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Modelling of fractal flow in dual media with fractional differentiation with power and generalized Mittag-Leffler laws kernels

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Abstract. The paper has considered the fractal flow in a dual media with fractal properties, where the media could be elastic, heterogeneous and visco-elastic. We argued that, the fractal flow within a geological formation with elastic property cannot be accurately described with the concept of differentiation with local operator, as this operator is unable to include into mathematical formulation the effect of elasticity. Thus to include into mathematical formula the observed facts, we have modified the model by replacing the local derivative with the non-local operator with power. A more complex problem was considered where the geological formation is considered to have visco-elastic and heterogeneity properties. We argued that, the flow within a matrix rock with these two properties cannot either be described with local derivative nor a non-local derivative with power law. In this case two non-local operators were considered, an operator with Mittag-Leffler kernel and Mittag-Leffler-Power law kernel [F. Ali et al., J. Magn. Magn. Mater. **423**, 327 (2017); F. Ali et al., Eur. Phys. J. Plus **131**, 310 (2016); F. Ali et al., Eur. Phys. J. Plus **131**, 377 (2016); F. Ali et al., Nonlinear Sci. Lett. A **8**, 101 (2017); N.A. Sheikh et al., Neural Comput. Appl. (2016) https://doi.org/10.1007/s00521-016-2815-5]. For each model, a detailed study of existence and uniqueness of the system solutions was presented using the fixed point theorem. We solved numerically each model using a more acculturate numerical scheme known as Upwind. Some numerical simulations are presented to underpin the effect of the suggested fractional differentiation.

1 Introduction

Many simple analytical models of groundwater flowing within a geological formation also known as aquifers were developed for the interpretation of drawdown curves and had been proven to yield values that are easily comparable to each other [1–4]. They provide quality criteria for the management of portion of the reservoir. Most are however based on the statement that a non-integer dimension prevails for flow,

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and flow takes place in a single homogenous/fractal medium. Originally introduced by Barenblatt et al. (1960) and Warren and Root (1963), the concept of dual media assumes homogenized flow (and transport) in a fracture field and accounts for the relationships with the porous matrix [5,6]. At relatively large scale (greater than matrix block), the aquifer is mostly drained by connected fractures. Dershowitz and Miller (1995) demonstrated the concept on dual porosity fracture flow and transport, using fracture networks models [7,8]. Quintard and Whitaker (1998) applied the concept in the spatial averaging of the macroscopic behaviour of heterogeneous/fractured porous media. But it is only in 2006 that Frederick et al. proposed a consistence tool for interpreting interference pumping tests based on the dual-medium approach. Considering the couple of equations at the Darcy scale describing flow in a dual medium as [9]:

$$\begin{cases} S_{S_f}\dfrac{\partial h_f}{\partial t} = \nabla\cdot(k_f\cdot\nabla h_f) + \alpha(h_m - h_f) + q_f \\ S_{S_m}\dfrac{\partial h_m}{\partial t} = \alpha(h_m - h_f), \end{cases} \tag{1}$$

where h (in m) is the hydraulic head, k_f (in m/day) is the hydraulic conductivity (which is basically a tensor in multidimensional flow) of fracture continuum, S_s (in m^{-1}) is the specific storage capacity, α (in $\mathrm{m}^{-1}\,\mathrm{day}^{-1}$) the exchange rate coefficient between fractures and the matrix, and q_f (in $\mathrm{m}^3\,\mathrm{day}^{-1}$) a sink–source term from a pumping well that is located in the fracture continuum. The hydraulic conductivity within the matrix (k_m) is considered negligible compared to k_f and is dropped. The application of power laws concept in space (fractal) on the hydrodynamic parameters of both the matrix and fractures is not new, and assumes that the parameters decrease with the lag distance r between the pumped well ($S_{S_{f_0}}$, k_{f_0}, $S_{S_{m_0}}$, α_0) and the observed ones. Many authors including Acuna and Yortsos (1995) and Delay and Porel (2004) have used this type of scaling laws (O'Shaughnessy and Procaccia, 1985) in interpreting interference pumping tests for single media [10–12]. However only few (Frederick and others in 1996), attend of its application to dual media has been recorded in the literature, as the application involves 8 parameters as follows:

$$S_{S_f}(r) = S_{S_{f_0}} r^{-b}, \quad k_f(r) = k_{f_0} r^{-a}, \quad S_{S_m}(r) = S_{S_{m_0}} r^{-c} \quad \text{and} \quad \alpha(r) = \alpha_0 r^{-d}, \tag{2}$$

where a, b, c and d power-law exponents, and considered to be positive. The above model does not account for the heterogeneity, elasticity, visco-elasticity of the geological formation within which the flow has taken place. The flow of water within the fractures system does not needs the heterogeneity effect to be included into mathematical formulation, this is also true for the elasticity and visco-elasticity as the fracture system is homogeneous everywhere in the aquifers. Nevertheless the flow within the matrix rock will encounter natural obstacles and those need to be included into mathematical formulation to obtain better prediction. Therefore, in order to include the physical properties into mathematical formulation the local classical operator of differentiation will be replaced in this paper with a non-local operator able to account for the heterogeneity, elasticity, visco-elasticity of the geological formation within which the flow has taken place. The non-local operator can be the convolution of the power law and the unknown function to account for elasticity, or it could be a convolution of exponential decay law to account for heterogeneity, or could be replaced by a convolution of the generalized Mittag-Leffler function and the unknown function. A more complex model will be adjusted where the classical local time derivative will be replaced by a fractional differential operator with two orders. Before all, we shall present some useful information about the concept of fractional differentiation with power, exponential decay, Mittag-Leffler law and also the concept of fractional derivative with two orders. These three new concepts of fractional differentiation have been introduced and used in many research papers and have been proved to be very efficient mathematical tools for modelling real world problems.

2 Fractional differentiation

We present in this section some useful information about the new trend of fractional differentiation [13–17,21–25]. However, we must first present the definition of existing fractional operator namely the Riemann-Liouville. The Riemann-Liouville fractional integral of a non-differentiable function f is given as:

$$ {}_0^{RL}D_x^{\alpha}(f(x)) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha} f(t)dt. \quad (3) $$

The Caputo-Fabrizio derivative in Riemann-Liouville sense of non-differentiable function f is given as:

$$ {}_0^{CF}D_x^{\alpha}(f(x)) = \frac{M(\alpha)}{(1-\alpha)}\frac{d}{dx}\int_0^x \exp\left[-\frac{\alpha}{1-\alpha}(x-t)\right] f(t)dt. \quad (4) $$

The Atangana-Baleanu fractional derivative in Riemann-Liouville sense of a non-differentiable function f is given as:

$$ {}_0^{ABR}D_x^{\alpha}(f(x)) = \frac{M(\alpha)}{(1-\alpha)}\frac{d}{dx}\int_0^x E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(x-t)^{\alpha}\right] f(t)dt. \quad (5) $$

The Riemann-Liouville fractional integral of a given continuous function f is given as:

$$ {}_0^{RL}J_x^{\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt. \quad (6) $$

The Caputo-Fabrizio fractional integral of a continuous function f is given as:

$$ {}_0^{CF}J_x^{\alpha}(f(x)) = \frac{1-\alpha}{M(\alpha)}f(x) + \frac{\alpha}{M(\alpha)}\int_0^x f(t)dt. \quad (7) $$

The Atangana-Baleanu fractional integral of a continuous function f is given as:

$$ {}_0^{AB}J_x^{\alpha}(f(x)) = \frac{1-\alpha}{AB(\alpha)}f(x) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt. \quad (8) $$

The Atangana fractional derivative with two orders of a continuous function f is given as:

$$ {}_0^{AR}D_x^{\alpha,\beta}(f(x)) = \frac{A(\alpha)}{1-\alpha}\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_0^x (x-t)^{-\beta} E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(x-t)^{\alpha+\beta}\right) f(t)dt. \quad (9) $$

The above nonlinear operators will be used in the following sections as mathematical tools to model the fractal flow within a dual media with inclusion of heterogeneity, elasticity, visco-elasticity and memory effect.

3 Model of fractal flow in dual media accounting for elasticity

In a dual media the water flows within the fracture network and also within the matrix soils. Within the matrix rock, it is without doubt that the media is non-viscous, homogeneous and but there is a memory effect. However the water flowing within the matrix rock flow within a geological formation that can have elastic property, this elasticity cannot be described with the time classical derivative but can efficiently being described with the non-local operator with a power law kernel known as Riemann-Liouville or Caputo fractional derivative. Thus in order to include into mathematical formula the effect of elasticity of the matrix rock, the time local derivative will be replaced by the Caputo fractional derivative to obtain:

$$\begin{cases} S_{S_f}\ {}_0^C D_t^{\alpha}(h_f(r,t)) = \nabla.(k_f.\nabla h_f(r,t)) + \eta(h_m(r,t) - h_f(r,t)) + q_f \\ S_{S_m}\ {}_0^C D_t^{\alpha}(h_m(r,t)) = \eta(h_m(r,t) - h_f(r,t)). \end{cases} \tag{10}$$

We shall first present the existence and uniqueness of the above system.

3.1 Existence of system solutions

The existence of a positive solution for a given fractional differential equation is a big concern to mathematician, because sometimes there exist some complex differential equations that cannot be solved analytically but the proof of existence helps us know that there exists a solution under some conditions within a well-constructed Sobolev space. In this paper, we consider the following Sobolev space:

$$H^1(0,T) = \{u \in L^2(a,b)/u' \in L^2(a,b)\}.$$

We also consider the following Hilbert space where

$$H \in \left\{ u, v / \int_0^t (t-y)^{\alpha} vu\, dy < \infty \right\}.$$

To prove the existence of equation (10), we express the change of hydraulic head within the matrix soil in terms of the change of hydraulic head within the fracture. To achieve this we employ the Laplace transform in time to obtain:

$$\begin{aligned}
&S_{S_m} p^{\alpha} h_m(r,p) = \eta(h_m(r,p) - h_f(r,p)) \\
&(S_{S_m} p^{\alpha} - \eta) h_m(r,p) = -\eta h_f(r,p), \\
&h_m(r,p) = \frac{-\eta h_f(r,p)}{S_{S_m}\left(p^{\alpha} - \frac{\eta}{S_{S_m}}\right)}, \\
&h_m(r,t) = L^{-1}\left(\frac{-\eta h_f(r,p)}{S_{S_m}\left(p^{\alpha} - \frac{\eta}{S_{S_m}}\right)}\right), \\
&h_m(r,t) = -\frac{\eta}{S_{S_m}} \int_0^t h_f(r,y) E_{\alpha}\left(\frac{\eta}{S_{S_m}}(t-y)\right) dy.
\end{aligned} \tag{11}$$

Equation (11) can now be replaced in system (10) to obtain

$$S_{S_f}\, {}_0^C D_0^{\alpha}(h_f(r,t)) = \nabla.(k_f.\nabla h_f(r,t)) + \eta\left(-\frac{\eta}{S_{S_m}}\int_0^t h_f(r,y)E_\alpha\left(\frac{\eta}{S_{S_m}}(t-y)\right)dy - h_f(r,t)\right) + q_f.$$

Let

$$\begin{aligned} &\Gamma_1 : H^1(0,T) \to H^1(0,T) \\ &h \to \Gamma_1 h = {}_0^{RL} I_t^{\alpha}\left(\begin{aligned}&\nabla.(k_f.\nabla h_f(r,t)) \\ &+\eta\left\{\frac{-\eta}{S_{sm}}\int_0^t h_f(r,y)E_\alpha\left(\frac{\eta}{S_{sm}}(t-y)d_y\right) - h_f(r,t)\right\} + q_f\end{aligned}\right), \end{aligned} \tag{12}$$

where

$${}_0^{RL} I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau. \tag{13}$$

We aim to prove that Γ possesses Lipchitz condition.

Let

$$K(r,t,h) = \nabla.(k_f.\nabla h_f(r,t)) - \frac{-\eta^2}{S_{sm}}\int_0^t h_f(r,y)E_\alpha\left(\frac{\eta}{S_{sm}}(t-y)dy\right). \tag{14}$$

Let $h_1, h_2 \in H^1(0,T)$ then

$$\begin{aligned} \|K(r,t,h_1) - K(r,t,h_2)\|_{H^1(0,T)} &\le \|\nabla.(k_f.\nabla h_f(r,t)) - \nabla(k_f\nabla h_2(r,t))\| + \left\|\frac{\eta^2}{S_{sm}}\right\|, \\ &\quad \times \int_0^t |h_1(r,y) - h_2(r,y)|\, E_\alpha\left(\frac{\eta}{S_{Sm}}(t-y)\right)dy \\ &\le \theta_1\theta_2\|k_f\|\,\|h_1 - h_2\| + \left\|\frac{\eta^2}{S_{Sm}}\right\|\,\|h_1 - h_2\|_{H^1(0,T)} M \\ &\quad + \|\eta\|\,\|h_1 - h_2\| \\ &\le \left(\theta_1\theta_2\|k_f\| + \left\|\frac{\eta^2}{S_{Sm}}\right\| M + \|\eta\|\right)\|h_1 - h_2\|_{H^1(0,T)} \\ &\le l\,\|h_1 - h_2\|. \end{aligned} \tag{15}$$

With the definition of K, we define the following function

$$\Gamma h = \frac{1}{\Gamma(\alpha)}\int_0^t K(r,\tau,h)(t-\tau^{\alpha-1})d\tau.$$

Let h_1 and h_2 be elements of $H^1(0,T)$

$$
\begin{aligned}
\|\Gamma_1 h_1 - \Gamma_1 h_2\|_{H^1(0,T)} &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (K(r,\tau,h_1) - K(r,\tau,h_2)(t-\tau)^{\alpha-1} d\tau \right\|_{H^1(0,T)} \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t \|K(r,\tau,h_1) - K(r,\tau,h_2)\| (t-\tau)^{\alpha-1} d\tau \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t l\, \|h_1 - h_2\| \, (t-\tau)^{\alpha-1} d\tau \\
&\le \frac{1}{\Gamma(\alpha)} \|h_1 - h_2\|_{H^1(0,T)} \frac{T^{\alpha-1}}{\alpha} \\
&\le \frac{lT^\alpha}{\Gamma(\alpha+1)} \|h_1 - h_2\|_{H^1(0,T)} \\
&\le \beta \|h_1 - h_2\|_{H^1(0,T)} \, .
\end{aligned} \tag{16}
$$

Let us consider the following recursive formula

$$
\begin{aligned}
h_f^{n+1}(r,t) &= \Gamma_1 h_f^n = h_f(r,0) + \frac{1}{\Gamma(\alpha)} \int_0^t K(h_f^n, r, \tau)(t-\tau)^{\alpha-1} d\tau \\
\left\| \Gamma_1 h_f^n - \Gamma_1 h_f^{n-1} \right\|_{H^1(0,T)} &= \left\| h_f^{n+1} - h_f^n \right\|_{H^1(0,T)} \\
&= \frac{1}{\Gamma(\alpha)} \left\| \int_0^t \{K(h_f^n, r, \tau) - K(h_f^{n-1}, r, \tau)\}(t-\tau)^{\alpha-1} d\tau \right\| \\
&\le \frac{l}{\Gamma(\alpha)} \left\| h_f^{n-1} - h_f^n \right\|_{H^1(0,T)} \frac{T^\alpha}{\alpha} \\
&= \frac{lT^\alpha}{\Gamma(\alpha+1)} \left\| h_f^{n-1} - h_f^n \right\|_{H^1(0,T)} .
\end{aligned} \tag{17}
$$

Recursively on n, we obtain

$$
\left\| \Gamma_1 h_f^n - \Gamma_1 h_f^{n-1} \right\| \le \left(\frac{lT^\alpha}{\Gamma(\alpha+1)} \right)^n \left\| h_f^1(r,t) \right\| . \tag{18}
$$

We chose $\frac{lT^\alpha}{\Gamma(\alpha+1)}$ such that $\frac{lT^\alpha}{\Gamma(\alpha+1)} < 1$ then for $n \to \infty$, $\left\| \Gamma_1 h_f^n - \Gamma_1 h_f^{n-1} \right\| \to 0$, thus $(h_f^n)_{n\in\infty}$ to a Cauchy sequence in a Banch space therefore converge toward h_f. Taking the limit on both sides, we obtain

$$
\lim_{x\to\infty} h_f^{n+1} = \Gamma_1 h_f^n \Leftrightarrow h_f = \Gamma h_f .
$$

This shows that Γ has a solution and is unique.

4 Numerical solution

In this section, we argue the fact that the storativity coefficients within the aquifer follow the power decay law as suggested in the equation. Here we suggest that the storativity coefficient may follow the exponential law with an upper boundary. With this in mind, we present the numerical solution of the system of equations using the upwind numerical scheme in space and the Crank-Nicholson in space. We, first, for each non-local operator, present its numerical approximation for time derivative.

We present first the numerical approximation of Caputo fractional derivative in time. Let $t_{n+1} - t_n = \Delta t, u(t_n, x_i) = u_i^n$

$$\begin{aligned}
{}_0^cD_t^\alpha u(x_i, t_n) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{\partial u(x_i,\tau)}{\partial t}(t_n - \tau)^{-\alpha} d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{u_i^{j+1} - u_i^j}{\Delta t}(t_n - \tau)^{-\alpha} d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \frac{u_i^{j+1} - u_i^j}{\Delta t} \int_{t_j}^{t_{j+1}} (t_n - \tau)^{-\alpha} d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \frac{u_i^{j+1} - u_i^j}{\Delta t} \left(- \frac{Y^{-\alpha+1}}{1-\alpha} \Big|_{t_n - t_j}^{t_n - t_{j+1}} \right) \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{n} \frac{u_i^{j+1} - u_i^j}{\Delta t} \left\{ ((n-j)\Delta t)^{1-\alpha} - ((n-j-1)\Delta t)^{1-\alpha} \right\} \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{n} u_i^{j+1} - u_i^j \left\{ ((n-j)\Delta t)^{1-\alpha} - ((n-j-1)\Delta t)^{1-\alpha} \right\}.
\end{aligned} \tag{19}$$

We consider the second order upwind scheme for first order space derivative

$$\begin{aligned}
u_x^- &= \frac{3u_i^n - 4u_{i-1}^n + 3u_{i-2}^n}{2(\Delta x)}, \\
u_x^+ &= \frac{-u_{i+2}^n - 4u_{i+1}^n + 3u_i^n}{2(\Delta x)}.
\end{aligned} \tag{20}$$

This numerical scheme has been recognized as a powerful mathematical scheme able to have less diffusive compared to the classical first order accurate scheme also it is recognized as a linear upwind differencing scheme. However one can have the third order. With the third order upwind numerical scheme, we have the following discretization.

$$\begin{aligned}
u_x^- &= \frac{2u_i^n + 3u_{i-1}^n - 6u_{i-2}^n + u_{i-3}^n}{6(\Delta x)}, \\
u_x^+ &= \frac{-2u_{i+3}^n + 6u_{i+2}^n - 3u_{i+1}^n - 2u_i^n}{6(\Delta x)}.
\end{aligned} \tag{21}$$

It is argued in the literature that this numerical scheme is less diffusive compared to the second-order scheme [18–20]. We shall note that, it is also known to introduce slight dispersive errors in the region where the gradient is elevated. Thus with second order upwind scheme, we have the following numerical formulas

$$\sum_{j=0}^{n} \frac{h_{fi}^{j+1} - h_{fi}^{j}}{(\Delta t)^{\alpha}\Gamma(2-\alpha)} \left\{(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\right\} S_{sf}(r_i)$$
$$= \frac{k_f^{j+1} - k_f^{j}}{(\Delta r)} \cdot \frac{3h_{fi}^{n} - 4h_{f(i-1)}^{n} + 3h_{f(i-2)}^{n}}{2(\Delta r)}$$
$$+k_f^{i}\left\{\frac{h_{f_1(i+1)}^{n+1} - 2h_{f_1(i)}^{n+1} + h_{f_1(i-1)}^{n+1}}{2(\Delta r)^2} + \frac{h_{f_1(i+1)}^{n} - 2h_{f_1(i)}^{n} + h_{f_1(i-1)}^{n}}{2(\Delta r)^2}\right\}$$
$$+\eta\{h_{ni}^{n} - h_{fi}^{n}\},$$
$$\sum_{j=0}^{n} \frac{h_{mi}^{j+1} - h_{mi}^{j}}{(\Delta t)^{\alpha}\Gamma(2-\alpha)} \left\{(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\right\} S_{sm}(r_i) = \eta(h_{mi}^{n} - h_{fi}^{n})$$
$$\text{if} \quad \frac{k_f^{j+1} - k_f^{j}}{(\Delta r)} > 0 \tag{22}$$

and

$$\sum_{j=0}^{n} \frac{h_{fi}^{j+1} - h_{fi}^{j}}{(\Delta t)^{\alpha}\Gamma(2-\alpha)} \{(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\} S_{Sf}(r_i)$$
$$= \frac{k_f^{j+1} - k_f^{j}}{(\Delta r)} \cdot \frac{-h_{f(i+2)}^{n} - 4h_{f(i+1)}^{n} + 3h_{fi}^{n}}{2(\Delta x)}$$
$$+k_f^{i}\left\{\frac{h_{f_1(i+1)}^{n+1} - 2h_{f_1(i)}^{n+1} + h_{f_1(i-1)}^{n+1}}{2(\Delta r)^2} + \frac{h_{f_1(i+1)}^{n} - 2h_{f_1(i)}^{n} + h_{f_1(i-1)}^{n}}{2(\Delta r)^2}\right\}$$
$$+\eta\{h_{mi}^{n} + h_{f_1}^{n}\},$$
$$\sum_{j=0}^{n} \frac{h_{mi}^{j+1} - h_{mi}^{j}}{(\Delta t)^{\alpha}\Gamma(2-\alpha)} \{(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\} S_{Sm}(r_i) = \eta\{h_{mi}^{n} + h_{f_1}^{n}\}$$
$$\text{if} \quad \frac{k_f^{j+1} - k_f^{j}}{(\Delta r)} < 0. \tag{23}$$

Thus with the third order upwind numerical scheme, we obtain

$$\sum_{j=0}^{n} \frac{h_{fi}^{j+1} - h_{fi}^{j}}{(\Delta t)^{\alpha}\Gamma(2-\alpha)} \{(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\} S_{sf}(r_i)$$
$$= \frac{k_f^{j+1} - k_f^{j}}{(\Delta r)} \cdot \frac{-h_{f(i+2)}^{n} - 4h_{f(i+1)}^{n} + 3h_{fi}^{n}}{2(\Delta x)}$$
$$+k_f^{i}\left\{\frac{h_{f_1(i+1)}^{n+1} - 2h_{f_1(i)}^{n+1} + h_{f_1(i+1)}^{n+1})}{2(\Delta r)^2} + \frac{h_{f_1(i+1)}^{n} - 2h_{f_1(i)}^{n} + h_{f_1(i+1)}^{n})}{2(\Delta r)^2}\right\}$$
$$+\eta\{h_{mi}^{n} - h_{fi}^{n}\},$$
$$\sum_{j=0}^{n} \frac{h_{mi}^{j+1} - h_{mi}^{j}}{(\Delta t)^{\alpha}\Gamma(2-\alpha)} \{(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\} S_{sm}(r_i) = \eta(h_{mi}^{n} - h_{fi}^{n})$$
$$\text{if} \quad \frac{k_f^{j+1} - k_f^{j}}{(\Delta r)} < 0 \tag{24}$$

and

$$\sum_{j=0}^{n} \frac{h_{fi}^{j+1} - h_{fi}^{j}}{(\Delta t)^{\alpha} \Gamma(2-\alpha)} \{(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\} S_{sf}(r_i)$$
$$= \frac{k_f^{j+1} - k_f^{j}}{(\Delta r)} \cdot \frac{-2h_{f(i+3)}^{n} + 6h_{f(i+2)}^{n} - 3h_{f(i+1)}^{n} - 2h_{fi}^{n}}{6(\Delta r)}$$
$$+ k_f^{i} \left\{ \frac{h_{f_1(i+1)}^{n+1} - 2h_{f_1(i)}^{n+1} + h_{f_1(i+1)}^{n+1})}{2(\Delta r)^2} + \frac{h_{f_1(i+1)}^{n} - 2h_{f_1(i)}^{n} + h_{f_1(i-1)}^{n})}{2(\Delta r)^2} \right\}$$
$$+ \eta\{h_{mi}^{n} - h_{fi}^{n}\},$$
$$\sum_{j=0}^{n} \frac{h_{mi}^{j+1} - h_{mi}^{j}}{(\Delta t)^{\alpha} \Gamma(2-\alpha)} \{(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\} S_{sm}(r_i) = \eta(h_{mi}^{n} - h_{fi}^{n})$$
$$\text{if} \quad \frac{k_f^{j+1} - k_f^{j}}{(\Delta r)} < 0. \tag{25}$$

5 Model of fractal flow in dual media accounting for visco-elasticity

In a dual media the water flows within the fracture network and also within the matrix rock as we said before. These matrix rocks possess different characteristics, in this section, we consider the matrix soil with the property of visco-elasticity. We shall note that a suitable or realistic representation of the subsurface may be achieved by putting together the mechanical properties of the elastic solids and that of the viscous fluids. In the resulting medium or material the stress depends both on the strain and the rate of strain together, as well as higher time derivatives of the strain. Such geological formation which combines solid-like and liquid-like behaviour is called visco-elastic. This section considers fractal dual flow simulation in a general heterogeneous inelastic geological formation within the framework of the theory of linear visco-elasticity. In this case, it is assumed that, water flowing within the matrix rock which has visco-elastic property. It is well documented that this real world observation cannot be described with the time classical derivative but can efficiently be described with the non-local operator with a Mittag-Leffler kernel known as Atangana-Baleanu fractional derivative in Caputo and Riemann-Liouville sense. Thus in order to include into mathematical formula the effect of elasticity of the matrix rock, the time local derivative will be replaced by the Atangana-Baleanu fractional derivative to obtain:

$$\begin{cases} S_{S_f}\, {}_{0}^{ABC}D_t^{\alpha}(h_f(r,t)) = \nabla.(k_f.\nabla h_f(r,t)) + \eta(h_m(r,t) - h_f(r,t)) + q_f \\ S_{S_m}\, {}_{0}^{ABC}D_t^{\alpha}(h_m(r,t)) = \eta(h_m(r,t) - h_f(r,t)). \end{cases} \tag{26}$$

We shall first present the existence and uniqueness of the above system.

5.1 Existence of system solutions

In this section, the well-constructed Sobolev space is considered. In this paper, we consider the following Sobolev space

$$H^1(0,T) = \{u \in L^2(a,b)/u' \in L^2(a,b)\}.$$

We also consider the following Hilbert space where

$$H \in \left\{ u, v / \int_0^t E_\alpha \left[-\frac{\alpha}{1-\alpha}(t-y)^\alpha \right] vudy < \infty \right\}. \tag{27}$$

To prove the existence of equation (26), we express the change of hydraulic head within the matrix soil in terms of the change of hydraulic head within the fracture as presented earlier in the case of power law. To achieve this we employ the Laplace transform in time to obtain:

$$\begin{aligned}
&\frac{S_{S_m}(r,\alpha)p^\alpha h_m(r,p)}{p^\alpha + \frac{\alpha}{1-\alpha}} = \eta(h_m(r,p) - h_f(r,p)) \\
&\left(\frac{S_{S_m}(r,\alpha)p^\alpha}{p^\alpha + \frac{\alpha}{1-\alpha}} - \eta \right) h_m(r,p) = -\eta h_f(r,p), \\
&h_m(r,p) = \frac{-\eta h_f(r,p)}{\frac{S_{S_m}(r,\alpha)p^\alpha}{p^\alpha + \frac{\alpha}{1-\alpha}} - \eta}, \\
&h_m(r,t) = L^{-1}\left(\frac{-\eta h_f(r,p)}{\frac{S_{S_m}(r,\alpha)p^\alpha}{p^\alpha + \frac{\alpha}{1-\alpha}} - \eta} \right), \\
&h_m(r,t) = \frac{\eta}{\eta - S_{S_m}(r,\alpha)} \int_0^t h_f(r,y) E_\alpha \left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)}(t-y)^\alpha \right) dy \\
&\quad + \frac{\eta\alpha}{(1-\alpha)(\eta - S_{S_m}(r,\alpha))} \int_0^t h_f(r,y) E_\alpha \left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)}(t-y) \right) dy.
\end{aligned} \tag{28}$$

Equation (28) can now be replaced in system (26) to obtain

$$S_{S_f}\, {}_0^{ABC}D_t^\alpha(h_f(r,t)) = \nabla.(k_f.\nabla h_f(r,t)) + \eta \left\{ \begin{array}{l} \left\{ \begin{array}{l} \frac{\eta}{\eta - S_{S_m}(r,\alpha)} \int_0^t h_f(r,y) E_\alpha \left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)}(t-y)^\alpha \right) dy \\ + \frac{\eta\alpha}{(1-\alpha)(\eta - S_{S_m}(r,\alpha))} \int_0^t h_f(r,y) E_\alpha \left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)}(t-y) \right) dy \end{array} \right\} \\ -h_f(r,t) \end{array} \right\} + q_f. \tag{29}$$

Let us consider the following function

$$\begin{aligned}
&T_1 : H^1(0,T) \to H^1(0,T) \\
&v \to T_1 v \\
&= {}_0^{AB} I_t^\alpha \left\{ \begin{array}{l} \nabla.(k_f.\nabla v(r,t)) \\ + \eta \left\{ \begin{array}{l} \left\{ \begin{array}{l} \frac{\eta}{\eta - S_{S_m}(r,\alpha)} \int_0^t h_f(r,y) E_\alpha \left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)}(t-y)^\alpha \right) dy \\ + \frac{\eta\alpha}{(1-\alpha)(\eta - S_{S_m}(r,\alpha))} \\ \times \int_0^t h_f(r,y) E_\alpha \left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)}(t-y) \right) dy \end{array} \right\} \\ -h_f(r,t) \end{array} \right\} + q_f \end{array} \right\}.
\end{aligned} \tag{30}$$

The fractional integral used here is known as Atangana-Baleanu fractional integral and is given as:

$$ {}_{0}^{AB}I_t^{\alpha} f(t) = \frac{1-\alpha}{AB(\alpha)} f(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{31} $$

We aim to prove that T_1 possesses Lipschitz condition.

Let us consider the following operator:

$$ B(r,t,v) = \nabla.(k_f.\nabla v(r,t)) + \eta \left\{ \begin{array}{l} \left\{ \begin{array}{l} \frac{\eta}{\eta - S_{S_m}(r,\alpha)} \int_0^t v(r,y) E_{\alpha}\left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)} (t-y)^{\alpha} \right) dy \\ + \frac{\eta\alpha}{(1-\alpha)(\eta - S_{S_m}(r,\alpha))} \int_0^t v(r,y) E_{\alpha}\left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)} (t-y) \right) dy \end{array} \right\} \\ -v(r,t) \end{array} \right\}. \tag{32} $$

Let $h_1, h_2 \in H^1(0,T)$ then

$$ \begin{aligned} &\|B(r,t,h_1) - B(r,t,h_2)\|_{H^1(0,T)} \\ &\leq \|\nabla.(k_f.\nabla h_1(r,t)) - \nabla.(k_f.\nabla h_2(r,t))\| \\ &\quad + \left\| \frac{\eta}{\eta - S_{S_m}(r,\alpha)} \right\| \int_0^t \|h_1(r,y) - h_2(r,y)\| E_{\alpha}\left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)} (t-y)^{\alpha} \right) dy \\ &\quad + \left\| \frac{\eta\alpha}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)} \right\| \\ &\quad \times \int_0^t \|h_1(r,y) - h_2(r,y)\| E_{\alpha}\left(\frac{\alpha\eta}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)} (t-y) \right) dy \\ &\leq \theta_1\theta_2 \|k_f\| \|h_1 - h_2\| \\ &\quad + \left\| \frac{\eta M_2}{\eta - S_{S_m}(r,\alpha)} + \frac{\eta\alpha M_1}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)} \right\| \|h_1 - h_2\|_{H^1(0,T)} + \|\eta\| \|h_1 - h_2\| \\ &\leq \left(\theta_1\theta_2 \|k_f\| + \left\| \frac{\eta M_2}{\eta - S_{S_m}(r,\alpha)} + \frac{\eta\alpha M_1}{(\eta - S_{S_m}(r,\alpha))(1-\alpha)} \right\| + \|\eta\| \right) \|h_1 - h_2\|_{H^1(0,T)} \\ &\leq l_1 \|h_1 - h_2\|. \end{aligned} \tag{33} $$

Using the definition of B presented previously, we consider the following operator

$$ Ph = \frac{1-\alpha}{AB(\alpha)} B(r,\tau,h) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t K(r,\tau,h)(t-\tau)^{\alpha-1} d\tau. \tag{34} $$

Let h_1 and h_2 be elements of $H^1(0,T)$

$$ \begin{aligned} \|Ph_1 - Ph_2\|_{H^1(0,T)} &= \left\| \begin{array}{l} \frac{1-\alpha}{AB(\alpha)} \{(B(r,\tau,h_1) - B(r,\tau,h_2))\} \\ + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t (B(r,\tau,h_1) - B(r,\tau,h_2))(t-\tau)^{\alpha-1} d\tau \end{array} \right\|_{H^1(0,T)} \\ &\leq \frac{1-\alpha}{AB(\alpha)} \|\{(B(r,\tau,h_1) - B(r,\tau,h_2))\}\|_{H^1(0,T)} \\ &\quad + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t \|B(r,\tau,h_1) - B(r,\tau,h_2)\|(t-\tau)^{\alpha-1} d\tau \end{aligned} $$

$$\leq \frac{(1-\alpha)l_1}{AB(\alpha)} + \frac{\alpha}{AB(\alpha)\Gamma^2(\alpha)} \int_0^t l_1 \|h_1 - h_2\| (t-\tau)^{\alpha-1} d\tau$$
$$\leq \left\{ \frac{(1-\alpha)l_1}{AB(\alpha)} + \frac{l_1 T^\alpha}{AB(\alpha)\Gamma(\alpha)} \right\} \|h_1 - h_2\|_{H^1(0,T)}$$
$$\leq \left\{ \frac{(1-\alpha)l_1}{AB(\alpha)} + \frac{l_1 T^\alpha}{AB(\alpha)\Gamma^2(\alpha)} \right\} \|h_1 - h_2\|_{H^1(0,T)}$$
$$\leq \beta_1 \|h_1 - h_2\|_{H^1(0,T)}. \tag{35}$$

In this section, the establishment of the existence of the system solutions will be achieved using the Picard iterative approach. Thus let us consider the following Volterra equation based on Atangana-Baleanu fractional integral.

$$h_f^{n+1}(r,t) = Ph_f^n = \frac{1-\alpha}{AB(\alpha)} B(h_f^n, r, \tau) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t B(h_f^n, r, \tau)(t-\tau)^{\alpha-1} d\tau. \tag{36}$$

Thus,

$$\left\| Ph_f^n - Ph_f^{n-1} \right\|_{H^1(0,T)} = \left\| h_f^{n+1} - h_f^n \right\|_{H^1(0,T)}$$
$$= \frac{1-\alpha}{AB(\alpha)} (B(h_f^n, r, \tau) - B(h_f^{n-1}, r, \tau))$$
$$+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \left\| \int_0^t \{B(h_f^n, r, \tau) - B(h_f^{n-1}, r, \tau)\} (t-\tau)^{\alpha-1} d\tau \right\|$$
$$\leq \left\{ \frac{(1-\alpha)l_1}{AB(\alpha)} + \frac{l_1 T^\alpha}{AB(\alpha)\Gamma(\alpha)} \right\} \left\| h_f^{n-1} - h_f^n \right\|_{H^1(0,T)}. \tag{37}$$

Recursively on n, we obtain

$$\left\| Ph_f^n - Ph_f^{n-1} \right\| \leq \left\{ \frac{(1-\alpha)l_1}{AB(\alpha)} + \frac{l_1 T^\alpha}{AB(\alpha)\Gamma(\alpha)} \right\}^n \|h_f^1\|_{H^1(0,T)}. \tag{38}$$

The choice of l_1 such that, for a very large n

$$\left\{ \frac{(1-\alpha)l_1}{AB(\alpha)} + \frac{l_1 T^\alpha}{AB(\alpha)\Gamma(\alpha)} \right\}^n \to 0. \tag{39}$$

$\left\| Ph_f^n - Ph_f^{n-1} \right\| \to 0$, thus $(h_f^n)_{n\in\infty}$ to a Cauchy sequence in a Banach space therefore converge toward h_f. Taking the limit on both sides, we obtain

$$\lim_{x\to\infty} h_f^{n+1} = Ph_f^n \Leftrightarrow h_f = Ph_f.$$

This shows that B has a solution and is unique and the unique solution is the solution of equation (26).

6 Numerical solution with Mittag-Leffler law

We present the numerical solution of the system of equation using the upwind numerical scheme in space and the Crank-Nicholson in space. We present first the numerical approximation of the Atangana-Baleanu fractional derivative in Caputo sense. Let $t_{n+1} - t_n = \Delta t, u(t_n, x_i) = u_i^n$.

Then the Atangana-Baleanu fractional derivative in Caputo sense is approximated as:

$$
\begin{aligned}
{}_0^{ABC}D_t^\alpha u(x_i, t_n) &= \frac{AB(\alpha)}{(1-\alpha)} \int_0^{t_n} \frac{\partial u(x_i, T)}{\partial t} E_\alpha \left\{ -\frac{\alpha}{1-\alpha}(t_n - \tau)^\alpha \right\} d\tau \\
&= \frac{AB(\alpha)}{(1-\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{u_i^{j+1} - u_i^j}{\Delta t} E_\alpha \left\{ -\frac{\alpha}{1-\alpha}(t_n - \tau)^\alpha \right\} d\tau \\
&= \frac{AB(\alpha)}{(1-\alpha)} \sum_{j=0}^{n} \frac{u_i^{j+1} - u_i^j}{\Delta t} \int_{t_j}^{t_{j+1}} E_\alpha \left\{ -\frac{\alpha}{1-\alpha}(t_n - \tau)^\alpha \right\} d\tau \\
&= \frac{AB(\alpha)}{(1-\alpha)} \sum_{j=0}^{n} \frac{u_i^{j+1} - u_i^j}{\Delta t} \left\{ -(t_n - t_{j+1}) E_{\alpha,2} \left(-\frac{\alpha}{1-\alpha}(t_n - t_{j+1}) \right) \right. \\
&\quad \left. + (t_n - t_j) E_{\alpha,2} \left(-\frac{\alpha}{1-\alpha}(t_n - t_j) \right) \right\} \\
&= \frac{AB(\alpha)}{(1-\alpha)} \sum_{j=0}^{n} \frac{u_i^{j+1} - u_i^j}{\Delta t} \left\{ -\Delta t(n-j-1) E_{\alpha,2} \left(-\frac{\alpha \Delta t}{1-\alpha}(n-j-1) \right) \right. \\
&\quad \left. + \Delta t(n-j) E_{\alpha,2} \left(-\frac{\alpha \Delta t}{1-\alpha}(n-j) \right) \right\} \\
&= \frac{AB(\alpha)}{(1-\alpha)} \sum_{j=0}^{n} \left(u_i^{j+1} - u_i^j \right) \left\{ (n-j) E_{\alpha,2} \left(-\frac{\alpha \Delta t}{1-\alpha}(n-j) \right) \right. \\
&\quad \left. - (n-j-1) E_{\alpha,2} \left(-\frac{\alpha \Delta t}{1-\alpha}(n-j-1) \right) \right\}. \qquad (40)
\end{aligned}
$$

Using the above numerical approximation and the Upwind second order in space, we obtain the below numerical formula

$$
\begin{aligned}
&\frac{AB(\alpha)}{(1-\alpha)} \sum_{j=0}^{n} (h_{fi}^{j+1} - h_{fi}^j) \left\{ \begin{array}{l} (n-j) E_{\alpha,2} \left(-\frac{\alpha \Delta t}{1-\alpha}(n-j) \right) \\ -(n-j-1) E_{\alpha,2} \left(-\frac{\alpha \Delta t}{1-\alpha}(n-j-1) \right) \end{array} \right\} S_{sf}(r_i) \\
&\quad = \frac{k_f^{j+1} - k_f^j}{(\Delta r)} \cdot \frac{3h_{fi}^n - 4h_{f(i-1)}^n + 3h_{f(i-2)}^n}{2(\Delta r)} \\
&\quad + k_f^i \left\{ \frac{h_{f_1(i+1)}^{n+1} - 2h_{f_1(i)}^{n+1} + h_{f_1(i-1)}^{n+1}}{2(\Delta r)^2} + \frac{h_{f_1(i+1)}^n - 2h_{f_1(i)}^n + h_{f_1(i-1)}^n}{2(\Delta r)^2} \right\} \\
&\quad + \eta \{ h_{mi}^n - h_{fi}^n \}, \\
&\frac{AB(\alpha)}{(1-\alpha)} \sum_{j=0}^{n} (h_{fi}^{j+1} - h_{fi}^j) \left\{ \begin{array}{l} (n-j) E_{\alpha,2} \left(-\frac{\alpha \Delta t}{1-\alpha}(n-j) \right) \\ -(n-j-1) E_{\alpha,2} \left(-\frac{\alpha \Delta t}{1-\alpha}(n-j-1) \right) \end{array} \right\} S_{S_m}(r_i) \\
&\quad = \eta (h_{mi}^n - h_{fi}^n) \\
&\quad \text{if} \quad \frac{k_f^{j+1} - k_f^j}{(\Delta r)} > 0 \qquad (41)
\end{aligned}
$$

and

$$\frac{AB(\alpha)}{(1-\alpha)}\sum_{j=0}^{n}(h_{fi}^{j+1}-h_{fi}^{j})\left\{\begin{array}{l}(n-j)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j)\right)\\ -(n-j-1)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j-1)\right)\end{array}\right\}S_{sf}(r_i)$$
$$=\frac{k_f^{j+1}-k_f^{j}}{(\Delta r)}\cdot\frac{-h_{f(i+2)}^{n}-4h_{f(i+1)}^{n}+3h_{fi}^{n}}{2(\Delta r)}$$
$$+k_f^{i}\left\{\frac{h_{f_1(i+1)}^{n+1}-2h_{f_1(i)}^{n+1}+h_{f_1(i-1)}^{n+1}}{2(\Delta r)^2}+\frac{h_{f_1(i+1)}^{n}-2h_{f_1(i)}^{n}+h_{f_1(i-1)}^{n}}{2(\Delta r)^2}\right\}$$
$$+\eta\{h_{mi}^{n}-h_{fi}^{n}\},$$
$$\frac{AB(\alpha)}{(1-\alpha)}\sum_{j=0}^{n}(h_{fi}^{j+1}-h_{fi}^{j})\left\{\begin{array}{l}(n-j)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j)\right)\\ -(n-j-1)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j-1)\right)\end{array}\right\}S_{S_m}(r_i)$$
$$=\eta(h_{mi}^{n}-h_{fi}^{n})$$
$$\text{if}\quad \frac{k_f^{j+1}-k_f^{j}}{(\Delta r)}>0. \tag{42}$$

Thus with the third order upwind numerical scheme, we obtain

$$\frac{AB(\alpha)}{(1-\alpha)}\sum_{j=0}^{n}(h_{fi}^{j+1}-h_{fi}^{j})\left\{\begin{array}{l}(n-j)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j)\right)\\ -(n-j-1)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j-1)\right)\end{array}\right\}S_{sf}(r_i)$$
$$=\frac{k_f^{j+1}-k_f^{j}}{(\Delta r)}\cdot\frac{2h_{fi}^{n}+3h_{f(i-1)}^{n}-6h_{f(i-2)}^{n}+h_{f(i-3)}^{n}}{6(\Delta r)}$$
$$+k_f^{i}\left\{\frac{h_{f_1(i+1)}^{n+1}-2h_{f_1(i)}^{n+1}+h_{f_1(i-1)}^{n+1}}{2(\Delta r)^2}+\frac{h_{f_1(i+1)}^{n}-2h_{f_1(i)}^{n}+h_{f_1(i-1)}^{n}}{2(\Delta r)^2}\right\}$$
$$+\eta\{h_{mi}^{n}-h_{fi}^{n}\},$$
$$\frac{AB(\alpha)}{(1-\alpha)}\sum_{j=0}^{n}(h_{fi}^{j+1}-h_{fi}^{j})\left\{\begin{array}{l}(n-j)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j)\right)\\ -(n-j-1)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j-1)\right)\end{array}\right\}S_{S_m}(r_i)$$
$$=\eta(h_{mi}^{n}-h_{fi}^{n})$$
$$\text{if}\quad \frac{k_f^{j+1}-k_f^{j}}{(\Delta r)}>0 \tag{43}$$

and

$$\frac{AB(\alpha)}{(1-\alpha)}\sum_{j=0}^{n}(h_{fi}^{j+1}-h_{fi}^{j})\left\{\begin{array}{l}(n-j)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j)\right)\\ -(n-j-1)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j-1)\right)\end{array}\right\}S_{sf}(r_i)$$
$$=\frac{k_f^{j+1}-k_f^{j}}{(\Delta r)}\cdot\frac{-2h_{f(i+3)}^{n}+6h_{f(i+2)}^{n}-3h_{f(i+1)}^{n}-2h_{fi}^{n}}{6(\Delta r)}$$
$$+k_f^{i}\left\{\frac{h_{f_1(i+1)}^{n+1}-2h_{f_1(i)}^{n+1}+h_{f_1(i-1)}^{n+1}}{2(\Delta r)^2}+\frac{h_{f_1(i+1)}^{n}-2h_{f_1(i)}^{n}+h_{f_1(i-1)}^{n}}{2(\Delta r)^2}\right\}$$
$$+\eta\{h_{mi}^{n}-h_{fi}^{n}\},$$

$$\frac{AB(\alpha)}{(1-\alpha)}\sum_{j=0}^{n}(h_{fi}^{j+1}-h_{fi}^{j})\left\{\begin{array}{l}(n-j)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j)\right)\\-(n-j-1)E_{\alpha,2}\left(-\frac{\alpha\Delta t}{1-\alpha}(n-j-1)\right)\end{array}\right\}S_{S_m}(r_i)$$
$$=\eta(h_{mi}^{n}-h_{fi}^{n})$$
$$\text{if}\quad \frac{k_f^{j+1}-k_f^{j}}{(\Delta r)}>0. \tag{44}$$

7 Model of fractal flow in dual media with heterogeneity and visco-elasticity properties

In this section, we consider the model with a more complex non-local operator. The considered operator here is a convolution of power-Mittag-Leffler with the unknown function. This non-local operator was recently proposed by Atangana on his paper with title "Derivative with two fractional orders: A new avenue of investigation toward revolution in fractional calculus" [21]. Therefore using the new established non-local fractional operator suggested by Atangana, the modified model is given as follows:

$$\begin{cases}S_{S_f}\,{}_0^{AC}D_t^{\alpha}(h_f(r,t))=\nabla.(k_f.\nabla h_f(r,t))+\eta(h_m(r,t)-h_f(r,t))+q_f\\S_{S_m}\,{}_0^{AC}D_t^{\alpha}(h_m(r,t))=\eta(h_m(r,t)-h_f(r,t)).\end{cases} \tag{45}$$

The discussion regarding the analysis of existence and uniqueness of exact solution will not be presented in this section. Rather, the model will be solved numerically. To do this we present first the numerical approximation of the Atangana fractional derivative with two order.

$$\begin{aligned}&{}_0^{AC}D_t^{\alpha,\beta}\left(f(x,t)\right)\\&=\frac{A(\beta)}{1-\beta}\frac{1}{\Gamma(1-\alpha)}\int_0^t\partial_t f(x,y)\,(t-y)^{-\alpha}\,E_\beta\left(-\frac{\beta}{1-\beta}(t-y)^{\alpha+\beta}\right)dy,\\&=\frac{A(\beta)}{1-\beta}\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{f(x,y+\Delta y)-f(x,y)}{\Delta t}(t-y)^{-\alpha}\,E_\beta\left(-\frac{\beta}{1-\beta}(t-y)^{\alpha+\beta}\right)dy,\\&=\frac{A(\beta)}{1-\beta}\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{f_i^{j+1}-f_i^{j}}{\Delta t}(t_n-y)^{-\alpha}\,E_\beta\left(-\frac{\beta}{1-\beta}(t_n-y)^{\alpha+\beta}\right)dy,\\&=\frac{A(\beta)}{1-\beta}\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\left(\frac{f_i^{j+1}-f_i^{j}}{\Delta t}\right)\int_{t_j}^{t_{j+1}}(t_n-y)^{-\alpha}\,E_\beta\left(-\frac{\beta}{1-\beta}(t_n-y)^{\alpha+\beta}\right)dy.\end{aligned} \tag{46}$$

In the above expression, the integral is given as follows:

$$\begin{aligned}&\int_{t_j}^{t_{j+1}}(t_n-y)^{-\alpha}\,E_\beta\left(-\frac{\beta}{1-\beta}(t_n-y)^{\alpha+\beta}\right)dy\\&\quad=(t_n-t_{j+1})^{1-\alpha}\,E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_{j+1})^{\beta+\alpha}\right)\\&\quad\quad+(t_n-t_j)^{1-\alpha}\,E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_j)^{\beta+\alpha}\right).\end{aligned} \tag{47}$$

Replacing (47) in (46) we obtain the following numerical approximation

$$
{}_{0}^{AC}D_t^{\alpha,\beta} f(x_i,t_n) = \frac{A(\beta)}{(1-\beta)\,\Gamma(1-\alpha)} \sum_{j=1}^{n} \left(\frac{f_i^{j+1} - f_i^{j}}{\Delta t} \right) \times \left\{ \begin{array}{l} (t_n - t_{j+1})^{1-\alpha} E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta} (t_n - t_{j+1})^{\beta+\alpha} \right) \\ + (t_n - t_j)^{1-\alpha} E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta} (t_n - t_j)^{\beta+\alpha} \right) \end{array} \right\}. \tag{48}
$$

Coupling the upwind for second order with the above the numerical solution of equation (45) is given as:

$$
\begin{aligned}
&\frac{A(\beta)}{(1-\beta)\,\Gamma(1-\alpha)} \sum_{j=1}^{n} \left(\frac{h_{fi}^{j+1} - h_{fi}^{j}}{\Delta t} \right) \\
&\quad \times \left\{ \begin{array}{l} (t_n - t_{j+1})^{1-\alpha} E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta} (t_n - t_{j+1})^{\beta+\alpha} \right) \\ + (t_n - t_j)^{1-\alpha} E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta} (t_n - t_j)^{\beta+\alpha} \right) \end{array} \right\} S_{Sf}(r_i) \\
&\quad = \frac{k_f^{i+1} - k_f^{i}}{\Delta r} \frac{3h_{f(i)}^{n} - 4h_{f(i-1)}^{n} + 3h_{f(i-2)}^{n}}{2(\Delta r)} \\
&\quad\quad + k_f^{i} \left\{ \frac{h_{f(i+1)}^{n+1} - 2h_{fi}^{n+1} + h_{f(i-1)}^{n+1}}{2(\Delta r)^2} + \frac{h_{f(i+1)}^{n} - 2h_{fi}^{n} + h_{f(i-1)}^{n}}{2(\Delta r)^2} \right\} \\
&\quad\quad + \eta \left(h_{fi}^{n} - h_{mi}^{n} \right), \\
&\frac{A(\beta)}{(1-\beta)\,\Gamma(1-\alpha)} \sum_{j=1}^{n} \left(\frac{h_{mi}^{j+1} - h_{mi}^{j}}{\Delta t} \right) \\
&\quad \times \left\{ \begin{array}{l} (t_n - t_{j+1})^{1-\alpha} E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta} (t_n - t_{j+1})^{\beta+\alpha} \right) \\ + (t_n - t_j)^{1-\alpha} E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta} (t_n - t_j)^{\beta+\alpha} \right) \end{array} \right\} S_{Sm}(r_i) \\
&\quad = \eta \left(h_{fi}^{n} - h_{mi}^{n} \right) \\
&\quad \text{if} \quad \frac{k_f^{i+1} - k_f^{i}}{\Delta r} > 0.
\end{aligned} \tag{49}
$$

Otherwise we have the following

$$
\begin{aligned}
&\frac{A(\beta)}{(1-\beta)\,\Gamma(1-\alpha)} \sum_{j=1}^{n} \left(\frac{h_{fi}^{j+1} - h_{fi}^{j}}{\Delta t} \right) \\
&\quad \times \left\{ \begin{array}{l} (t_n - t_{j+1})^{1-\alpha} E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta} (t_n - t_{j+1})^{\beta+\alpha} \right) \\ + (t_n - t_j)^{1-\alpha} E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta} (t_n - t_j)^{\beta+\alpha} \right) \end{array} \right\} S_{Sf}(r_i) \\
&\quad = \frac{k_f^{i+1} - k_f^{i}}{\Delta r} \frac{-h_{f(i+2)}^{n} - 4h_{f(i+1)}^{n} + 3h_{f(i)}^{n}}{2(\Delta r)} \\
&\quad\quad + k_f^{i} \left\{ \frac{h_{f(i+1)}^{n+1} - 2h_{fi}^{n+1} + h_{f(i-1)}^{n+1}}{2(\Delta r)^2} + \frac{h_{f(i+1)}^{n} - 2h_{fi}^{n} + h_{f(i-1)}^{n}}{2(\Delta r)^2} \right\} \\
&\quad\quad + \eta \left(h_{fi}^{n} - h_{mi}^{n} \right),
\end{aligned}
$$

$$\frac{A(\beta)}{(1-\beta)\,\Gamma(1-\alpha)}\sum_{j=1}^{n}\left(\frac{h_{mi}^{j+1}-h_{mi}^{j}}{\Delta t}\right)$$
$$\times\left\{\begin{array}{l}(t_n-t_{j+1})^{1-\alpha}E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_{j+1})^{\beta+\alpha}\right)+\\(t_n-t_j)^{1-\alpha}E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_j)^{\beta+\alpha}\right)\end{array}\right\}S_{Sm}(r_i)$$
$$=\eta\left(h_{fi}^{n}-h_{mi}^{n}\right)$$
$$\text{if}\quad\frac{k_f^{i+1}-k_f^{i}}{\Delta r}<0. \tag{50}$$

Coupling the derived numerical approximation with the upwind for third order then the numerical solution of equation (45) is given as

$$\frac{A(\beta)}{(1-\beta)\,\Gamma(1-\alpha)}\sum_{j=1}^{n}\left(\frac{h_{fi}^{j+1}-h_{fi}^{j}}{\Delta t}\right)$$
$$\times\left\{\begin{array}{l}(t_n-t_{j+1})^{1-\alpha}E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_{j+1})^{\beta+\alpha}\right)\\+(t_n-t_j)^{1-\alpha}E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_j)^{\beta+\alpha}\right)\end{array}\right\}S_{Sf}(r_i)$$
$$=\frac{k_f^{i+1}-k_f^{i}}{\Delta r}\,\frac{2h_{f(i)}^{n}+6h_{f(i-1)}^{n}-6h_{f(i-2)}^{n}+6h_{f(i-3)}^{n}}{6(\Delta r)}$$
$$+k_f^{i}\left\{\frac{h_{f(i+1)}^{n+1}-2h_{fi}^{n+1}+h_{f(i-1)}^{n+1}}{2(\Delta r)^2}+\frac{h_{f(i+1)}^{n}-2h_{fi}^{n}+h_{f(i-1)}^{n}}{2(\Delta r)^2}\right\}$$
$$+\eta\left(h_{fi}^{n}-h_{mi}^{n}\right),$$
$$\frac{A(\beta)}{(1-\beta)\,\Gamma(1-\alpha)}\sum_{j=1}^{n}\left(\frac{h_{mi}^{j+1}-h_{mi}^{j}}{\Delta t}\right)$$
$$\times\left\{\begin{array}{l}(t_n-t_{j+1})^{1-\alpha}E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_{j+1})^{\beta+\alpha}\right)\\+(t_n-t_j)^{1-\alpha}E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_j)^{\beta+\alpha}\right)\end{array}\right\}S_{Sm}(r_i)$$
$$=\eta\left(h_{fi}^{n}-h_{mi}^{n}\right)$$
$$\text{if}\quad\frac{k_f^{i+1}-k_f^{i}}{\Delta r}>0. \tag{51}$$

Otherwise we have the following

$$\frac{A(\beta)}{(1-\beta)\,\Gamma(1-\alpha)}\sum_{j=1}^{n}\left(\frac{h_{fi}^{j+1}-h_{fi}^{j}}{\Delta t}\right)$$
$$\times\left\{\begin{array}{l}(t_n-t_{j+1})^{1-\alpha}E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_{j+1})^{\beta+\alpha}\right)\\+(t_n-t_j)^{1-\alpha}E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_j)^{\beta+\alpha}\right)\end{array}\right\}S_{Sf}(r_i)$$
$$=\frac{k_f^{i+1}-k_f^{i}}{\Delta r}\,\frac{-h_{f(i+3)}^{n}+6h_{f(i+2)}^{n}-3h_{f(i+1)}^{n}-2h_{f(i)}^{n}}{2(\Delta r)}$$
$$+k_f^{i}\left\{\frac{h_{f(i+1)}^{n+1}-2h_{fi}^{n+1}+h_{f(i-1)}^{n+1}}{2(\Delta r)^2}+\frac{h_{f(i+1)}^{n}-2h_{fi}^{n}+h_{f(i-1)}^{n}}{2(\Delta r)^2}\right\}$$
$$+\eta\left(h_{fi}^{n}-h_{mi}^{n}\right),$$

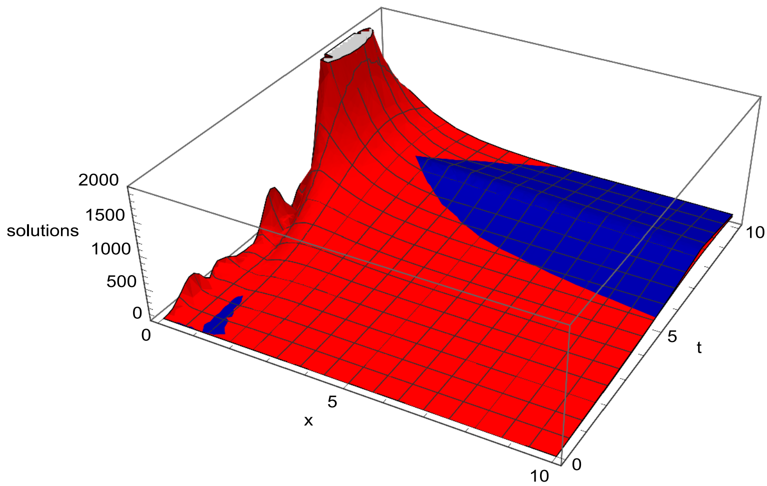

Fig. 1. Numerical simulation of system solution with red the quantity of water in a fracture system and blue in a matrix rock $\alpha = 0.5$.

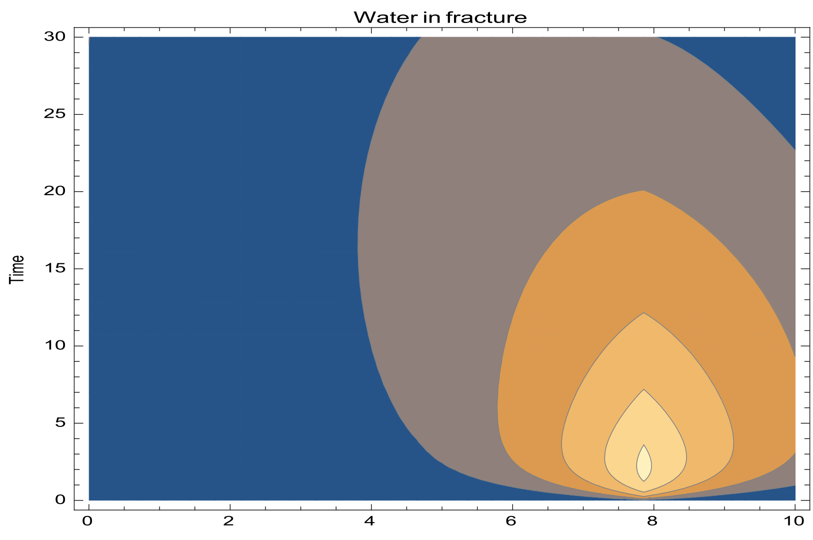

Fig. 2. Numerical simulation of water in the fracture network for $\alpha = 0.5$.

$$\begin{aligned}
&\frac{A(\beta)}{(1-\beta)\,\Gamma(1-\alpha)}\sum_{j=1}^{n}\left(\frac{h_{mi}^{j+1}-h_{mi}^{j}}{\Delta t}\right)\\
&\qquad\times\left\{\begin{array}{l}(t_n-t_{j+1})^{1-\alpha}\,E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_{j+1})^{\beta+\alpha}\right)\\ +(t_n-t_j)^{1-\alpha}\,E_{\beta,2-\alpha}\left(-\frac{\beta}{1-\beta}(t_n-t_j)^{\beta+\alpha}\right)\end{array}\right\}S_{Sm}(r_i)\\
&\qquad= \eta\left(h_{fi}^{n}-h_{mi}^{n}\right)\\
&\qquad\text{if}\quad \frac{k_f^{i+1}-k_f^{i}}{\Delta r}<0. \qquad (52)
\end{aligned}$$

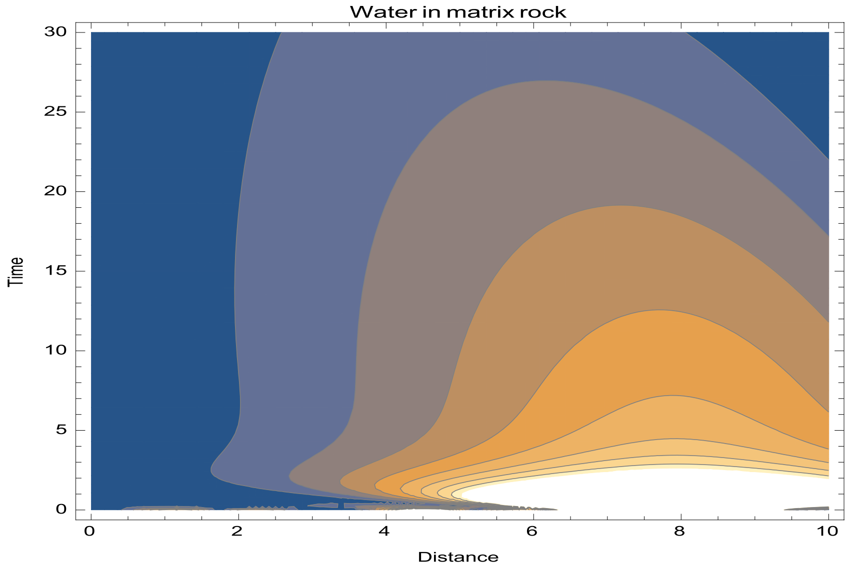

Fig. 3. Numerical simulation of water within the matrix rock for $\alpha = 0.5$.

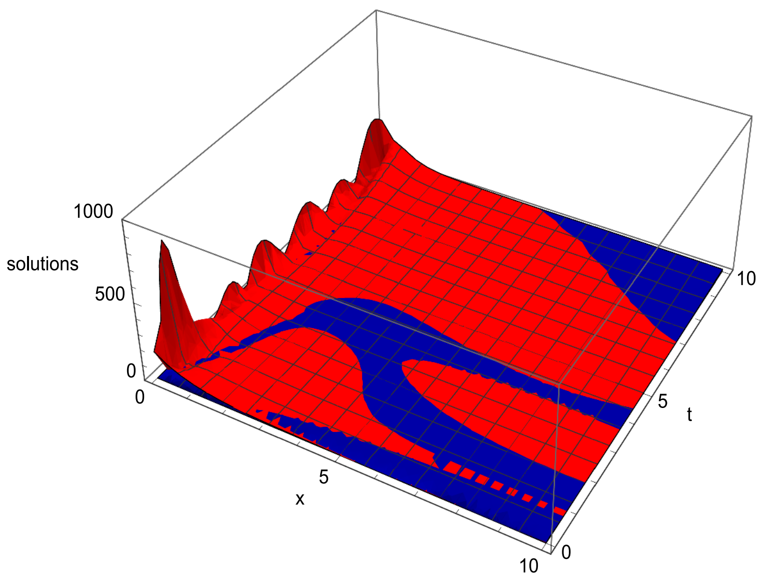

Fig. 4. Numerical simulation of system solution with red the quantity of water in a fracture system and blue in a matrix rock $\alpha = 0.75$.

8 Numerical simulation for different values of fractional order

In this section, the numerical simulations of the modified groundwater fractal flow in dual media are presented for different values of fractional order. We consider the contour plot of the solutions to see the solution in space and time for a given value of alpha and beta. The aquifer parameters used here are theoretical not measured from the field however, this section is designed to show readers more scenarios that can be described using the concept of fractional differentiation. In these simulations, we will

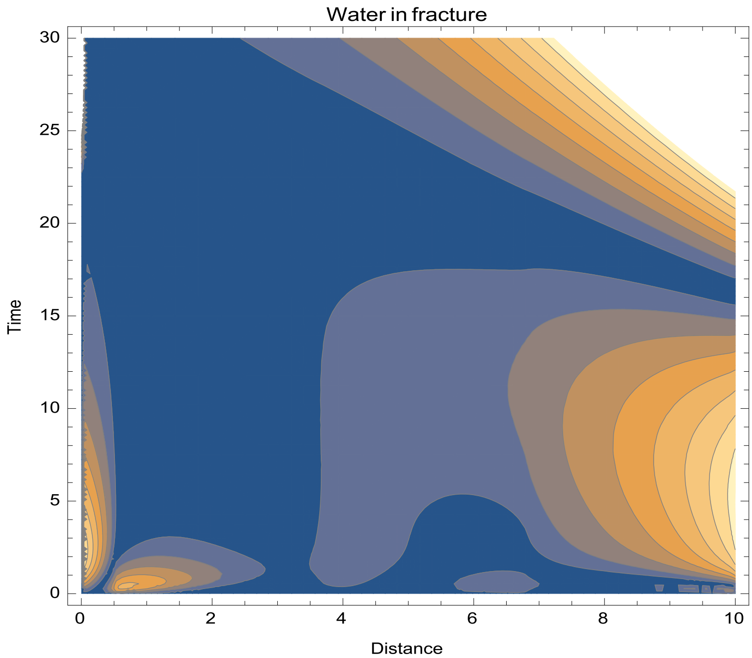

Fig. 5. Numerical simulation of water in the fracture network for $\alpha = 0.75$.

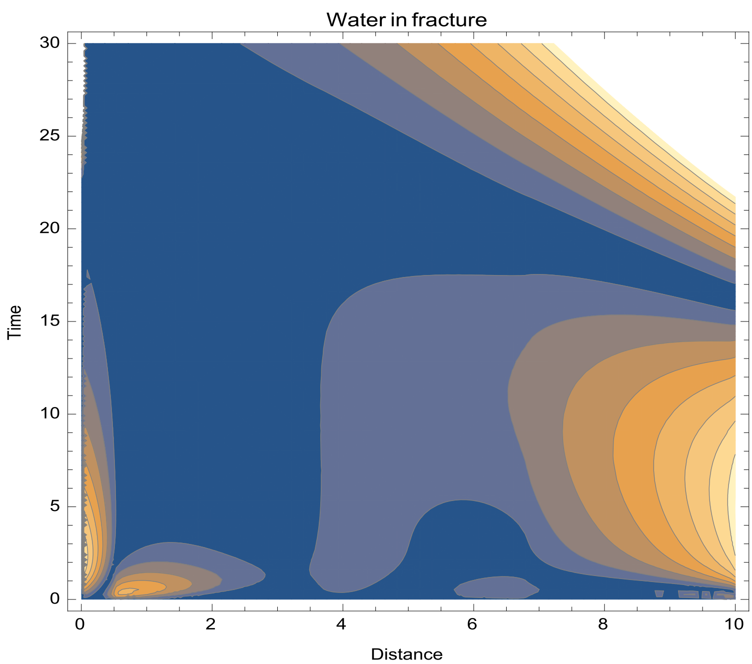

Fig. 6. Numerical simulation of water within the matrix rock for $\alpha = 0.75$.

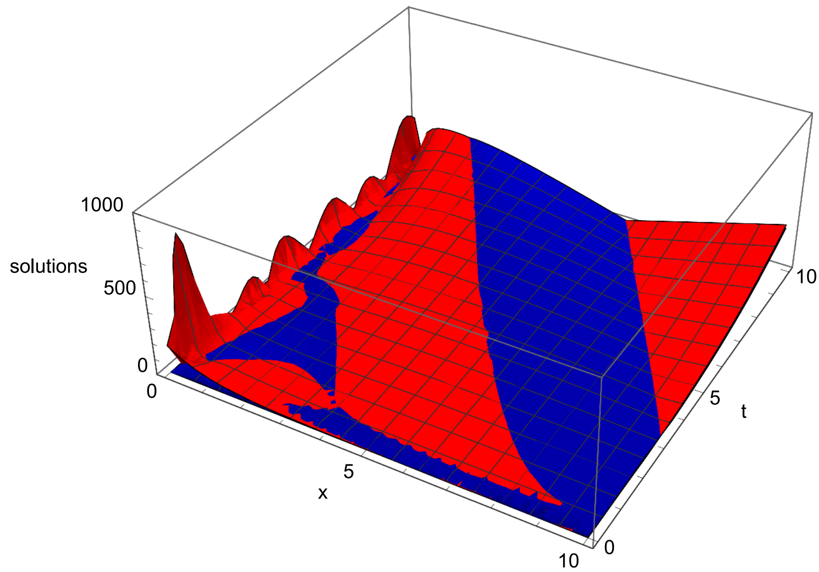

Fig. 7. Numerical simulation of system solution with red the quantity of water in a fracture system and blue in a matrix rock $\alpha = 1$.

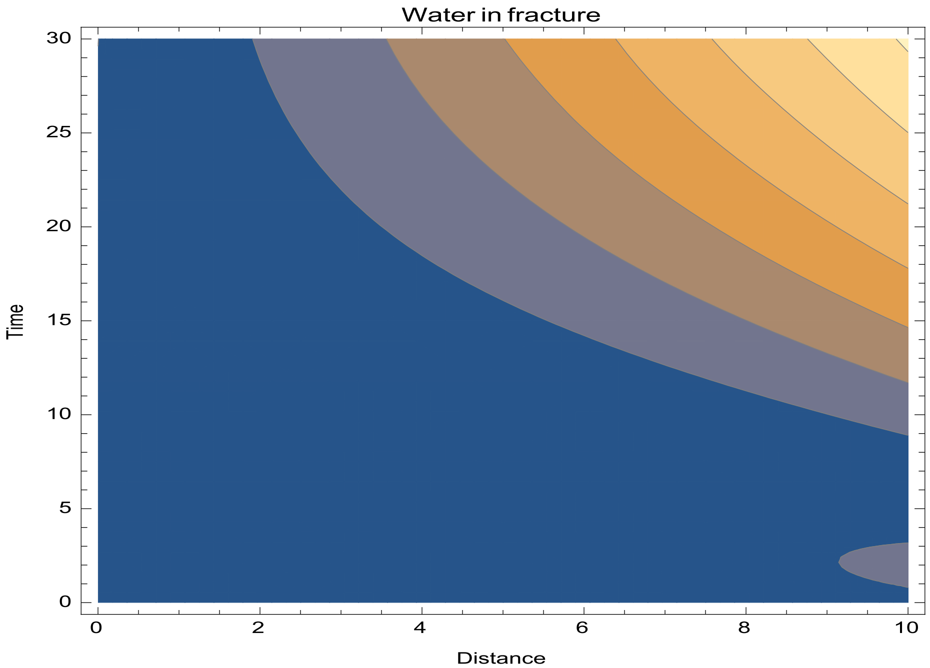

Fig. 8. Numerical simulation of water in the fracture network for $\alpha = 1$.

not only consider the aquifer parameters in power decay law form, we will suggest other form and see the effect. The numerical simulations are depicted in Figures 1–9. The numerical graphics show some interesting real world observations. The numerical simulation is generated based on the non-local operator with Mittag-Leffler kernel.

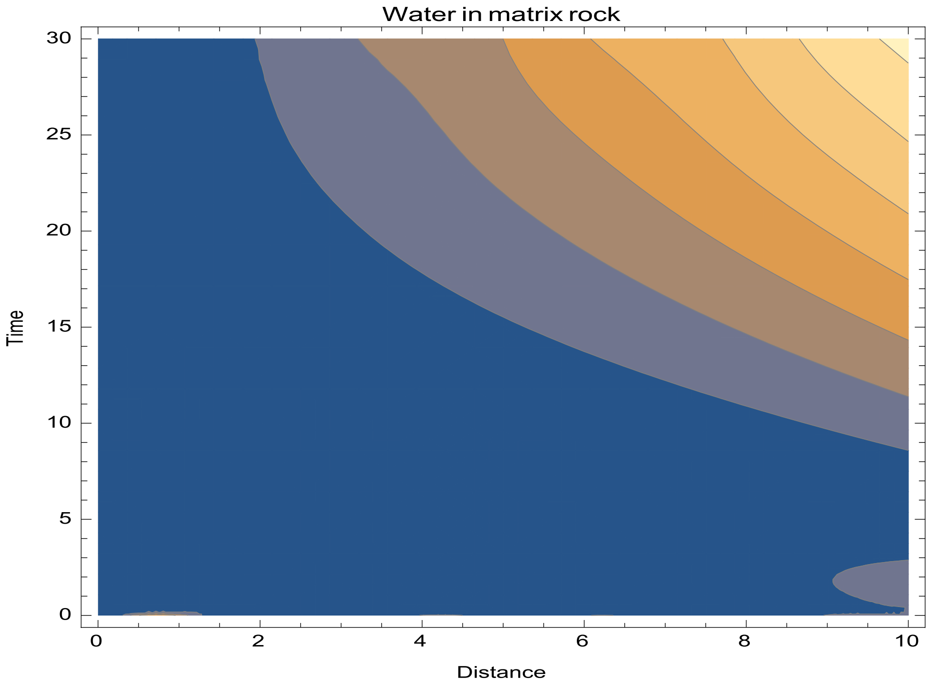

Fig. 9. Numerical simulation of water within the matrix rock for $\alpha = 1$.

It is important to notice that, the numerical simulation of hydraulic head of water within the matrix rock and fracture network depend on the fractional order. When the fractional order is one that is we are dealing with the classical model the hydraulic change within the fracture network and matrix rock are homogeneous which is correct because the classical derivative is unable to portray a non-homogeneous scenario. The numerical simulation when the fractional order is less than 1 show new features that could not have been pointed out with the classical differentiation. In Figures 1–6 we can see the flow within a visco-elastic media. More importantly we observe that when using the fractional differentiation, the total amount of water within the matrix rock is less than in the fracture network. In addition, one can observe that the water within the matrix rock is moving toward the fracture and this scenario is always observed in the real world problem.

9 Conclusion

The flow of groundwater within a given geological formation has been a focus of many researchers in the last decades due to the importance of the groundwater which is in many countries a source of fresh water on one hand, in another hand this water is estimated to constitute about twenty percent of the world's fresh water supply, which is about 0.61% of the entire world's water, including oceans and permanent ice. One of the top challenge is the geological formation via which this water is moving. To monitor the flow of these water, one need to construct a mathematical equation that accounts for some parameters and properties of these aquifers. The fractal flow model in dual media is perhaps one of the most complex groundwater flow model as its account for the flow within the fractures network and also the flow within the matrix rock. One can easily conclude that the flow within a fractional does not require more complex mathematical formulas as the medium is considered to be homogeneous, non-elastic and non-viscoelastic. Nevertheless, the flow within the matrix rock needs to be modelled with care as the matrix rock may be heterogeneous, elastic or visco-elastic. The mathematical tools used to construct the partial differential equation (1) cannot account for heterogeneous, elastic or visco-elastic therefore a suitable operator of differentiation needs to be used. Thus to include into mathematical formula the observed facts, we modified the classical model by replacing the local derivative with the non-local operator with power kernel, Mittag-Leffler law and finally with combined both law to obtain power-Mittag-Leffler law. The modified models were analysed numerically using the upwind for second and third order approximation in space. Some numerical simulations are presented to see the effect of power, Mittag-Leffler and Mittag-Leffler-Power laws.

Author contribution statement

All authors contributed equally.

References

1. C.V. Theis, Eos Trans. Am. Geophys. Union **16**, 519 (1935)
2. M.S. Hantush, J. Hydraul. Div. **87**, 83 (1961)
3. M.S. Hantush, J. Hydraul. Div. **87**, 171 (1961)
4. H.H. Cooper, C.E. Jacob, Eos Trans. Am. Geophys. Union **27**, 526 (1946)
5. J.E. Warren, P.J. Root, SPE J. **3**, 245 (1963)
6. G.E. Barenblatt, I.P. Zheltov, I.N. Kochina, J. Appl. Math. Mech. **24**, 1286 (1960)
7. K. Serra, A.C. Reynolds, R. Raghavan, J. Pet. Technol. **35**, 2271 (1983)
8. A. de Swaan O, SPE J. **16**, 117 (1976)
9. D. Bourdet et al., *New Type Curves Aid Analysis of Fissured Zone Well Tests* (World Oil, 1984)
10. B. O'Shaughnessy, I. Procaccia, Phys. Rev. A **32**, 3073 (1985)
11. J. Leveinen, J. Hydrol. **234**, 116 (2000)
12. T. Hirata, Pure Appl. Geophys. **121**, 157 (1989)
13. K.B. Oldham, J. Spanier, in *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*, Mathematics in Science and Engineering (Academic Press, 1974), Vol. V
14. B. Ross, A brief history and exposition of the fundamental theory of fractional calculus, in *Fractional Calculus and Its Applications*, Lecture Notes in Mathematics (1975), Vol. 457, pp. 1–36
15. L. Debnath. Int. J. Math. Educ. Sci. Technol. **35**, 487 (2004)
16. B.S.T. Alkahtani, Chaos Solitons Fractals **89**, 547 (2016)
17. A. Atangana, B. Dumitru, Therm. Sci. **20**, 763 (2016)
18. R. Courant, E. Isaacson, M. Rees, Commun. Pure Appl. Math. **5**, 243 (1952)
19. H.K. Versteeg, W. Malalasekera, in *An Introduction to Computational Fluid Dynamics* (1995), Chap. 5, p. 104
20. H.K. Versteeg, W. Malalasekera, in *An Introduction to Computational Fluid Dynamics* (1995), Chap. 5, p. 105
21. F. Ali, N.A. Sheikh, I. Khan, M. Saqib, J. Magn. Magn. Mater. **423**, 327 (2017)
22. F. Ali, S.A.A. Jan, I. Khan, M. Gohar, N.A. Sheikh, Eur. Phys. J. Plus **131**, 310 (2016)
23. F. Ali, M. Saqib, I. Khan, N.A. Sheikh, Eur. Phys. J. Plus **131**, 377 (2016)
24. F. Ali, N.A. Sheikh, M. Saqib, A. Khan, Nonlinear Sci. Lett. A **8**, 101 (2017)
25. N.A. Sheikh, F. Ali, I. Khan, M. Saqib, Neural Comput. Appl. (2016) https://doi.org/10.1007/s00521-016-2815-5