Eur. Phys. J. Special Topics **226**, 1911–1920 (2017)

DOI: 10.1140/epjst/e2017-70060-2

THE EUROPEAN
PHYSICAL JOURNAL
SPECIAL TOPICS

Regular Article

Synchronization in network motifs of delay-coupled map-based neurons

J.M. Sausedo-Solorio[1] and A.N. Pisarchik[2,3,4,a]

[1] Universidad Autónoma del Estado de Hidalgo, Carretera Pachuca-Tulancingo Km. 4.5, 42074 Pachuca, Hidalgo, Mexico
[2] Center for Biomedical Technology, Technical University of Madrid, Campus Montegancedo, Pozuelo de Alarcón, 28223 Madrid, Spain
[3] Centro de Investigaciones en Óptica, Loma del Bosque, 115, Col. Lomas del Campestre, 37150 León, Guanajuato, Mexico
[4] Yuri Gagarin Saratov State Technical University, Politehnicheskaya, 77, Saratov, 410056, Russia

Received 17 February 2017 / Received in final form 26 February 2017
Published online 21 June 2017

Abstract. We study the influence of delayed coupling on synchronization in neural network motifs. Numerical simulations based on the Rulkov map reveal different behavior in the presence and in the absence of the delay. While without delay, synchronization improves as the coupling strength is increased, in the presence of a delay, synchronization becomes worse. We also study how a feedback loop affects synchronization. An increase in the number of neurons involved in the loop leads to desynchronization in the motifs, saturating at a certain value of the synchronization index.

1 Introduction

Information transmission through a neural network is an important brain function, where synchronization plays a key role in processing information in the brain [1,2]. This process is characterized by a certain delay due to a finite velocity of the action potential propagating along the neuron's axon as well as time lapses in dendritic and synaptic processes [3]. The delay in synapses is caused by a neurotransmitter to be released from a presynaptic membrane, diffusing across the synaptic cleft, and finally binding to a receptor site on the postsynaptic membrane [4]. The presence of a time delay in the feedback loops is a common structural feature of neural networks, long predicted to be responsible for short-term memory [5].

The interest in mathematical modeling of neuronal synchronization has significantly increased after neurobiological experiments with two electrically coupled neurons [6,7], where various synchronous states have been identified. In order to simulate cooperative neuron dynamics, numerous models based on either iterative maps or differential equations in various coupling configurations have been developed [6,8–22].

[a] e-mail: alexander.pisarchik@ctb.upm.es

Depending on the coupling strength and synaptic delay time, coupled neurons generate spike sequences that are matching in their timings, or bursts either with lag or anticipation [21,23–26].

When three or more oscillators are accounted for, a large number of coupling configurations can be realized. In the theory of graphs or complex networks, these basic configurations are called *network motifs*. A motif is a pattern of interconnections (links), that occurs significantly more often than in randomized versions of the graph, i.e., in graphs with the same number of nodes, links and degree distribution as the original one, but where the links are randomly distributed. The notion of network motifs was used by Uri Alon and his colleagues [27] to characterize patterns of interconnections in the gene regulation (transcription) network of the bacterium *Escherichia coli*. Later, other researchers focused on the computational theory of network motifs [28].

Synchronization properties of network motifs were first studied by Lodato et al. [29]. Using the master stability function approach they found that in directed graphs the correlation between neuronal dynamics exists only for some specific motifs. In this work, we are interested in how network motifs synchronize in the presence of a synaptic delay and a feedback loop. We explore a simple neural model, the Rulkov map [23,30,31]. Although this model is not explicitly inspired by physiological processes in the membrane, it is capable of generating extraordinary complexity and quite specific neural dynamics (silence, periodic spiking, and chaotic bursting), thus replicating to a great extent most of the experimentally observed regimes [6,7,23], including spike adaptation [12], routes from silence to bursting mediated by subthreshold oscillations [32], emergent bursting [30], phase and antiphase synchronization with chaos regularization [23,31], and complete and burst synchronization [33–35]. A simple model of only two delay-coupled Rulkov neurons with a single feedback loop was studied in reference [36]. It was shown that this system displays different synchronous states, including phase, lag and anticipating synchronization depending on the delay times. Here, we are interested in understanding how synchronization arises in network motifs in the presence of delay in coupling. We start with small motifs formed by three neurons and study how an increase in the number of interconnected neurons affects synchronization in the whole ensemble of the coupled neurons.

The paper is organized as follows. In Section 2 we review the theoretical framework of the Rulkov neuron and describe model parameters. Section 3 is devoted to synchronization in network motifs; we show how synchronization depends on the coupling strength and synaptic delay. Finally, in Section 4 we conclude our results.

2 Model equations

We consider the network motifs formed by three, five and more unidirectionally coupled neurons, as shown in Figure 1.

For every pair of coupled neurons, we can write the following Rulkov equations [23, 30,31]

$$\begin{aligned} x_{n+1}^{(i)} &= f(x_n^{(i)}, y_n^{(i)}), \\ y_{n+1}^{(i)} &= y_n^{(i)} - \mu(x_n^{(i)} + 1) + \mu\sigma, \\ x_{n+1}^{(j)} &= f(x_n^{(j)}, y_n^{(j)} + \beta_n), \\ y_{n+1}^{(j)} &= y_n^{(j)} - \mu(x_n^{(j)} + 1) + \mu\sigma + \mu\sigma_n, \end{aligned} \tag{1}$$

where superindices (i) and (j) belong to master and slave neurons, respectively, x_n is a fast variable associated with the membrane potential, y_n is a slow variable which

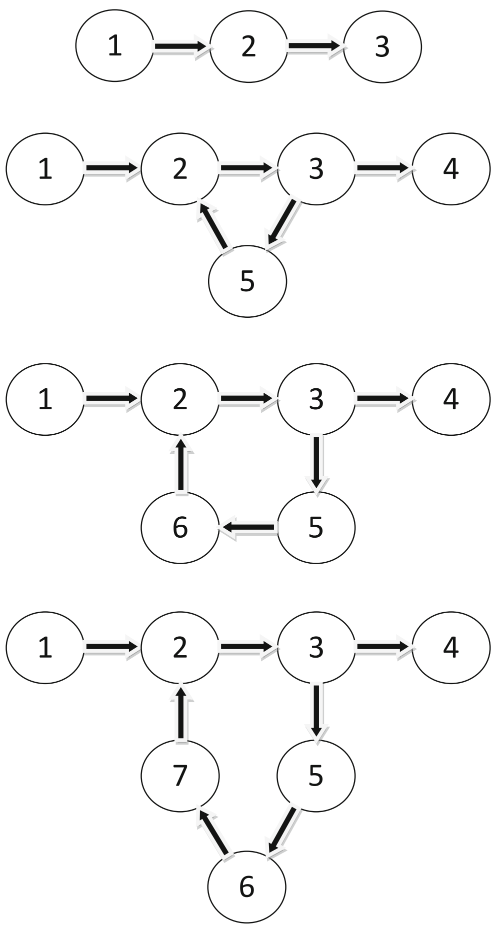

Fig. 1. Network motifs of three, five, six and seven neurons. First row: three neurons in a chain configuration, second row: five neurons with a feedback loop formed by three neurons, third row: six neurons with a feedback loop formed by four neurons, fourth row: seven neurons with a feedback loop formed by five neurons.

has some analogy with gating variables, β_n and σ_n are related to external stimuli, μ and σ are intrinsic parameters, and f is a piecewise function defined as

$$f(x_n, y_n) = \begin{cases} \alpha/(1 - x_n) + y_n & \text{for } x_n \leq 0, \\ \alpha + y_n & \text{for } 0 < x_n < \alpha + y_n \text{ and } x_{n-1} \leq 0, \\ -1 & \text{for } x_n \geq \alpha + y_n \text{ or } x_{n-1} > 0, \end{cases} \tag{2}$$

where α is a parameter. In this paper, we use the following fixed parameters for each neuron in all motifs: $\alpha = 4.2$, $\mu = 0.001$, and $\sigma = -0.025$. For this parameters the neuron is in a periodic spiking regime as shown in Figure 2.

When the physiological response of the postsynaptic neuron to a signal is assumed to be immediate, the coupling between the cells can be defined as

$$\beta_n = \sigma_n = \eta(x_{n-s}^{(i)} - x_n^{(j)}), \tag{3}$$

where s is a synaptic delay time (in units of number of iterations of the map) and η is a coupling strength.

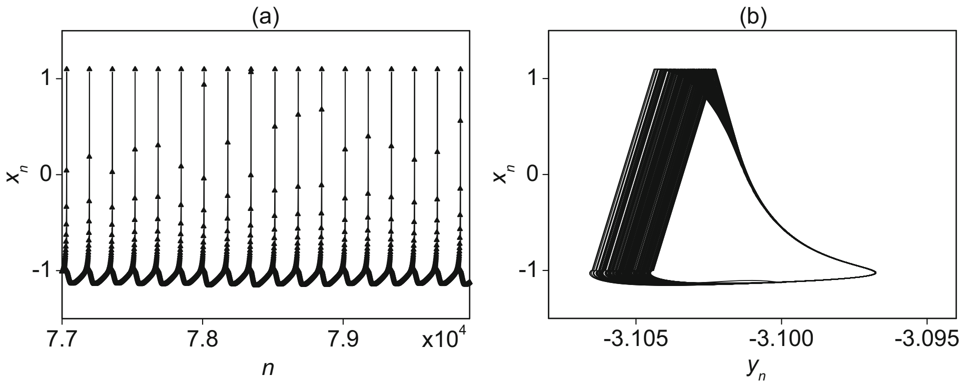

Fig. 2. (a) Time series and (b) phase portrait of a periodic spiking regime generated by a solitary Rulkov neuron. The period of spikes (inter-spike interval) is ISI = 164 iterations.

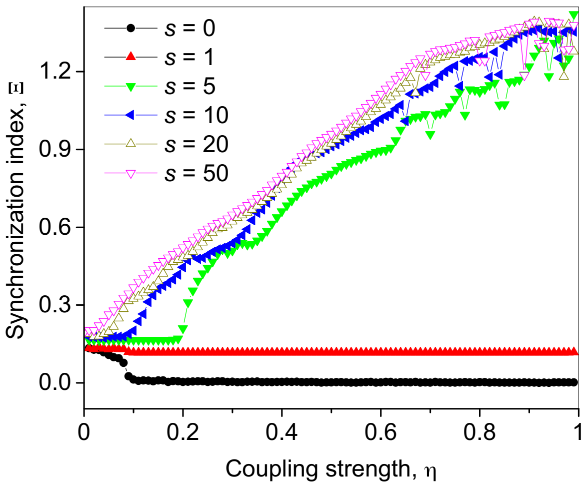

Fig. 3. Synchronization index in the motif of three Rulkov neurons coupled in a chain as a function of coupling for different delay times.

3 Synchronization in network motifs

Synchronization in a network of N oscillators can be quantitatively described by synchronization index Ξ given as [37]

$$\Xi = \sqrt{\frac{1}{T-n_0}\sum_{n=n_0+1}^{T}\xi_n}, \quad \xi_n = \frac{1}{N}\sum_{i=1}^{N}\left(x_n^{(i)}\right)^2 - \left(\frac{1}{N}\sum_{i=1}^{N}x_n^{(i)}\right)^2, \tag{4}$$

where T is the total number of iterations and n_0 is the duration of transients. The smaller Ξ, the better synchronization. $\Xi = 0$ means complete synchronization.

3.1 Synchronization in motifs without feedback loops

First, we consider the simplest network motif formed by only three neurons, coupled in a chain, as shown in the first row of Figure 1, and study how synchronization depends on both the coupling strength and delay time. The results are shown in Figure 3.

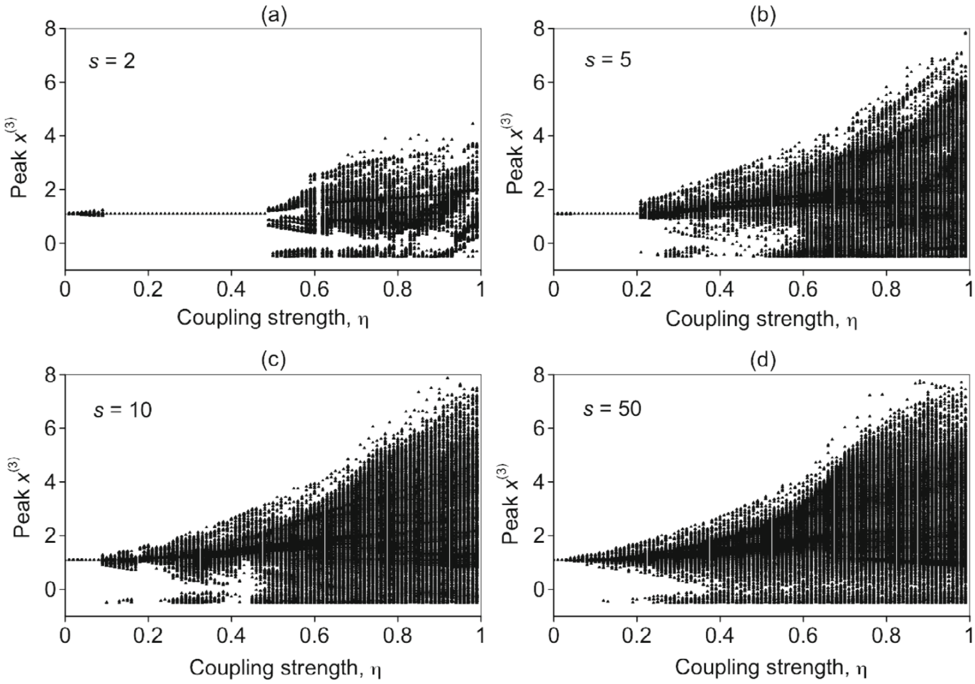

Fig. 4. Bifurcation diagrams of spike amplitude $x^{(3)}$ of neuron 3 in the motif of three neurons versus coupling strength for delays (a) $s = 2$, (b) $s = 5$, (c) $s = 10$, and (d) $s = 50$.

One can see that without delay ($s = 0$) synchronization improves as the coupling is increased, and for $\eta > 0.1$ the neurons are completely synchronized. When the delay is very small ($s = 1$), the neurons are in lag synchronization for $\eta > 0.1$ independently of the coupling. However, if the delay is sufficiently large ($s \geq 2$), the increasing coupling worsens synchronization after a Hopf bifurcation which position depends on the delay. For $s = 2$ the Hopf bifurcation arises at $\eta \approx 0.5$ and for $s = 5$ at $\eta \approx 0.2$, as seen from the bifurcation diagrams in Figures 4a and b, respectively. The increasing delay shifts the Hopf bifurcation toward lower values of the coupling strength (see Figs. 4c, 4d). For very large delays, synchronization is almost independent of the delay time. In this case the delay-coupled states act like random noise.

In a chain of several neurons without delay in coupling, synchronization gradually improves as the coupling strength is increased, independently of the number of neurons in the chain. This situation is demonstrated in Figure 5, where we plot the synchronization index versus the coupling for different number of neurons in the chain. One can see that in the absence of delay, the neurons are completely synchronized for $\eta > 0.2$ regardless of the number of neurons in the chain.

3.2 Synchronization in motifs with a feedback loop

A different situation occurs in the presence of a feedback loop. In Figure 6 we plot the synchronization index as a function of the coupling in the motif of five neurons with a feedback loop formed by three neurons (the configuration shown in the second row of Fig. 1). Similarly to the case of three neurons, complete synchronization is only possible in the absence of delay ($s = 0$) for sufficiently strong coupling ($\eta > 0.5$).

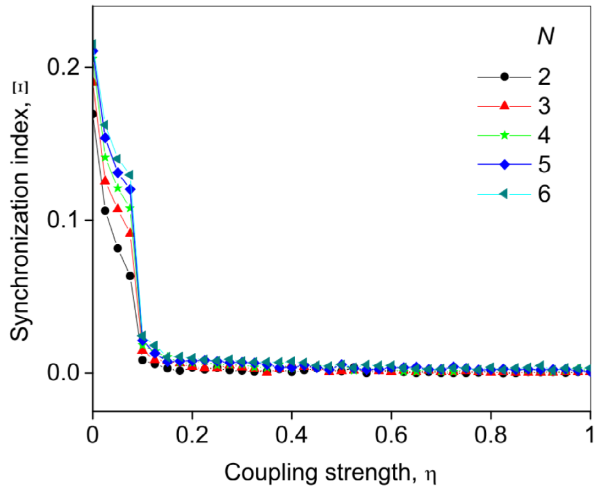

Fig. 5. Synchronization index in the neuron chain without delay as a function of coupling for different number of neurons in the chain.

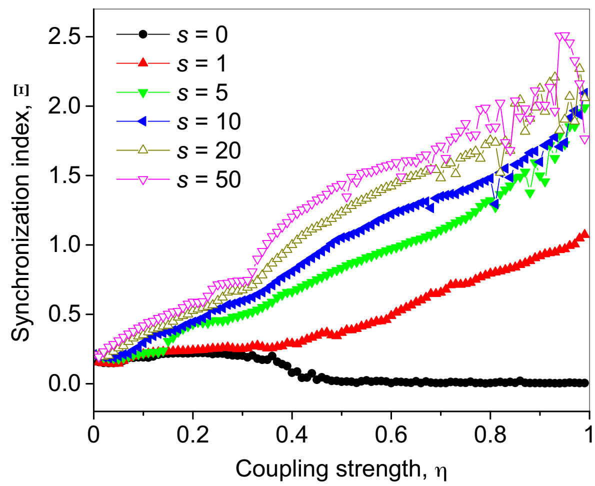

Fig. 6. Synchronization index in the motif of five neurons with a feedback loop versus coupling for different delay times.

However, even with a very small delay ($s = 1$) synchronization worsens as the coupling is increased.

The time series in Figures 7a and 7b illustrate synchronization between the neurons 1 and 4 in configuration shown in the second row of Figure 1 for weak ($\eta = 0.3$) and strong ($\eta = 0.9$) coupling strengths. While for small coupling, the neurons generate asynchronous spikes (Fig. 7a), for strong coupling they are completely synchronized (Fig. 7b).

In contrast to the chain configuration, the presence of a feedback loop makes synchronization drastically dependent on the number of coupled neurons in the loop even without any delay. These dependences are displayed in Figure 8, where we plot the synchronization index versus the number of neurons formed a feedback loop, for two coupling strengths without delay (Fig. 8a) and with different delay times (Fig. 8b). We consider configurations where four neurons are coupled in a chain and

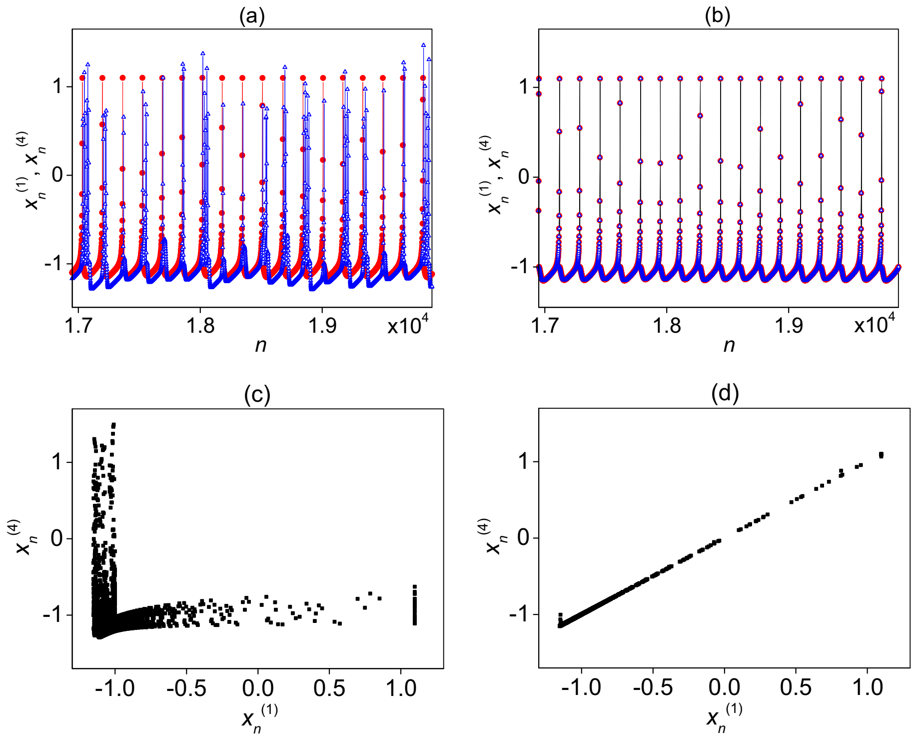

Fig. 7. (a,b) Time series and (c,d) phase portraits of (a,c) asynchronous (at $\eta = 0.3$) and (b,d) synchronous (at $\eta = 0.9$) regimes of neuron 4 (blue triangles) and neuron 1 (red dots) in the motif of five neurons with a feedback loop without delay.

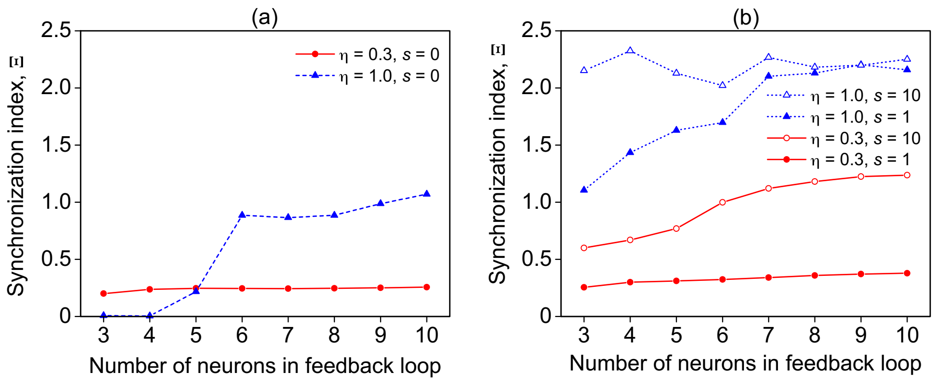

Fig. 8. Synchronization index in network motifs with a feedback loop as a function of the number of neurons in the loop (a) in the absence of delay and (b) with delays $s = 1$ and $s = 10$, for two different coupling strengths, $\eta = 0.3$ (blue triangles) and $\eta = 1$ (red dots).

other neurons (together with two neurons in the chain) form a feedback loop, as shown in Figure 1. The minimum number of neurons to form the feedback loop is three.

One can see that complete synchronization is only achieved for three and four neurons in the loop (for configurations shown in the second and third rows in Fig. 1) if the coupling is very strong (blue dotted line in Fig. 8a), whereas for small couplings, synchronization is almost independent of the number of neurons in the loop (red solid line in Fig. 8). In the presence of delay, complete synchronization is

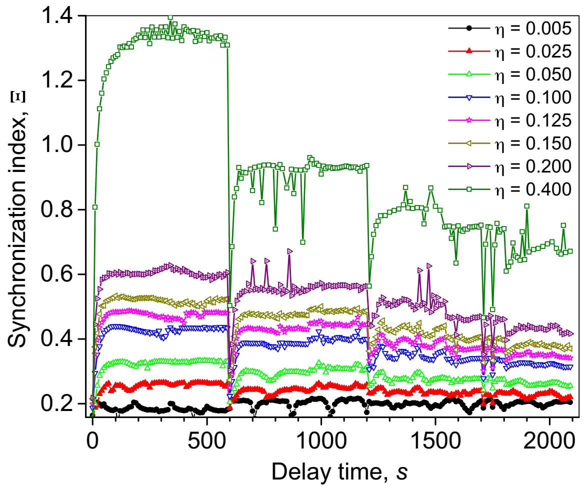

Fig. 9. Synchronization index in the network motif of five neurons with a feedback loop of three neurons as a function of the delay time for different coupling strengths.

never achieved. Interestingly, synchronization worsens as the number of neurons in the loop increases and saturates to a certain level depending on the coupling strength.

The dependence of synchronization on the delay time for large delays is shown in Figure 9. This dependence has a periodic character with a period of 600 iterations, at least for the first two periods. The origin of this periodicity is not clear, because the spiking period (ISI) of the master neuron (neuron 1) is equal to 164 iterations. The ISI of other neurons is not constant, it is fluctuated depending on the coupling strength, as seen in Figure 7a. The periodicity in the synchronization index related to ISI of the master neuron only appears in the weakly coupled motif (the lowest trace in Fig. 9 for $\eta = 0.005$). When the delay is shorter than ISI, synchronization worsens as the delay increases, and saturates to a certain value. When the delay further increases, the neurons synchronize again (at $s = 600$), and then again synchronization worsens with increasing delay. Surprisingly, every subsequent period of the delay time (600 iterations) the motif synchronizes better than for shorter delays. Finally, for very large delay times synchronization is independent of the delay. Similar delay-induced synchronization transitions were observed in scale-free neuronal networks [37,38].

4 Conclusion

We have studied synchronization in network motifs of Rulkov neurons formed by three and more neurons. We have shown how synchronization depends on the coupling strength and delay times in different configurations in the presence and in the absence of a feedback loop. In both cases, an increase in coupling improves synchronization only in the absence of delay. However, in the presence of delay synchronization worsens as the coupling is increased, especially for large delays. The presence of a feedback loop also worsens synchronization. When the number of neurons in the feedback loop increases, synchronization index saturates at a certain value. We have demonstrated delay-induced synchronization transitions which are manifested as well-pronounced minima in the synchronization index with respect to the delay time.

This work has been supported by the Russian Science Foundation (Grant No. 16-12-10100).

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