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Time analysis of forced variable-order fractional Van der Pol oscillator

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Abstract. This paper presents a new discretization technique for the forced Van der Pol oscillator with variable order derivatives. The study introduces the variable-order fractional time derivatives into the state-space model and investigates their influence upon the system dynamics. The resulting model of the variable-order fractional Van der Pol oscillator is solved and analyzed in the time domain.

1 Introduction

Fractional calculus deals with derivatives and integrals of arbitrary order and has found a wide range of applications covering distinct many fields, such as applied mathematics, engineering, and physics [1,2]. In [3,4], the advantages of using fractional-order algorithms in control problems were discussed. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [5–7]. In [8,9], it was shown that the behavior of a viscoelastic material can be correctly described by a fractional model with a small number of model parameters, in contrast with a classical integer order model requiring a large number of model parameters.

In the last decades, fractional order chaotic systems were a focal point of renewed interest for many researchers [10–12]. Recently, many efforts were devoted to the study of chaotic dynamics and control of systems utilizing fractional calculus techniques [13–15].

The study of the nonlinear oscillators is an important topic in the development of the theory of dynamical systems [16–19]. The Van der Pol oscillator (VPO) was first introduced in 1920 with a model for describing the oscillation of a triode vacuum tube in an electrical circuit [20]. The VPO has a long history in the study of various dynamical phenomena, such as heartbeats [21], the action potential of neurons [22] or the radiation of mobile phones [23]. The VPO was also extended to the Burridge–Knopoff model that characterizes earthquake faults with viscous friction [24] and phonation to model the right and left vocal fold oscillators [25]. This state of affairs motivates further study and possible generalizations of the VPO due to its widespread applications.

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The VPO is given by a second-order nonlinear differential equation of type:

$$y''(t)+\varepsilon(y^2(t)-1)y'(t)+y(t)=0, \tag{1}$$

where $\varepsilon \in \mathbb{R}^+$ is a control parameter that reflects the degree of nonlinearity of the system. If $\varepsilon = 0$, then equation (1) represents the simple linear oscillator, else for $\varepsilon \gg 1$ it represents a relaxation oscillator. Equation (1) produces a limit cycle that attracts other solutions except the only trivial one at the unique equilibrium point $y = y' = 0$.

The state-space model of the system, with $y'(t) = x(t)$ is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon(y^2(t)-1) \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{2}$$

The concept of non-integer derivative and integral is used to model the behavior of systems with memory effects. Fractional dynamics revealed a considerable success in the analysis of anomalous diffusion [26], steady heat-conduction [27], viscoelastic rheology [28], control systems [29], double pendulum systems [30], wave dissipation in human tissue, and electrochemical processes [31]. More recently, a number of theoretical and applied studies in physics have attracted increasing attention to variable-order (VO) fractional dynamics, which is a natural extension of the fixed-order fractional dynamics [32–36].

There are several definitions of the VO fractional operators [35–40]. In this paper, we adopt the Riemann-Liouville VO fractional integral operator

$${}_{0+}\mathcal{J}_t^{\alpha(t)}y(t)=\frac{1}{\Gamma(\alpha(t))}\int_{0+}^{t}(t-\xi)^{\alpha(t)-1}y(\xi)d\xi, \quad \mathrm{Re}(\alpha(t))\geq 0, \tag{3}$$

and the definition of the VO fractional derivative operator formulated by Coimbra [41],

$${}_{0+}\mathscr{D}_t^{\alpha(t)}y(t)=\frac{1}{\Gamma(1-\alpha(t))}\int_{0+}^{t}\frac{y'(\xi)d\xi}{(t-\xi)^{\alpha(t)}}+\frac{(y(0+)-y(0-))t^{-\alpha(t)}}{\Gamma(1-\alpha(t))}, \tag{4}$$

where $0 < \alpha(t) \leq 1$, $y(t)$ is continuously differentiable, $y'(t)$ is integrable, and $\Gamma(\cdot)$ is the gamma function.

We can find phenomena of the multi-system interaction, such as temperatures field, stress field, and electromagnetic field that coupled together with dynamic systems. In some cases, the order of the dynamical system is a function of the output of another dynamical system, such as temperature, mass or other parameters. Diaz and Coimbra [40] discussed the dynamical behavior of a viscous-viscoelastic oscillator. The dynamic order formulation of the VPO based on dynamic order operators was proposed in [41]. The main motivation in this paper was to study the influence of VO fractional operators in the dynamical behavior of a forced VPO system.

Inspired on these concepts and studies, the remainder of the paper is organized as follows. Section 2 presents the numerical approximation Riemann-Liouville VO fractional integral operator. Section 3 formulates the procedure for studying the effect of DFDs in the VPO dynamics and analyses the results. Finally, Section 4 outlines the main conclusions.

2 The approximation of the system response

Consider the VO fractional differential equation system

$$\begin{cases} {}_{0+}\mathscr{D}_t^{\alpha(t)}x(t) = f(t, x(t), y(t)), & 0 < \alpha(t) \leq 1 \\ {}_{0+}\mathscr{D}_t^{\beta(t)}y(t) = g(t, x(t), y(t)), & 0 < \beta(t) \leq 1, \end{cases} \tag{5}$$

with initial conditions $x(0+) = x(0-) = x_0$ and $y(0+) = y(0-) = y_0$, or equally

$$\begin{cases} x(t) = x_0 + {}_{0+}\mathcal{J}_t^{\alpha(t)} f(t, x(t), y(t)), & 0 < \alpha(t) \leq 1 \\ y(t) = y_0 + {}_{0+}\mathcal{J}_t^{\beta(t)} g(t, x(t), y(t)), & 0 < \beta(t) \leq 1, \end{cases} \tag{6}$$

where the values of variable-orders, $\alpha(t)$ and $\beta(t)$ are determined simultaneously by the output of another synchronized dynamic system.

The functions $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ are continuous, with $\Omega = (0, T]$ and $T \in \mathbb{R}^+$. Let us define the mesh points $t_n = nh$, $n = 0, 1, \ldots, N$, where h denotes the uniform step size. A discrete model of equation (6) given by

$$\begin{cases} x_{n+1} = F(t_n, x_n, y_n, \alpha_n, h) \\ y_{n+1} = G(t_n, x_n, y_n, \beta_n, h), \end{cases} \tag{7}$$

where F and G are such that the solution of the equation (6) exists and it is unique. In addition, x_n and y_n denote approximations of $x(t_n)$ and $y(t_n)$, respectively.

To obtain the discrete solution of equation (6), we need to obtain its Riemann–Liouville VO fractional integral. At any time t_n, we have

$$\begin{aligned} \left({}_{0+}\mathcal{J}_t^{\alpha(t)} y(t)\right)(t_n) &= \int_0^{t_n} \frac{1}{\Gamma(\alpha_n)} (t_n - \xi)^{\alpha_n - 1} y(\xi) d\xi \\ &= \sum_{j=0}^{n-2} \int_{jh}^{(j+2)h} \frac{1}{\Gamma(\alpha_n)} (t_n - \xi)^{\alpha_n - 1} y(\xi) d\xi. \end{aligned}$$

The function $y(\xi)$ in integrand can be replaced by a second order Newton interpolation polynomial yielding

$$\begin{aligned} \left({}_{0+}\mathcal{J}_t^{\alpha(t)} y(t)\right)(t_n) = \sum_{j=0}^{n-2} \int_{jh}^{(j+2)h} & \frac{1}{\Gamma(\alpha_n)} \\ & \times (t_n - \xi)^{\alpha_n - 1} \Bigg[y(t_j) \\ & + \frac{y(t_{j+1}) - y(t_j)}{h} (\xi - t_j) \\ & + \frac{y(t_{j+2}) - 2y(t_{j+1}) + y(t_j)}{2h^2} (\xi - t_j)(\xi - t_{j+1}) \Bigg] d\xi, \end{aligned}$$

where ξ is an auxiliary variable belong to the interval $(0, t_n]$.

Then, approximating the Riemann–Liouville VO fractional integral, yields

$$\left({}_{0+}\mathcal{J}_t^{\alpha(t)} y(t)\right)(t_n) = \sum_{j=0}^{n-2} \left(\tilde{a}_{n,j} y(t_j) + \tilde{b}_{n,j} y(t_{j+1}) + c_{n,j} y(t_{j+2}) \right), \tag{8}$$

where $\tilde{a}_{n,j} = a_{n,j} - b_{n,j} + c_{n,j}$, $\tilde{b}_{n,j} = b_{n,j} - 2c_{n,j}$,

$$a_{n,j} = \frac{h^{\alpha_n}}{2\Gamma(\alpha_n+1)}\Big((n-j)^{\alpha_n} - (n-j-2)^{\alpha_n}\Big),$$
$$b_{n,j} = \frac{h^{\alpha_n}}{2\Gamma(\alpha_n+2)}\Big((n-j)^{\alpha_n+1} - (n-j+2\alpha_n)(n-j-2)^{\alpha_n}\Big),$$

and

$$c_{n,j} = \frac{h^{\alpha_n}}{2\Gamma(\alpha_n+3)}\Big((n-j-\frac{1}{2}\alpha_n-1)(n-j)^2)^{\alpha_n} - \Big(j^2-(2n-1+\frac{3}{2}\alpha_n)j+n^2-(1-\frac{3}{2}\alpha_n)n+\alpha_n^2\Big)(n-j-2)^{\alpha_n}\Big).$$

Proposition 1. *If we suppose $y(t) \in C^3(\Omega)$ and $Re(\alpha(t)) \geq 0$, then there exists a constant $C_{\alpha(\cdot)} > 0$, such that*

$$\|{}_{0+}\mathcal{J}_{t_n}^{\alpha(t)}y(t) - {}_{0+}\mathcal{J}_{t_n}^{\alpha(t)}\tilde{y}(t)\|_\infty \leq C_{\alpha(\cdot)}\|y^{(3)}(\eta)\|_\infty h^{3+\alpha_n}, \tag{9}$$

where $t_n = nh$ and η are arbitrary values belong to Ω.

Proof. The second order Newton interpolation polynomial $\tilde{y}(t)$, satisfies

$$y(t) - \tilde{y}(t) = \frac{y^{(3)}(t)}{3!}(t-t_k)(t-t_{k+1})(t-t_{k+2}); \quad t \in (t_k, t_{k+2}),$$

and

$$\|y(t) - \tilde{y}(t)\|_\infty \leq Mh^3\|y^{(3)}(\eta)\|_\infty, \quad \forall t \in \Omega,\ \exists \eta \in (0,T), \tag{10}$$

for some constant $M \in \mathbb{R}^+$. Thus, we have

$$\begin{aligned}\|{}_{0+}\mathcal{J}_{t_n}^{\alpha(t)}y(t) - {}_{0+}\mathcal{J}_{t_n}^{\alpha(t)}\tilde{y}(t)\|_\infty &\leq \frac{(nh)^{\alpha_n}}{\Gamma(1+\alpha_n)}Mh^3\|y^{(3)}(\eta)\|_\infty \\ &= C_{\alpha(\cdot)}\|y^{(3)}(\eta)\|_\infty h^{3+\alpha_n} = \mathcal{O}(h^{3+\alpha_n})\end{aligned}$$

using above inequality yields

$$\begin{aligned}\|{}_{0+}\mathcal{J}_{t_n}^{\alpha(t)}y(t) - {}_{0+}\mathcal{J}_{t_n}^{\alpha(t)}\tilde{y}(t)\|_\infty &\leq \frac{(nh)^{\alpha_n}}{\Gamma(1+\alpha_n)}Mh^3\|y^{(3)}(\eta)\|_\infty \\ &= C_{\alpha(\cdot)}\|y^{(3)}(\eta)\|_\infty h^{3+\alpha_n} = \mathcal{O}(h^{3+\alpha_n}),\end{aligned}$$

where $C_{\alpha(\cdot)} = \dfrac{n^{\alpha_n}M}{\Gamma(1+\alpha_n)}$.

3 Variable-order fractional Van der Pol oscillator

The fractional-order VPO have been studied in the literature [42–47]. Nevertheless, these reports differ in the numerical methods adopted to approximate the fractional

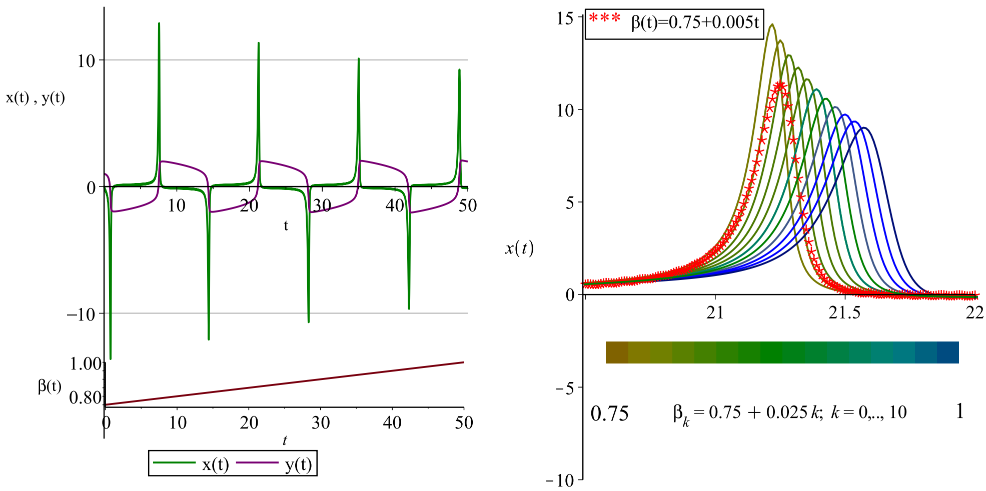

Fig. 1. (a) Time response for $(y(t), x(t))$ of the VO fractional VPO, for $f = 1.5$, $\varepsilon = 5$, $w_f = 2.46\,\mathrm{rad/s}$, with $\beta(t) = 0.75 + 0.005t$. (b) Comparison of the numerical solutions of VO fractional VPO with sequence of the FO, $\beta_k = 0.75 + 0.025k$, $k = 0, \ldots, 10$.

derivative. In this paper, we introduce modified version of equation (1) as follows:

$$\begin{cases} {}_{0+}\mathscr{D}_t^{\gamma(t)} y(t) + \varepsilon(y^2(t) - 1)y'(t) + y(t) = f(t) \\ y(0+) = x_0, \qquad y'(0+) = y_0, \end{cases} \tag{11}$$

where $1 < \gamma(t) \leq 2$, and $f(t)$ denotes the forcing function.

We consider the forced version of the VO fractional VPO (11). For converting the equation (11) into a state-space model, if we assume $y'(t) = x(t)$ then we need an arbitrary function such as $\beta(t)$. Considering $\beta(t) = \gamma(t) - 1$ the state-space model of equation (11) can be written as follows:

$$\begin{cases} y'(t) = x(t) \\ {}_{0+}\mathscr{D}_t^{\beta(t)} x(t) = f(t) - y(t) + \varepsilon(1 - y^2(t))x(t), \quad 0 < \beta(t) \leq 1, \end{cases} \tag{12}$$

with initial conditions $x(0+) = x_0$ and $y(0+) = y_0$.

We simulate the system (12) for $f(t) = f\cos(w_f t)$, where f and w_f are the amplitude and angular frequency of the forcing sinusoidal input.

It is well known that the classical and fractional forced VPO exhibits chaos for the numerical values $\varepsilon = 6$, $w_f = 2.46\,\mathrm{rad/s}$, $f = 1.5$, and $\beta(t) = 1$ or 0.75 in the case of fractional order. In this perspective, let us assume numerical values $\varepsilon = 6$, $w_f = 2.46\,\mathrm{rad/s}$, $f = 1.5$, and $\beta(t) = 0.75 + 0.005t$ in $t \in (0, 50]$.

Figure 1 shows the plot of the time response of system (12) with $\beta(t) = 0.75 + 0.025t$ in the interval $t \in (0, 50]$ (right) including, for comparison, a series of time responses of (12) with sequence of FO, $\beta_k = 0.75 + 0.025k$, $k = 0, \ldots, 10$, in the interval $t \in (20, 22]$ (left). Figure 2 depicts the phase-space solutions of system (12) with $\beta(t) = 0.75, 1, 0.75 + 0.025t$. In Figure 2, one can observe two different periodic motions, so that the phase-space exhibits period doubling with $\beta(t) = 0.75$ and $\beta(t) = 1$ and, moreover, quasi-periodic motion with $\beta(t) = 0.75 + 0.025t$. The time response of system (12) with $\beta(t) = 0.75 + 0.025t$ intersects all the time responses when the FO increases from $\beta(t) = 0.75$ to $\beta(t) = 1$.

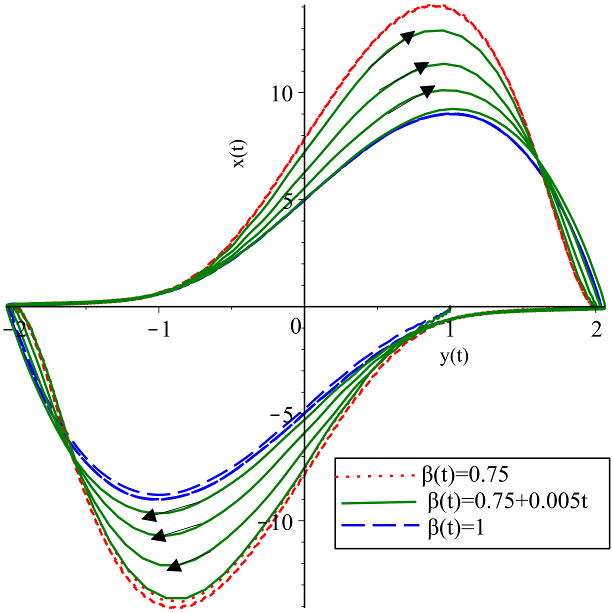

Fig. 2. Phase-space solution of the VO fractional VPO (11), with $\beta(t) = 0.75, 1, 0.75 + 0.005t$, $f = 1.5$, $\varepsilon = 5$, and $w_f = 2.46\,\text{rad/s}$, $t \in (0, 50]$.

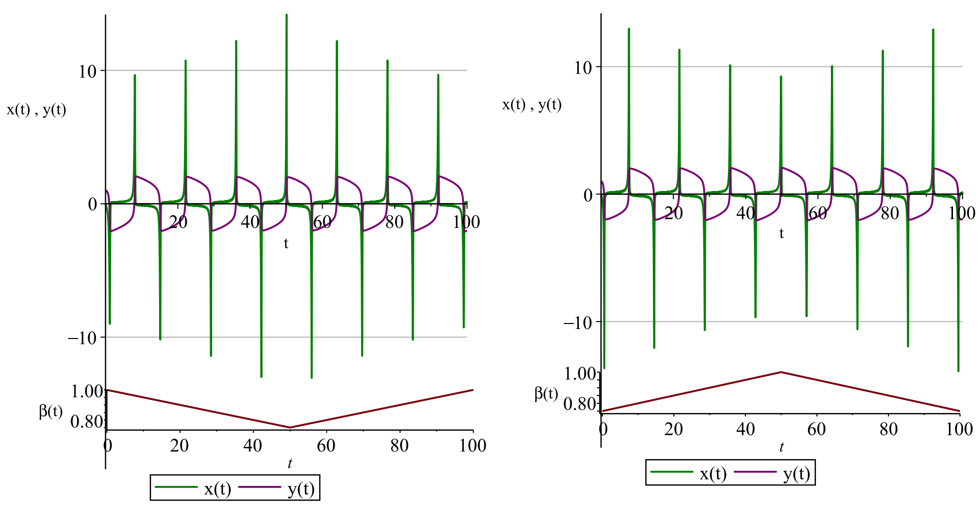

Fig. 3. (a) Time response of the VO fractional VPO (11) with $\beta(t) = \kappa(t)$, and (b) time response with $\beta(t) = \nu(t)$, for $f = 1.5$, $\varepsilon = 5$, $w_f = 2.46\,\text{rad/s}$, $t \in (0, 100]$.

The VO fractional operators represent a good tool for capturing the memory that changes with time. Therefore, by applying VO fractional operators, we obtain controlled output of dynamical systems. To demonstrate this property, we study the time behavior of the response of system (12) with $\beta(t) = 0.75$ and $\beta(t) = 1$, while assuming that $\beta(t) = \kappa(t) = \begin{cases} 1 - 0.005t, & 0 < t \leq 50 \\ 0.5 + 0.005t, & 50 < t \leq 100 \end{cases}$ and

$$\beta(t) = \nu(t) = \begin{cases} 0.75 + 0.005t, & 0 < t \leq 50 \\ 1.25 - 0.005t, & 50 < t \leq 100. \end{cases}$$

The results in Figure 3 illustrate the time response of the VO fractional VPO (11) with $\beta(t) = \kappa(t)$ and $\nu(t)$. Figure 3a, for $\beta(t) = \kappa(t)$, shows that the variation of the time response of (12) increases in $t \in (0, 50]$ and decreases in the interval $t \in (50, 100]$. Figure 3b for $\beta(t) = \nu(t)$ shows that the variation of time response of (12) decreases in the interval $t \in (0, 50]$ and increases in the interval $t \in (50, 100]$.

4 Conclusion

A numerical approximation of Riemann–Liouville VO fractional operator was introduced and applied in the solution of the VPO with VO fractional operators. The forced version of VO fractional VPO was analyzed in the time domain and its response, namely the phase portrait plot was illustrated. The results show that the VO fractional VPO can exhibit a richer behavior than the one obtained with the standard VPO. Furthermore, the results motivate further studies on the modelling with variable order operators. In fact, the VO operators allow more flexibility in the modelling leading to an improved description of the system dynamics.

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