

$$u_0(t) = \lambda_1 M_1 \int_0^1 G_1(t,s) ds = \frac{\lambda_1 M_1}{\Gamma(\alpha_1 + 1)} t^{\alpha_1 - 1}(1 - t),$$
$$v_0(t) = \lambda_2 M_2 \int_0^1 G_2(t,s) ds = \frac{\lambda_2 M_2}{\Gamma(\alpha_2 + 1)} t^{\alpha_2 - 1}(1 - t).$$

We shall prove that there exists a solution (x, y) for the problem (9)–(10) with $x(t) \ge u_0(t)$ and $y(t) \ge v_0(t)$ on $[0,1]$, $x(t) > u_0(t)$ and $y(t) > v_0(t)$ on $(0,1)$. In this case, then (u,v) with $u(t)=x(t)-u_0(t)$ and $v(t)=y(t)-v_0(t)$ is a nonnegative solution (positive on $(0,1)$) of the problem (1)–(2). Since for $t\in(0,1)$, we have

$$\begin{aligned}
D_{0+}^{\alpha_1}u(t) &= D_{0+}^{\alpha_1}x(t)-D_{0+}^{\alpha_1}u_0(t)\\
&= -\lambda_1(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_1)+\lambda_1 M_1\\
&= -\lambda_1 f_1(t,u(t),v(t)),\\
D_{0+}^{\alpha_1}v(t) &= D_{0+}^{\alpha_1}y(t)-D_{0+}^{\alpha_1}v_0(t)\\
&= -\lambda_2(f_2(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_2)+\lambda_2 M_2\\
&= -\lambda_2 f_2(t,u(t),v(t)),
\end{aligned}$$

and

$$\begin{aligned}
&u(0)=x(0)-u_0(0)=h_1\left(\int_0^1(\beta_1 u(t)+\gamma_1 v(t))d\phi_1(s)\right),\\
&u'(0)=x'(0)-u_0'(0)=0,\ldots,u^{(n-2)}(0)=x^{(n-2)}(0)-u_0^{(n-2)}(0)=0,\\
&u(1)=x(1)-u_0(1)=g_1\left(\int_0^1(\delta_1 u(t)+\xi_1 v(t))d\theta_1(s)\right),\\
&v(0)=y(0)-v_0(0)=h_2\left(\int_0^1(\beta_2 u(t)+\gamma_2 v(t))d\phi_2(s)\right),\\
&v'(0)=y'(0)-v_0'(0)=0,\ldots,v^{(n-2)}(0)=y^{(n-2)}(0)-v_0^{(n-2)}(0)=0,\\
&v(1)=y(1)-v_0(1)=g_2\left(\int_0^1(\delta_2 u(t)+\xi_2 v(t))d\theta_2(s)\right).
\end{aligned}$$

Thus, in what follows, we will concentrate our study on the problem (9)–(10).

Let the Banach space $E=C[0,1]$ be endowed with the norm $\|x\|=\max\limits_{0\le t\le 1}|x(t)|$ and $\|(x,y)\|^*=\|x\|+\|y\|$, for $(x,y)\in E\times E$. Define the cones $K_1,K_2\subset E$ by

$$\begin{aligned}
K_1&=\{x\in E: x(t)\ge k_1(t)\|x\|,\forall t\in[0,1]\},\\
K_2&=\{y\in E: y(t)\ge k_2(t)\|y\|,\forall t\in[0,1]\},
\end{aligned}$$

and $K=K_1\times K_2\subset E\times E$. Define the operators $T_i:K_i\to E$ $(i=1,2)$ by

$$\begin{aligned}
T_1(x,y)(t)={}&\lambda_1\int_0^1 G_1(t,s)(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_1)ds\\
&+t^{\alpha_1-1}g_1\left(\int_0^1(\delta_1[x(t)-u_0(t)]^*+\xi_1[y(t)-v_0(t)]^*)d\theta_1(s)\right)\\
&+(t^{\alpha_1-n}-t^{\alpha_1-1})h_1\left(\int_0^1(\beta_1[x(t)-u_0(t)]^*+\gamma_1[y(t)-v_0(t)]^*)d\phi_1(s)\right),\\
T_2(x,y)(t)={}&\lambda_2\int_0^1 G_2(t,s)(f_2(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_2)ds\\
&+t^{\alpha_2-1}g_2\left(\int_0^1(\delta_2[x(t)-u_0(t)]^*+\xi_2[y(t)-v_0(t)]^*)d\theta_2(s)\right)\\
&+(t^{\alpha_2-m}-t^{\alpha_2-1})h_2\left(\int_0^1(\beta_2[x(t)-u_0(t)]^*+\gamma_2[y(t)-v_0(t)]^*)d\phi_2(s)\right),
\end{aligned}$$

and an operator $T : E \times E \to E \times E$ by

$$T(x,y) = (T_1(x,y), T_2(x,y)), \quad \text{for} \quad (x,y) \in E \times E.$$

It is clear that the existence of a positive solution to the system (9)–(10) is equivalent to the existence of fixed points of the operator T.

Lemma 3.1. *If* (H_0)*–*(H_2)*, then* $T : K \to K$ *is completely continuous.*

Proof. For all $(x,y) \in K$, by view of Lemma 2.5 and (H_2), we have

$$\begin{aligned} T_1(x,y)(t) = {} & \lambda_1 \int_0^1 G_1(t,s)(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*) + M_1)ds \\ & + t^{\alpha_1-1} g_1\left(\int_0^1 (\delta_1[x(t)-u_0(t)]^* + \xi_1[y(t)-v_0(t)]^*)d\theta_1(s)\right) \\ & + (t^{\alpha_1-n} - t^{\alpha_1-1}) h_1\left(\int_0^1 (\beta_1[x(t)-u_0(t)]^* + \gamma_1[y(t)-v_0(t)]^*)d\phi_1(s)\right) \\ \le {} & \lambda_1 \int_0^1 \varphi_1(s)(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*) + M_1)ds \\ & + g_1\left(\int_0^1 (\delta_1[x(t)-u_0(t)]^* + \xi_1[y(t)-v_0(t)]^*)d\theta_1(s)\right) \\ & + h_1\left(\int_0^1 (\beta_1[x(t)-u_0(t)]^* + \gamma_1[y(t)-v_0(t)]^*)d\phi_1(s)\right). \end{aligned} \tag{11}$$

On the other hand, for all $(x,y) \in K$, from (3.3), Lemma 8 and $n-1 < \alpha_1 \le n$, $n \ge 3$, we get

$$\begin{aligned} T_1(x,y)(t) \ge {} & \lambda_1 k_1(t) \int_0^1 \varphi_1(s)(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*) + M_1)ds \\ & + \frac{t^{\alpha_1-1}}{\alpha_1-1} g_1\left(\int_0^1 (\delta_1[x(t)-u_0(t)]^* + \xi_1[y(t)-v_0(t)]^*)d\theta_1(s)\right) \\ & + \frac{t^{\alpha_1-n} - t^{\alpha_1-1}}{\alpha_1-1} h_1\left(\int_0^1 (\beta_1[x(t)-u_0(t)]^* + \gamma_1[y(t)-v_0(t)]^*)d\phi_1(s)\right) \\ \ge {} & \lambda_1 k_1(t) \int_0^1 \varphi_1(s)(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*) + M_1)ds \\ & + \frac{t^{\alpha_1-1}}{\alpha_1-1} g_1\left(\int_0^1 (\delta_1[x(t)-u_0(t)]^* + \xi_1[y(t)-v_0(t)]^*)d\theta_1(s)\right) \\ & + \frac{t^{\alpha_1-2} - t^{\alpha_1-1}}{\alpha_1-1} h_1\left(\int_0^1 (\beta_1[x(t)-u_0(t)]^* + \gamma_1[y(t)-v_0(t)]^*)d\phi_1(s)\right) \\ \ge {} & k_1(t)\Bigg[\lambda_1 \int_0^1 \varphi_1(s)(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*) + M_1)ds \\ & + g_1\left(\int_0^1 (\delta_1[x(t)-u_0(t)]^* + \xi_1[y(t)-v_0(t)]^*)d\theta_1(s)\right) \\ & + h_1\left(\int_0^1 (\beta_1[x(t)-u_0(t)]^* + \gamma_1[y(t)-v_0(t)]^*)d\phi_1(s)\right)\Bigg] \\ \ge {} & k_1(t)\|T_1(x,y)\|. \end{aligned}$$

Consequently, $T_1(x,y) \in K_1$. Similarly, we can show that $T_2(x,y) \in K_2$. Hence, $(T_1(x,y), T_2(x,y)) \in K$, and so $T(K) \subset K$. In addition, standard arguments in the literature guarantee that T is a completely continuous operator.

Theorem 3.2. *Assume that* (H_0)*–*(H_2) *hold. Suppose the following conditions hold:*

(H_3*) there exist nondecreasing functions* $\psi_i \in C([0,\infty),[0,\infty))$*, (*$i=1,2$*) and* $\psi_i(x) > 0$*, for* $x > 0$*,* $i = 1,2$ *such that*

$$f_i(t,u,v)+M_i \leq \psi_i(u+v), \quad \textit{for all} \quad (t,u,v) \in [0,1]\times[0,\infty)\times[0,\infty), \quad i=1,2,$$

(H_4*) there exists* $\tau \in (0,\frac{1}{2})$ *such that* $\lim\limits_{u\to+\infty} \min\limits_{t\in[\tau,1-\tau]} \frac{f_1(t,u,v)}{u} = \infty$*,*
$\lim\limits_{v\to+\infty} \min\limits_{t\in[\tau,1-\tau]} \frac{f_2(t,u,v)}{v} = \infty$.

Thus there exist constants $\lambda_i^* > 0$*, (*$i=1,2$*) such that for any* $\lambda_i \in (0,\lambda_i^*]$*, (*$i=1,2$*), the system of RSVP (1)–(2) has at least one positive solution.*

Proof. We choose $\Omega_1 = \{(x,y) \in E\times E : \|x\| < r, \|y\| < r\}$ with

$$r > \max\{2h_i(r(\beta_i+\gamma_i)(\phi_i(1)-\phi_i(0)))+2g_i(r(\delta_i+\xi_i)(\theta_i(1)-\theta_i(0))),1\}, \quad \text{for } i=1,2.$$

We introduce

$$\lambda_i^* = \min\left\{ \begin{array}{c} \frac{\Gamma(\alpha_i+1)}{M_i(\alpha_i-1)}, \\ \frac{[\frac{r}{2}-h_i(r(\beta_i+\gamma_i)(\phi_i(1)-\phi_i(0)))-g_i(r(\delta_i+\xi_i)(\theta_i(1)-\theta_i(0)))]\Gamma(\alpha_i+2)}{(\alpha_i-1)\psi_i(2r)} \end{array} \right\}, \text{ for } i=1,2.$$

Then for all $(x,y) \in K\cap\partial\Omega_1$, we get $\|x\| = r$ or $\|y\| = r$, and

$$[x(t)-u_0(t)]^* = \begin{cases} x(t)-u_0(t) \leq x(t) \leq r, & x(t)-u_0(t) \geq 0, \\ 0, & x(t)-u_0(t) < 0, \end{cases}$$
$$[y(t)-v_0(t)]^* = \begin{cases} y(t)-v_0(t) \leq x(t) \leq r, & y(t)-v_0(t) \geq 0, \\ 0, & y(t)-v_0(t) < 0. \end{cases}$$

Thus, $[x(t)-u_0(t)]^* \in [0,r]$, $[y(t)-v_0(t)]^* \in [0,r]$, then, for all $i=1,2$, we get

$$f_i(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_1 \leq \psi_i([x(t)-u_0(t)]^*+[y(t)-v_0(t)]^*) \leq \psi_i(2r), \tag{12}$$

$$g_1\left(\int_0^1 (\delta_1[x(t)-u_0(t)]^*+\xi_1[y(t)-v_0(t)]^*)d\theta_1(s)\right) \leq g_1\left(r(\delta_1+\xi_1)(\theta_1(1)-\theta_1(0))\right), \tag{13}$$

$$h_1\left(\int_0^1 (\beta_1[x(t)-u_0(t)]^*+\gamma_1[y(t)-v_0(t)]^*)d\phi_1(s)\right) \leq h_1\left(r(\beta_1+\gamma_1)(\phi_1(1)-\phi_1(0))\right), \tag{14}$$

$$g_1\left(\int_0^1 (\delta_2[x(t)-u_0(t)]^*+\xi_2[y(t)-v_0(t)]^*)d\theta_2(s)\right) \leq g_2\left(r(\delta_2+\xi_2)(\theta_2(1)-\theta_2(0))\right), \tag{15}$$

$$h_2\left(\int_0^1(\beta_2[x(t)-u_0(t)]^*+\gamma_2[y(t)-v_0(t)]^*)d\phi_2(s)\right)$$
$$\leq h_2\left(r(\beta_2+\gamma_2)(\phi_2(1)-\phi_2(0))\right). \tag{16}$$

If $\|x\|=r$, by (12)–(14) and Lemma 2.5, we have

$$\begin{aligned}
T_1(x,y)(t) &= \lambda_1\int_0^1 G_1(t,s)(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_1)ds\\
&\quad+t^{\alpha_1-1}g_1\left(\int_0^1(\delta_1[x(t)-u_0(t)]^*+\xi_1[y(t)-v_0(t)]^*)d\theta_1(s)\right)\\
&\quad+(t^{\alpha_1-n}-t^{\alpha_1-1})h_1\left(\int_0^1(\beta_1[x(t)-u_0(t)]^*+\gamma_1[y(t)-v_0(t)]^*)d\phi_1(s)\right)\\
&\leq \lambda_1\int_0^1 \varphi_1(s)(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_1)ds\\
&\quad+g_1\left(\int_0^1(\delta_1[x(t)-u_0(t)]^*+\xi_1[y(t)-v_0(t)]^*)d\theta_1(s)\right)\\
&\quad+h_1\left(\int_0^1(\beta_1[x(t)-u_0(t)]^*+\gamma_1[y(t)-v_0(t)]^*)d\phi_1(s)\right)\\
&\leq \lambda_1\psi_1(2r)\int_0^1\frac{s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1-1)}ds+g_1\left(r(\delta_1+\xi_1)(\theta_1(1)-\theta_1(0))\right)\\
&\quad+h_1\left(r(\beta_1+\gamma_1)(\phi_1(1)-\phi_1(0))\right)\\
&\leq \lambda_1^*\frac{\psi_1(2r)(\alpha_1-1)}{\Gamma(\alpha_1+2)}+g_1\left(r(\delta_1+\xi_1)(\theta_1(1)-\theta_1(0))\right)\\
&\quad+h_1\left(r(\beta_1+\gamma_1)(\phi_1(1)-\phi_1(0))\right)\\
&\leq \frac{r}{2}.
\end{aligned}$$

If $\|y\|=r$, from (12), (15), (16)and Remark 2.6, we also obtain

$$\begin{aligned}
T_2(x,y)(t) &= \lambda_2\int_0^1 G_2(t,s)(f_2(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_2)ds\\
&\quad+t^{\alpha_2-1}g_2\left(\int_0^1(\delta_2[x(t)-u_0(t)]^*+\xi_2[y(t)-v_0(t)]^*)d\theta_2(s)\right)\\
&\quad+(t^{\alpha_2-m}-t^{\alpha_2-1})h_2\left(\int_0^1(\beta_2[x(t)-u_0(t)]^*+\gamma_2[y(t)-v_0(t)]^*)d\phi_2(s)\right)\\
&\leq \lambda_2\int_0^1 \varphi_2(s)(f_2(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_2)ds\\
&\quad+g_2\left(\int_0^1(\delta_2[x(t)-u_0(t)]^*+\xi_2[y(t)-v_0(t)]^*)d\theta_2(s)\right)\\
&\quad+h_2\left(\int_0^1(\beta_2[x(t)-u_0(t)]^*+\gamma_2[y(t)-v_0(t)]^*)d\phi_2(s)\right)\\
&\leq \lambda_2^*\frac{\psi_2(2r)(\alpha_2-1)}{\Gamma(\alpha_2+2)}+g_2\left(r(\delta_2+\xi_2)(\theta_2(1)-\theta_2(0))\right)\\
&\quad+h_2\left(r(\beta_2+\gamma_2)(\phi_2(1)-\phi_2(0))\right)\\
&\leq \frac{r}{2}.
\end{aligned}$$

Then,

$$\begin{aligned}\|T(x,y)\|^* &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &\le r \le \|x\| + \|y\| = \|(x,y)\|^*, \quad \text{for all} \quad (x,y) \in K \cap \partial\Omega_1. \end{aligned} \tag{17}$$

It follows from condition (H_4), that there exist constants $B > 0$ and $N > 1$ with $\frac{\lambda_i N \tau^{\alpha_i+1}(1-\tau)[(1-\tau)^{\alpha_i-1}-\tau^{\alpha_i-1}]}{\Gamma(\alpha_i+2)2^{\alpha_i}} \ge 1$, for $i = 1, 2$ such that

$$f_1(t,u,v) \ge Nu, \quad \text{for} \quad t \in [\tau, 1-\tau], \quad u \ge B, \quad v \ge 0, \tag{18}$$
$$f_2(t,u,v) \ge Nv, \quad \text{for} \quad t \in [\tau, 1-\tau], \quad u \ge 0, \quad v \ge B. \tag{19}$$

Let $\Omega_2 = \{(x,y) \in E \times E : \|x\| < R, \|y\| < R\}$ with $R = \max\left\{2r, \frac{B(\alpha_1-1)}{\tau^{\alpha_1}}, \frac{B(\alpha_2-1)}{\tau^{\alpha_2}}\right\}$, then for all $(u,v) \in K \cap \partial\Omega_2$, we get $\|x\| = R$ or $\|y\| = R$. If $\|x\| = R$, we have

$$\begin{aligned} x(t) - u_0(t) &= x(t) - \frac{\lambda_1 M_1}{\Gamma(\alpha_1+1)} t^{\alpha_1-1}(1-t) \ge k_1(t)\|x\| - \frac{\lambda_1 M_1}{\Gamma(\alpha_1+1)} t^{\alpha_1-1}(1-t) \\ &= \frac{t^{\alpha_1-1}}{\alpha_1-1} R - \frac{\lambda_1 M_1}{\Gamma(\alpha_1+1)} t^{\alpha_1-1}(1-t) = \frac{t^{\alpha_1-1}}{\alpha_1-1} R\left[1 - \frac{\lambda_1 M_1(\alpha_1-1)}{R\Gamma(\alpha_1+1)}(1-t)\right] \\ &\ge \frac{\tau^{\alpha_1-1}}{\alpha_1-1} R\left[1 - \frac{\lambda_1^* M_1(\alpha_1-1)}{\Gamma(\alpha_1+1)}(1-\tau)\right] \ge \frac{\tau^{\alpha_1} R}{\alpha_1-1}, \quad \text{for} \quad t \in [\tau, \frac{1}{2}], \end{aligned} \tag{20}$$

$$\begin{aligned} x(t) - u_0(t) &= x(t) - \frac{\lambda_1 M_1}{\Gamma(\alpha_1+1)} t^{\alpha_1-1}(1-t) \ge k_1(t)\|x\| - \frac{\lambda_1 M_1}{\Gamma(\alpha_1+1)} t^{\alpha_1-1}(1-t) \\ &= \frac{t^{\alpha_1-2}(1-t)}{\alpha_1-1} R - \frac{\lambda_1 M_1}{\Gamma(\alpha_1+1)} t^{\alpha_1-1}(1-t) = \frac{t^{\alpha_1-2}(1-t)}{\alpha_1-1} R\left[1 - \frac{\lambda_1 M_1(\alpha_1-1)}{R\Gamma(\alpha_1+1)} t\right] \\ &\ge \frac{(\frac{1}{2})^{\alpha_1-2}\tau}{\alpha_1-1} R\left[1 - \frac{\lambda_1^* M_1(\alpha_1-1)}{\Gamma(\alpha_1+1)}(1-\tau)\right] > \frac{\tau^{\alpha_1} R}{\alpha_1-1}, \quad \text{for} \quad t \in [\frac{1}{2}, 1-\tau]. \end{aligned} \tag{21}$$

It follows that (20) and (21), we deduce

$$x(t) - u_0(t) \ge \frac{\tau^{\alpha_1} R}{\alpha_1-1}, \quad \text{for} \quad t \in [\tau, 1-\tau]. \tag{22}$$

By (18) and (22), we have

$$f_1(t, [x(t)-u_0(t)]^*, [y(t)-v_0(t)]^*) \ge N[x(t)-u_0(t)]^* = N[x(t)-u_0(t)] \ge \frac{\tau^{\alpha_1} NR}{\alpha_1-1}, \tag{23}$$

for $t \in [\tau, 1-\tau]$, $[x(t)-u_0(t)]^* = x(t)-u_0(t) \ge \frac{\tau^{\alpha_1} R}{\alpha_1-1} \ge B$, $[y(t)-v_0(t)]^* \ge 0$.

It view of Lemma 2.5 and (23), we have

$$\begin{aligned} \|T_1(u,v)\| = \max_{0\le t\le 1} \Bigg[& \lambda_1 \int_0^1 G_1(t,s)(f_1(t,[x(t)-u_0(t)]^*, [y(t)-v_0(t)]^*) + M_1) ds \\ & + t^{\alpha_1-1} g_1\left(\int_0^1 (\delta_1[x(t)-u_0(t)]^* + \xi_1[y(t)-v_0(t)]^*) d\theta_1(s)\right) \\ & + (t^{\alpha_1-n} - t^{\alpha_1-1}) h_1\left(\int_0^1 (\beta_1[x(t)-u_0(t)]^* + \gamma_1[y(t)-v_0(t)]^*) d\phi_1(s)\right)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&\geq \lambda_1 \max_{0\leq t\leq 1} k_1(t)\int_\tau^{1-\tau}\varphi_1(s)f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)ds\\
&\geq \lambda_1\frac{\tau^{\alpha_1}NR}{\alpha_1-1}\max_{0\leq t\leq 1} k_1(t)\int_\tau^{1-\tau}\varphi_1(s)ds\\
&= \frac{\lambda_1\tau^{\alpha_1}NR}{\alpha_1-1}\cdot\frac{1}{2^{\alpha_1-1}(\alpha_1-1)}\\
&\quad\cdot\frac{\alpha_1\tau(1-\tau)^{\alpha_1}-\alpha_1\tau^{\alpha_1}(1-\tau)+(1-\tau)^{\alpha_1}-\tau^{\alpha_1}}{\alpha_1(\alpha_1+1)\Gamma(\alpha_1-1)}\\
&\geq \frac{\lambda_1\tau^{\alpha_1}NR}{2^{\alpha_1-1}}\cdot\frac{\tau(1-\tau)^{\alpha_1}-\tau^{\alpha_1}(1-\tau)}{(\alpha_1+1)\alpha_1(\alpha_1-1)\Gamma(\alpha_1-1)}\\
&= \frac{\lambda_1 N\tau^{\alpha_1+1}(1-\tau)[(1-\tau)^{\alpha_1-1}-\tau^{\alpha_1-1}]}{\Gamma(\alpha_1+2)2^{\alpha_1}}2R\geq 2R.
\end{aligned}
$$

So, $\|T_1(u,v)\|\geq 2R,\quad$ for all $(u,v)\in K\cap\partial\Omega_2$.

If $\|y\|=R$, we have

$$
\begin{aligned}
y(t)-v_0(t) &= y(t)-\frac{\lambda_2M_2}{\Gamma(\alpha_2+1)}t^{\alpha_2-1}(1-t)\geq k_2(t)\|y\|-\frac{\lambda_2M_2}{\Gamma(\alpha_2+1)}t^{\alpha_2-1}(1-t)\\
&= \frac{t^{\alpha_2-1}}{\alpha_2-1}R-\frac{\lambda_2M_2}{\Gamma(\alpha_2+1)}t^{\alpha_2-1}(1-t)=\frac{t^{\alpha_2-1}}{\alpha_2-1}R\left[1-\frac{\lambda_2M_2(\alpha_2-1)}{R\Gamma(\alpha_2+1)}(1-t)\right]\\
&\geq \frac{\tau^{\alpha_2-1}}{\alpha_2-1}R\left[1-\frac{\lambda_2^*M_2(\alpha_2-1)}{\Gamma(\alpha_2+1)}(1-\tau)\right]\geq\frac{\tau^{\alpha_2}R}{\alpha_2-1},\quad\text{for}\quad t\in[\tau,\frac{1}{2}],\\
y(t)-v_0(t) &= y(t)-\frac{\lambda_2M_2}{\Gamma(\alpha_2+1)}t^{\alpha_2-1}(1-t)\geq k_2(t)\|y\|-\frac{\lambda_2M_2}{\Gamma(\alpha_2+1)}t^{\alpha_2-1}(1-t)\\
&= \frac{t^{\alpha_2-2}(1-t)}{\alpha_2-1}R-\frac{\lambda_2M_2}{\Gamma(\alpha_2+1)}t^{\alpha_2-1}(1-t)=\frac{t^{\alpha_2-2}(1-t)}{\alpha_2-1}R\left[1-\frac{\lambda_2M_2(\alpha_2-1)}{R\Gamma(\alpha_2+1)}t\right]\\
&\geq \frac{(\frac{1}{2})^{\alpha_2-2}\tau}{\alpha_2-1}R\left[1-\frac{\lambda_2^*M_2(\alpha_2-1)}{\Gamma(\alpha_2+1)}(1-\tau)\right]>\frac{\tau^{\alpha_2}R}{\alpha_2-1},\quad\text{for}\quad t\in[\frac{1}{2},1-\tau],
\end{aligned}
$$

which implies that

$$
y(t)-v_0(t)\geq\frac{\tau^{\alpha_2}R}{\alpha_2-1},\quad\text{for}\quad t\in[\tau,1-\tau]. \tag{24}
$$

From (19) and (24), we have

$$
f_2(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)\geq N[y(t)-v_0(t)]^*=N[y(t)-v_0(t)]\geq\frac{\tau^{\alpha_2}NR}{\alpha_2-1}, \tag{25}
$$

for $t\in[\tau,1-\tau]$, $[x(t)-u_0(t)]^*\geq 0$, $[y(t)-v_0(t)]^*=y(t)-v_0(t)\geq\frac{\tau^{\alpha_2}R}{\alpha_2-1}\geq B$.

It view of Remark 2.6 and (25), we have

$$
\begin{aligned}
\|T_2(u,v)\| = \max_{0\leq t\leq 1}\Bigg[&\lambda_2\int_0^1 G_2(t,s)(f_2(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_2)ds\\
&+t^{\alpha_2-1}g_2\left(\int_0^1(\delta_2[x(t)-u_0(t)]^*+\xi_2[y(t)-v_0(t)]^*)d\theta_2(s)\right)\\
&+(t^{\alpha_2-m}-t^{\alpha_2-1})h_2\left(\int_0^1(\beta_2[x(t)-u_0(t)]^*+\gamma_2[y(t)-v_0(t)]^*)d\phi_2(s)\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\geq \lambda_2 \max_{0\leq t\leq 1} k_2(t)\int_\tau^{1-\tau} \varphi_2(s)f_2(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)ds\\
&\geq \lambda_2\frac{\tau^{\alpha_2}NR}{\alpha_2-1}\max_{0\leq t\leq 1} k_2(t)\int_\tau^{1-\tau}\varphi_2(s)ds\\
&= \frac{\lambda_2\tau^{\alpha_2}NR}{\alpha_2-1}\cdot\frac{1}{2^{\alpha_2-1}(\alpha_2-1)}\cdot\frac{\alpha_2\tau(1-\tau)^{\alpha_2}-\alpha_2\tau^{\alpha_2}(1-\tau)+(1-\tau)^{\alpha_2}-\tau^{\alpha_2}}{\alpha_2(\alpha_2+1)\Gamma(\alpha_2-1)}\\
&\geq \frac{\lambda_2\tau^{\alpha_2}NR}{2^{\alpha_2-1}}\cdot\frac{\tau(1-\tau)^{\alpha_2}-\tau^{\alpha_2}(1-\tau)}{(\alpha_2+1)\alpha_2(\alpha_2-1)\Gamma(\alpha_2-1)}\\
&= \frac{\lambda_2 N\tau^{\alpha_2+1}(1-\tau)[(1-\tau)^{\alpha_2-1}-\tau^{\alpha_2-1}]}{\Gamma(\alpha_2+2)2^{\alpha_2}}2R\geq 2R.
\end{aligned}
$$

So, $\|T_2(u,v)\|\geq 2R,\quad$ for all $(u,v)\in K\cap\partial\Omega_2$. Then,

$$
\begin{aligned}
\|T(x,y)\|^* &= \|T_1(x,y)\|+\|T_2(x,y)\|\\
&\geq 2R\geq \|x\|+\|y\| = \|(x,y)\|^*,\quad \text{for}\quad (x,y)\in K\cap\partial\Omega_2. \qquad (26)
\end{aligned}
$$

Therefore, it follows from Lemma 2.7, relations (17) and (26) that T has a fixed point $(x,y)\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$. That is, $(x,y)=T(x,y)\Leftrightarrow x=T_1(x,y), y=T_2(x,y)$ and $r\leq\|x\|\leq R$, $r\leq\|y\|\leq R$ with $x(t)\geq k_1(t)\|x\|$, $y(t)\geq k_2(t)\|y\|$, for all $t\in[0,1]$. Let $u(t)=x(t)-u_0(t)$, $v(t)=y(t)-v_0(t)$, for all $t\in[0,1]$. Thus we have

$$
\begin{aligned}
u(t) &= x(t)-u_0(t)\\
&\geq k_1(t)\|x\|-\frac{\lambda_1 M_1}{\Gamma(\alpha_1+1)}t^{\alpha_1-1}(1-t)\\
&\geq k_1(t)r-\frac{\lambda_1 M_1}{\Gamma(\alpha_1+1)}t^{\alpha_1-1}(1-t)\\
&= \begin{cases}\frac{t^{\alpha_1-1}}{\alpha_1-1}r-\frac{\lambda_1M_1}{\Gamma(\alpha_1+1)}t^{\alpha_1-1}(1-t), & 0<t\leq\frac{1}{2},\\ \frac{t^{\alpha_1-2}(1-t)}{\alpha_1-1}r-\frac{\lambda_1M_1}{\Gamma(\alpha_1+1)}t^{\alpha_1-1}(1-t), & \frac{1}{2}\leq t<1,\end{cases}\\
&= \begin{cases}\frac{t^{\alpha_1-1}}{\alpha_1-1}\left[r-\frac{\lambda_1M_1(\alpha_1-1)}{\Gamma(\alpha_1+1)}(1-t)\right], & 0<t\leq\frac{1}{2},\\ \frac{t^{\alpha_1-2}(1-t)}{\alpha_1-1}\left[r-\frac{\lambda_1M_1(\alpha_1-1)}{\Gamma(\alpha_1+1)}t\right], & \frac{1}{2}\leq t<1,\end{cases}\\
&\geq \begin{cases}\frac{t^{\alpha_1-1}}{\alpha_1-1}\left[r-\frac{\lambda_1M_1(\alpha_1-1)}{\Gamma(\alpha_1+1)}\right], & 0<t\leq\frac{1}{2},\\ \frac{t^{\alpha_1-2}(1-t)}{\alpha_1-1}\left[r-\frac{\lambda_1M_1(\alpha_1-1)}{\Gamma(\alpha_1+1)}\right], & \frac{1}{2}\leq t<1,\end{cases}\\
&= k_1(t)\left[r-\frac{\lambda_1M_1(\alpha_1-1)}{\Gamma(\alpha_1+1)}\right]\\
&\geq k_1(t)\left[r-\frac{\lambda_1^*M_1(\alpha_1-1)}{\Gamma(\alpha_1+1)}\right]\\
&\geq k_1(t)\,(r-1)>0,\quad \text{for all}\quad t\in(0,1), \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
v(t) &= y(t)-v_0(t)\\
&\geq k_2(t)\|y\|-\frac{\lambda_2 M_2}{\Gamma(\alpha_2+1)}t^{\alpha_2-1}(1-t)\\
&\geq k_2(t)r-\frac{\lambda_2 M_2}{\Gamma(\alpha_2+1)}t^{\alpha_2-1}(1-t)
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{cases} \frac{t^{\alpha_2-1}}{\alpha_2-1} r - \frac{\lambda_2 M_2}{\Gamma(\alpha_2+1)} t^{\alpha_2-1}(1-t), & 0 < t \le \frac{1}{2}, \\ \frac{t^{\alpha_2-2}(1-t)}{\alpha_2-1} r - \frac{\lambda_2 M_2}{\Gamma(\alpha_2+1)} t^{\alpha_2-1}(1-t), & \frac{1}{2} \le t < 1, \end{cases} \\
&= \begin{cases} \frac{t^{\alpha_2-1}}{\alpha_2-1} \left[r - \frac{\lambda_2 M_2(\alpha_2-1)}{\Gamma(\alpha_2+1)}(1-t) \right], & 0 < t \le \frac{1}{2}, \\ \frac{t^{\alpha_2-2}(1-t)}{\alpha_2-1} \left[r - \frac{\lambda_2 M_2(\alpha_1-1)}{\Gamma(\alpha_2+1)} t \right], & \frac{1}{2} \le t < 1, \end{cases} \\
&\ge \begin{cases} \frac{t^{\alpha_2-1}}{\alpha_2-1} \left[r - \frac{\lambda_2 M_2(\alpha_2-1)}{\Gamma(\alpha_2+1)} \right], & 0 < t \le \frac{1}{2}, \\ \frac{t^{\alpha_2-2}(1-t)}{\alpha_2-1} \left[r - \frac{\lambda_2 M_2(\alpha_2-1)}{\Gamma(\alpha_2+1)} \right], & \frac{1}{2} \le t < 1, \end{cases} \\
&= k_2(t) \left[r - \frac{\lambda_2 M_2(\alpha_2-1)}{\Gamma(\alpha_2+1)} \right] \\
&\ge k_2(t) \left[r - \frac{\lambda_2^* M_2(\alpha_2-1)}{\Gamma(\alpha_2+1)} \right] \\
&\ge k_2(t)\,(r-1) > 0, \quad \text{for all} \quad t \in (0,1). \qquad (28)
\end{aligned}
$$

By (27) and (28), we have

$$
\begin{aligned}
u(t) + u_0(t) &= x(t) = T_1(x,y)(t) \\
&= \lambda_1 \int_0^1 G_1(t,s)(f_1(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*) + M_1)ds \\
&\quad + t^{\alpha_1-1} g_1 \left(\int_0^1 (\delta_1 [x(t)-u_0(t)]^* + \xi_1 [y(t)-v_0(t)]^*) d\theta_1(s) \right) \\
&\quad + (t^{\alpha_1-n} - t^{\alpha_1-1}) h_1 \left(\int_0^1 (\beta_1 [x(t)-u_0(t)]^* + \gamma_1 [y(t)-v_0(t)]^*) d\phi_1(s) \right) \\
&= \lambda_1 \int_0^1 G_1(t,s) f_1(t,[u(t)]^*,[v(t)]^*) ds + \lambda_1 M_1 \int_0^1 G_1(t,s) ds \\
&\quad + t^{\alpha_1-1} g_1 \left(\int_0^1 (\delta_1 [u(t)]^* + \xi_1 [v(t)]^*) d\theta_1(s) \right) \\
&\quad + (t^{\alpha_1-n} - t^{\alpha_1-1}) h_1 \left(\int_0^1 (\beta_1 [u(t)]^* + \gamma_1 [v(t)]^*) d\phi_1(s) \right) \\
&= \lambda_1 \int_0^1 G_1(t,s) f_1(t,u(t),v(t)) ds + u_0(t) \\
&\quad + t^{\alpha_1-1} g_1 \left(\int_0^1 (\delta_1 u(t) + \xi_1 v(t)) d\theta_1(s) \right) \\
&\quad + (t^{\alpha_1-n} - t^{\alpha_1-1}) h_1 \left(\int_0^1 (\beta_1 u(t) + \gamma_1 v(t)) d\phi_1(s) \right),
\end{aligned}
$$

this implies that

$$
\begin{aligned}
u(t) = \lambda_1 \int_0^1 G_1(t,s) f_1(t,u(t),v(t)) ds + t^{\alpha_1-1} g_1 \left(\int_0^1 (\delta_1 u(t) + \xi_1 v(t)) d\theta_1(s) \right) \\
+ (t^{\alpha_1-n} - t^{\alpha_1-1}) h_1 \left(\int_0^1 (\beta_1 u(t) + \gamma_1 v(t)) d\phi_1(s) \right). \qquad (29)
\end{aligned}
$$

From (27) and (28), we also obtain

$$\begin{aligned}
v(t)+v_0(t) &= y(t) = T_2(x,y)(t)\\
&= \lambda_2\int_0^1 G_2(t,s)(f_2(t,[x(t)-u_0(t)]^*,[y(t)-v_0(t)]^*)+M_2)ds\\
&\quad + t^{\alpha_2-1}g_2\left(\int_0^1(\delta_2[x(t)-u_0(t)]^*+\xi_2[y(t)-v_0(t)]^*)d\theta_2(s)\right)\\
&\quad +(t^{\alpha_2-m}-t^{\alpha_2-1})h_2\left(\int_0^1(\beta_2[x(t)-u_0(t)]^*+\gamma_2[y(t)-v_0(t)]^*)d\phi_2(s)\right)\\
&= \lambda_2\int_0^1 G_2(t,s)f_2(t,[u(t)]^*,[v(t)]^*)ds+\lambda_2M_2\int_0^1 G_2(t,s)ds\\
&\quad + t^{\alpha_2-1}g_2\left(\int_0^1(\delta_2[u(t)]^*+\xi_2[v(t)]^*)d\theta_2(s)\right)\\
&\quad + (t^{\alpha_2-m}-t^{\alpha_2-1})h_2\left(\int_0^1(\beta_2[u(t)]^*+\gamma_2[v(t)]^*)d\phi_2(s)\right)\\
&= \lambda_2\int_0^1 G_2(t,s)f_2(t,u(t),v(t))ds+v_0(t)\\
&\quad + t^{\alpha_2-1}g_2\left(\int_0^1(\delta_2u(t)+\xi_2v(t))d\theta_2(s)\right)\\
&\quad + (t^{\alpha_2-m}-t^{\alpha_2-1})h_2\left(\int_0^1(\beta_2u(t)+\gamma_2v(t))d\phi_2(s)\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
v(t) &= \lambda_2\int_0^1 G_2(t,s)f_2(t,u(t),v(t))ds+t^{\alpha_2-1}g_2\left(\int_0^1(\delta_2u(t)+\xi_2v(t))d\theta_2(s)\right)\\
&\quad + (t^{\alpha_2-m}-t^{\alpha_2-1})h_2\left(\int_0^1(\beta_2u(t)+\gamma_2v(t))d\phi_2(s)\right). \qquad (30)
\end{aligned}$$

Thus, from (27)–(30), we deduce that (u,v) is a nonnegative solution (positive on $(0,1)$) of the system of RSVP (1)–(2).

4 Example

Consider the system of fractional differential equations VP

$$\begin{cases} D_{0^+}^{\frac{5}{2}}u(t)+\lambda_1[(u+v)^a+\ln((e^{-M_1}-1)t+1)]=0, & t\in(0,1),\\ D_{0^+}^{\frac{7}{2}}v(t)+\lambda_2[(u+v)^b+\ln((1-e^{-M_2})t+e^{-M_2})]=0, & t\in(0,1), \end{cases} \qquad (31)$$

with the boundary conditions

$$\begin{cases} u(0)=\int_0^1(\frac{1}{3}u(s)+\frac{2}{3}v(s))d\frac{s}{8}, \quad u'(0)=0, \quad u(1)=\int_0^1(\frac{1}{4}u(s)+\frac{3}{4}v(s))d\frac{s^2}{8},\\ v(0)=\int_0^1(\frac{2}{5}u(s)+\frac{3}{5}v(s))d\frac{s^3}{8}, \quad v'(0)=v''(0)=0, \quad v(1)=\int_0^1(\frac{1}{6}u(s)+\frac{5}{6}v(s))d\frac{s^4}{8}, \end{cases} \qquad (32)$$

where $a>1$, $b>1$, $M_1>0$, $M_2>0$.

We note that $\alpha_1=\frac{5}{2}$, $\alpha_2=\frac{7}{2}$, $n=3$, $m=4$, $\beta_1=\frac{1}{3}$, $\gamma_1=\frac{2}{3}$, $\delta_1=\frac{1}{4}$, $\xi_1=\frac{3}{4}$, $\beta_2=\frac{2}{5}$, $\gamma_2=\frac{3}{5}$, $\delta_2=\frac{1}{6}$, $\xi_2=\frac{5}{6}$, $g_1=g_2=h_1=h_2=t$, $f_1(t,u,v)=(u+v)^a+$

$\ln((e^{-M_1}-1)t+1)$, $f_2(t,u,v)=(u+v)^b+\ln((1-e^{-M_2})t+e^{-M_2})$, $\phi_1(t)=\frac{t}{8}$, $\phi_2(t)=\frac{t^2}{8}$, $\theta_1(t)=\frac{t^3}{8}$, $\theta_2(t)=\frac{t^4}{8}$.

Let $\psi_1(x)=x^a+M_1$, $\psi_2(x)=x^b+M_2$, for $x\geq 0$, then $f_i(t,u,v)$ for $i=1,2$, $(t,u,v)\in[0,1]\times[0,\infty)\times[0,\infty)$ satisfied the following conditions:

$$\begin{aligned}
&f_1(t,u,v)=(u+v)^a+\ln((e^{-M_1}-1)t+1)\geq -M_1;\\
&f_1(t,u,v)+M_1=(u+v)^a+\ln((e^{-M_1}-1)t+1)+M_1\leq (u+v)^a+M_1=\psi_1(u+v);\\
&f_2(t,u,v)=(u+v)^b+\ln((1-e^{-M_2})t+e^{-M_2})\geq -M_2;\\
&f_2(t,u,v)+M_2=(u+v)^b+\ln((1-e^{-M_2})t+e^{-M_2})+M_1\leq (u+v)^b+M_2=\psi_2(u+v);\\
&\lim_{u\to+\infty}\min_{t\in[\tau,1-\tau]}\frac{f_1(t,u,v)}{u}=+\infty \quad\text{and}\quad \lim_{v\to+\infty}\min_{t\in[\tau,1-\tau]}\frac{f_2(t,u,v)}{v}=+\infty,
\end{aligned}$$

which implies that (H_2)–(H_4) hold.

Let $r=1.4$, an easy computation shows that

$$\begin{aligned}
\lambda_1^*&=\min\left\{\frac{\Gamma(3.5)}{1.5M_1},\frac{0.2\Gamma(4.5)}{1.5(2.8^a+M_1)}\right\}=\frac{0.2\Gamma(4.5)}{1.5(2.8^a+M_1)}\approx\frac{1.5509}{2.8^a+M_1};\\
\lambda_2^*&=\min\left\{\frac{\Gamma(4.5)}{2.5M_2},\frac{0.2\Gamma(5.5)}{2.5(2.8^b+M_2)}\right\}=\frac{0.2\Gamma(5.5)}{2.5(2.8^b+M_2)}\approx\frac{4.1874}{2.8^b+M_2}.
\end{aligned}$$

Thus, all the hypotheses of Theorem 3.2 are satisfied. Hence, the system of RSVP (31)–(32) has at least one nonnegative solution (positive on $(0,1)$) for any $\lambda_i\in(0,\lambda_i^*]$, $(i=1,2)$.

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Author contribution statement

All authors contributed equally in drafting this manuscript and giving the main proofs. All authors read and approved the final manuscript.

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