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Kalman filters for fractional discrete-time stochastic systems along with time-delay in the observation signal

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Abstract. This paper investigates fractional Kalman filters when time-delay is entered in the observation signal in the discrete-time stochastic fractional order state-space representation. After investigating the common fractional Kalman filter, we try to derive a fractional Kalman filter for time-delay fractional systems. A detailed derivation is given. Fractional Kalman filters will be used to estimate recursively the states of fractional order state-space systems based on minimizing the cost function when there is a constant time delay (d) in the observation signal. The problem will be solved by converting the filtering problem to a usual d-step prediction problem for delay-free fractional systems.

1 Introduction

Fractional Calculus (FC) is a generalization of classical calculus that produces similar meanings and effects, but with more applications, by using the real (non-integer) order derivative and integral operations [1]. It is already known that non-integer order systems or fractional order systems can model dynamical behavior of different systems and processes in the time and frequency domain more accurately than traditional integer models [1–3]. Nowadays, the applications of fractional calculus have extended to various approaches, including control theory [4–9]. Regarding state and signal estimation and Kalman filter in [10–14], one of the most popular areas of fractional calculus is chaos theory and chaotic oscillations in nonlinear dynamic systems [15–23]. It is well known that chaos occurs in nonlinear systems with a total order greater than or equal to three, and that chaotic systems can be modeled with three fractional order differential equations while the total order of these fractional differential equations is less than three [24,25]. A Kalman filter is an optimal recursive estimator introduced by R. Kalman [26]. It infers the states of state-space representations of linear and nonlinear systems from a series of measurements which are observed

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over time, and these observations are contained with statistical noise and other inaccuracies. If all noise is Gaussian, a Kalman filter for a linear system can find an optimal solution by minimizing the mean square error (MSE) of the estimated states. State estimation and Kalman filter for classical integer systems have been investigated widely, for example in [27–30]. The Kalman filter for time-delay systems has often been investigated using the augmented state method in [31,32], we know this method produces a heavier computational burden when the dimension of the system is high and the time-delay in the measurement equation is large The first attempt to design Kalman filters for discrete-time fractional order systems was done by Sierociuk and Dzielinski [12]. Sierociuk considered the square of the estimation error as the cost function and, with some simplifying assumptions, estimated accurately the states of linear and nonlinear fractional discrete-time systems and also the order of commensurate fractional systems. To get rid of some simplifying assumptions, Sierociuk [13] improved the method by augmenting the states, and designed the fractional Kalman filter for linear and nonlinear systems. Although the augmented state vector provides a simple method of estimating the states of systems, one important drawback of this method is that the order of the system and the size of the different matrices are increased.

In this article, Grunwald-Letnikov's definition of the fractional derivative is used and, by applying a fractional Kalman filter, we try to estimate the states of discrete time fractional order systems with time-delay observations from a sequence of noisy measurements. For this purpose, we convert the filtering fractional problem to the d-step ahead prediction fractional problem which is free of time delay (i.e. a delay free observation).

2 Preliminaries

2.1 Fractional calculus

Basic definitions required in the following sections are given below: [33,34]

Definition 21. The Grunwald-Letnikov fractional derivative of function f, based on the generalization of backward difference, is defined as:

$$ {}_{a}^{GL}D_t^{\alpha} f(t) = \lim \frac{1}{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh) \quad (1)$$

$$\times \left[\frac{t-a}{h}\right] \to \text{integer}$$

$$\binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha - j)!} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha - j + 1)} \quad (2)$$

$$\text{where the Euler Gama function is given by } \Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt. \quad (3)$$

Definition 22. The continuous time state-space model of linear fractional order systems is given by:

$$\begin{aligned} D^{\alpha} x(t) &= Ax(t) + Bu(t) \\ y(t) &= cx(t) + Du(t) \end{aligned} \quad (4)$$

where $\alpha = [\alpha_1, \ldots, \alpha_n]$ is a fractional derivative order.

Discrete time state-space representations of (4) using a Grunwald-Letnikov definition are given by:

$$\begin{aligned} D^{\alpha}x(KT) &\approx \tfrac{1}{T^{\alpha}} \sum_{i=0}^{k+1} (-1)^i \binom{\alpha}{i} x((k+1-i)T) \\ &= \tfrac{1}{T^{\alpha}} \left(x(k+1)T - \binom{\alpha}{1} x(kT) + \sum_{i=2}^{k+1} (-1)^i \binom{\alpha}{i} x((k+1-i)T) \right) \end{aligned} \tag{5}$$

where T is sampling time. Substituting (5) into (5) yields (6):

$$\begin{aligned} &x(k+1) = (AT^{\alpha} + \alpha I)\, x(k) - \sum_{i=2}^{k+1} (-1)^i \binom{\alpha}{i} x(k+1-i) + BT^{\alpha}u(k) \\ &y(k) = cx(k) + Du(k) \qquad\qquad k \geq 1. \end{aligned} \tag{6}$$

Definition 23. The one-step backward difference is defined as $\Delta^1 x(k+1) = x(k+1) - x(k)$ and if we define $A_d = A - I$, where I is the identity matrix, we have:

$$\begin{aligned} &\Delta^1 x(k+1) = A_d x(k) + Bu(k) + \omega(k) \\ &x(k+1) = \Delta^1 x(k+1) + x(k) \\ &y(k) = Cx(k) + v(k). \end{aligned} \tag{7}$$

Definition 24. The generalized discrete-time fractional stochastic system in a state-space representation is given by

$$\begin{aligned} &\Delta^n x(k+1) = A_d x(k) + Bu(k) + \omega(k) \\ &x(k+1) = \Delta^n x(k+1) - \sum_{j=1}^{k+1} (-1)^j \binom{n}{j} x(k+1-j) \\ &y(k) = Cx(k) + v(k). \end{aligned} \tag{8}$$

3 System description

For the case when equation orders are not identical in the fractional order system, the following equations are introduced [12]

$$\begin{aligned} &\Delta^{\Psi} x(k+1) = A_d x(k) + Bu(k) + \omega(k) \\ &y(k) = Cx(k) + v(k) \end{aligned} \tag{9}$$

$$\Psi_j = diag\left[\binom{n_1}{j} \cdots \binom{n_n}{j} \right], \quad \Delta^{\Psi} x_{k+1} = \begin{bmatrix} \Delta^{n_1} x_{1,k+1} \\ \vdots \\ \Delta^{n_N} x_{N,k+1} \end{bmatrix} \tag{10}$$

$[n_1, \ldots, n_N]$ are orders of the fractional order system. The number of equations is N, as stated before the dynamic equation (9) for the case when the equation orders are not equal in (9), k is discrete time, Ψ is fractional order, $x(k) \in R^n$, $y(k) \in R^m$ and $u(k)$ are the state vector, measurement or observation signal and known control

Fig. 1. Optimal filter.

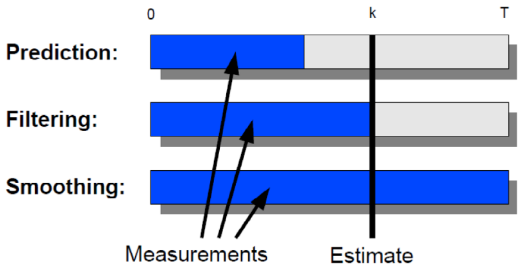

Fig. 2. Three types of estimation problem [35].

input, respectively. $\omega(k)$and $\upsilon(k)$are the system noise and measurement noise. For the system described in (9) we also consider some assumptions as follows:

Assumption 1. $x(0)$is uncorrelated with $\omega(k)$ and $\upsilon(k)$, the mean and covariance matrix in k=0 are

$$E\left[x(0)\right] = m \textit{ and } E\left[(x(0)-m)\left(x(0)-m\right)^T\right] = P_0. \tag{11}$$

Assumption 2. $\omega(k)$ and $\upsilon(k)$ are white noises with zero mean and variances $Q(k)$ and $R(k)$as follow:

$$E\left\{\begin{bmatrix}\omega(k)\\ \upsilon(k)\end{bmatrix}\cdot\left[\omega^T(j)\ \ \upsilon^T(j)\right]\right\} = \begin{bmatrix}Q(k) & 0\\ 0 & R(k)\end{bmatrix}\delta(k-j). \tag{12}$$

Where E denotes mathematical expectation, T is transpose and $\delta(k-j) = 1(k=j)$ is the impulse function.

Assumption 3. $E[(x(l)-\widehat{x}(l))(x(m)-\widehat{x}(m))^T] = 0$ when $l \neq m$ which means that there is no correlation between past state vectors.

4 Estimation, optimal filter and Kalman filter

The optimal filter, shown in Fig. 1, computes the (marginal) posterior distribution of the state given the measurements $p\left(x\left(t_k\right)|y_1, y_2, \ldots, y_k\right)$, we know that the filtered state $\hat{x}\left(t_k\right)$ is often the posterior mean $E\left(x\left(t_k\right)|y_1, y_2, \ldots, y_k\right)$. The estimation problem is divided into three categories, namely prediction, filtering and smoothing. Figure 2 displays these three modes and their differences completely [35].

Prediction:
Prediction distributions are the marginal distributions of the future states, n steps after the current time step, k:

$$p\left(x_{k+n}|y_1, y_2, \ldots, y_k\right), \quad k = 1, 2, \ldots, T,\ n = 1, 2, ..$$

Filtering:
The purpose of filtering is to compute the marginal posterior distribution of the state on the current time step, k:

$$p\left(x_k \left| y_1, y_2, \ldots, y_k \right.\right), \quad k = 1, 2, \ldots, T$$

Smoothing:
The purpose of smoothing is to calculate the marginal posterior distribution of the state at the time step k after receiving the measurements up to a time step T, where T > k:

$$p\left(x_k \left| y_1, y_2, \ldots, y_T \right.\right), \quad k < T.$$

A Kalman filter is a recursive algorithm that uses a sequence of measurements observed over time, containing statistical noise $Z_k = \{y_0, y_1, \ldots, y_k; u_0, u_1, \ldots u_k\}$, and produces estimates of unknown states $\hat{x}(k)$. The Kalman filter is the closed form solution to the optimal filtering equations of the discrete-time filtering model, where the dynamic and measurements models are linear Gaussian [35].

5 State estimation for fractional order stochastic system with time delay in the measurement equation

Before we state our problem, we need to review two main lemmas, which we will use afterwards in the proof of our problem.

Lemma 1 [12]. For the discrete-time fractional order stochastic system introduced in definition 24, with the assumptions 1–3, the simplified Kalman predictor and filter (called the fractional Kalman filter) are given by the following set of equations

$$\begin{aligned}
&\Delta^{\Psi}\tilde{x}(k+1|k) = A_d\hat{x}(k|k) + Bu(k)\\
&\tilde{x}(k+1\,|k) = A_d\hat{x}(k\,|k) + Bu(k) - \sum_{j=1}^{k+1}(-1)^j\,\Psi_j\hat{x}(k+1-j)\\
&\hat{x}(k\,|k) = \tilde{x}(k\,|k-1) + K_k\left(y(k) - c\tilde{x}(k\,|k-1)\right)\\
&K_k = \tilde{P}(k\,|k-1)\,c^T(c\tilde{P}(k\,|k-1)\,c^T + R)^{-1}\\
&\tilde{P}(k\,|k-1) = (A_d + \Psi_1)\,P(k-1)\,(A_d+\Psi_1)^T + Q_{k-1} + \sum_{j=2}^{k}\Psi_jP(k-j)\Psi_j^T\\
&P(k\,|k) = (I - K_kc)\,\tilde{P}(k\,|k-1)
\end{aligned}$$

where $\tilde{P}(k\,|k-1)$ and $P(k\,|k)$ are prediction and filtering error covariance matrices, respectively.

$$\begin{cases} x(k+1) = A_dx(k) + Bu(k) + \omega(k) - \sum_{j=1}^{k+1}(-1)^j\,\Psi_jx(k+1-j)\\ y(k) = Cx(k) + v(k). \end{cases} \tag{13}$$

The Fractional order Kalman filter algorithm for system (9) was derived in [12,13,36], we know that $\tilde{x}(k\,|k-1) = E\left[x(k)\,|Z_{k-1}\right]$ is the predicted state and obtains as:

$$\begin{aligned}\tilde{x}(k+1\,|k) &= E\left[x(k+1)|Z_k\right]\\ &= E\left[A_d x(k) + Bu(k) + \omega(k) - \sum_{j=1}^{k+1}(-1)^j\,\Psi_j x(k+1-j)|Z_k\right]\\ \tilde{x}(k+1\,|k) &= A_d\hat{x}(k\,|k) + Bu_k - \sum_{j=1}^{k+1}(-1)^j\,\Psi_j E\left[x(k+1-j)|Z_k\right]\end{aligned} \tag{14}$$

by using the simplifying assumption below [12]:

$$\begin{aligned}&E\left[x(k+1-j)|Z_k\right] \cong E\left[x(k+1-j)|Z_{k+j-1}\right]\\ &j = 1\ldots(k+1).\end{aligned} \tag{15}$$

This simplification means discarding of some measurements; consequently we obtain a suboptimal solution for the fractional order Kalman filter. So the one-step predicted state obtains as:

$$\tilde{x}(k+1\,|k) = A_d\hat{x}(k\,|k) + Bu(k) - \sum_{j=1}^{k+1}(-1)^j\,\Psi_j\hat{x}(k+1-j) \tag{16}$$

so the state estimation obtains from the following relation:

$$\hat{x}(k\,|k) = \tilde{x}(k\,|k-1) + K_k\left(y(k) - c\tilde{x}(k\,|k-1)\right) \tag{17}$$

where K_k is the Kalman gain matrix and $y(k)$ is the measurement in time step k, Kalman gain matrix obtains as:

$$K_k = \tilde{P}(k\,|k-1)\;c^T(c\tilde{P}(k\,|k-1)\;c^T + R)^{-1}. \tag{18}$$

The prediction error covariance matrix obtains as follows:

$$\begin{aligned}\tilde{P}(k\,|k-1) &= E\left(\tilde{x}(k\,|k-1) - x(k)\right)\left(\tilde{x}(k\,|k-1) - x(k)\right)^T \rightarrow\\ \tilde{P}(k\,|k-1) &= \left(A_d + \Psi_1\right)P(k-1)\left(A_d + \Psi_1\right)^T + Q_{k-1} + \sum_{j=2}^{k}\Psi_j P(k-j)\Psi_j^T\end{aligned} \tag{19}$$

and the estimation error covariance matrix is:

$$\begin{aligned}P(k\,|k) &= E\left(\hat{x}(k\,|k) - x(k)\right)\left(\hat{x}(k\,|k) - x(k)\right)^T \rightarrow\\ P(k\,|k) &= \left(I - K_k c\right)\tilde{P}(k\,|k-1)\end{aligned} \tag{20}$$

The details of this proof are found in [12].

Therefore, Lemma 1 represented the filtering problem in the fractional order systems while in Lemma 2, the smoothing problem in the fractional order systems is stated.

So the fractional Kalman smoother for the linear discrete-time fractional systems is presented as:

Lemma 2 [37]. For a linear discrete-time fractional order stochastic system, the fixed point Kalman smother is given by

$$\hat{x}(t\,|t+N\,) = \hat{x}(t\,|t+N-1\,) + K(t\,|t+N)\;\varepsilon(t+N),\; N = 1, 2, \ldots,$$

$$\begin{aligned} K(t|t+N) = &\left\{ P(t\,|t-1) \left\{ \prod_{j=0}^{N-1} \Phi^T\,(t+j) \right\} - \left[\sum_{j=1}^{t+1} (-1)^j\;\Psi_j P\,(t+1-j\,|t+1-j\,) \right] \right. \\ &\left. \times \left\{ I_n - \sum_{k=1}^{N-1} \left[\prod_{j=k}^{N-1} \Phi^T(t+j) \right] \right\} \right\} H^T \\ &\times \left[HP\,(t+N\,|t+N-1\,)\,H^T + R(t+N) \right]^{-1} \end{aligned}$$

$$K(t\,|t) = P\,(t\,|t-1\,)\;H^T\left[HP\,(t\,|t-1\,)\,H^T + R(t+N) \right]^{-1} = K(t)$$

where Φ is the transition matrix. $K(t+j)$ is the Kalman filtering gain and $\hat{x}(k\,|k\,)$ is the Kalman filter state estimation, which are calculated from lemma 1.

$K(t\,|t+N)$ is smoothing gain and $\varepsilon(t+N)$ is the innovation signal obtains as:

$$\varepsilon(t+N) = y(t+N) - H\hat{x}(t+N\,|t+N-1).$$

The smoothing error $\tilde{x}\,(t\,|t+N\,) = x(t) - \hat{x}(t\,|t+N)$ and the smoothing covariance matrix $P\,(t\,|t+N\,) = E\left[\tilde{x}\,(t\,|t+N\,) .\tilde{x}^T\,(t\,|t+N\,)\right]$ can be recursively calculated as:

$$\begin{aligned} &P\,(t\,|t+N\,) = P\,(t\,|t+N-1\,) - K(t\,|t+N) \\ &\times \left[HP\,(t+N\,|t+N-1\,)\,H^T + R(t+N) \right] K^T(t\,|t+N). \end{aligned}$$

Proof: The proof of Lemma 2 is given in [37], which is omitted.

Now, by extending the method in [29] we state our problem which is filtering in the fractional order system with time-delay d in the observation signal.

Problem 1. The problem is to find the state estimation $\hat{x}(k\,|k\,) = E\,[x(k)\,|Z_k\,]$ for a fractional system with time-delay, presented in (21), based on a sequence of measurements $Z_k = \{y_d, y_{d+1}, \ldots, y_k; u_d, u_{d+1}, \ldots u_k\}$,

$$\begin{aligned} &\Delta^\Psi x(k+1) = A_d x(k) + Bu(k) + \omega(k) \\ &y(k) = Cx(k-d) + v(k) \end{aligned} \tag{21}$$

where d is time-delay in the measurement equation.

Proof:
If we define the new variables as:

$$\begin{cases} Y(k) = y(k+d) \\ V(k) = v(k+d) \end{cases} \tag{22}$$

so the new state-space representation for fractional system can be achieved as follows:

$$\begin{aligned} \Delta^{\Psi} x(k+1) &= A_d x(k) + Bu(k) + \omega(k) \\ Y(k) &= Cx(k) + V(k). \end{aligned} \tag{23}$$

It is obvious from (22) that the sequence of measurements $\{y(d), y(d+1), \ldots, y(k)\}$ is equal to $\{Y(0), Y(1), , Y(k-d)\}$, in other words the spanned space with these two sets are equivalent. Therefore we can conclude:

$$\widehat{x}\left(k\left|k\right.\right) = E\left\{x(k)\left|y(d), y(d+1), \ldots, y(k)\right.\right\} \tag{24}$$
$$\tilde{x}_d\left(k\left|k-d\right.\right) = E\left\{x(k)\left|Y(0), Y(1), \ldots, Y(k-d)\right.\right\}$$

since the sequences of measurements are equivalent, we have:

$$\begin{aligned} \tilde{x}_d\left(k\left|k-d\right.\right) &= \widehat{x}\left(k\left|k\right.\right) \\ Q_V(k) &= Q(k+d) \end{aligned} \tag{25}$$

so the problem will be solved if we can find $\tilde{x}_d\left(k\left|k-d\right.\right)$, accordingly the filtering problem will be converted to the d-step ahead prediction problem. From (13) and (14) we find the one-step ahead prediction for a fractional order system:

$$\begin{aligned} \tilde{x}_d\left(k+1\left|k\right.\right) &= E\left[x_d(k+1)\left|Z_k\right.\right] \\ &= A_d E\left[x_d\left(k\right)\left|Z_k\right.\right] + G.E\left[\omega(k)\left|Z_k\right.\right] - \sum_{j=1}^{k+1} E\left[x_d\left(k+1-j\right)\left|Z_k\right.\right]. \end{aligned} \tag{26}$$

Since $\omega(k)$ is white noise with mean zero and is also independent of Z_k, we have $E\left[\omega(k)\left|Z_k\right.\right] = E\left[\omega(k)\right] = 0$, and with regard to (14) we can obtain:

$$\tilde{x}_d(k+1|k) = (A_d + \Psi_1)\widehat{x}_d\left(k\left|k\right.\right) - \sum_{j=2}^{k+1} (-1)^j\, \Psi_j x_d(k+1-j|k) \tag{27}$$

where $\widehat{x}_d\left(k\left|k\right.\right) = x\left(k\left|k+d\right.\right)$ and $x\left(k\left|k+d\right.\right)$ is the smoothing in time step k when the measurements are available up to time step k+d, that can be computed from lemma 2, and from (26) we have $x_d(k+1-j|k) = x\left(k+1-j|k+d\right)$, which is calculated from lemma 2.

The covariance matrix of the one−step ahead prediction error is calculated from (27) as follows:

$$P_d(k+1|k) = (A_d + \Psi_1)P(k|k+d)(A_d + \Psi_1)^T + \sum_{j=2}^{k+1} \Psi_j P(k+1-j|k+d)\Psi_j^T. \tag{28}$$

By continuing this approach and calculating the two-step ahead prediction:

$$\tilde{x}_d\left(k+2\left|k\right.\right) = (A_d + \Psi_1 I)\tilde{x}_d\left(k+1\left|k\right.\right) - \sum_{j=2}^{k+2} (-1)^j \Psi_j E\left[x_d\left(k+2-j\right)\left|Z_k\right.\right]$$
$$\tilde{x}_d\left(k+2\left|k\right.\right) = (A_d + \Psi_1 I)\tilde{x}_d\left(k+1\left|k\right.\right) - \sum_{j=2}^{k+2} (-1)^j \Psi_j x_d\left(k+2-j|k\right) \tag{29}$$

and the two-step ahead prediction error covariance matrix:

$$P_d(k+2|k) = (A_d + \Psi_1)P_d(k+1\,|k)\ (A_d + \Psi_1)^T + \sum_{j=2}^{k+2} \Psi_j P(k+2-j|k)\Psi_j^T \tag{30}$$

where $\tilde{x}_d\,(k+1\,|k\,)$ in the (29) is a one-step ahead prediction that obtains from (27), and $x_d(k+2-j|k) = x(k+2-j|k+d)$ obtains from (25) and lemma 2, the calculations continue in the same way, so for a d-step ahead prediction we have:

$$\begin{aligned}
\tilde{x}_d(k+d|k) &= (A_d + \Psi_1 I)\tilde{x}_d(k+d-1|k) - \sum_{j=2}^{k+d} (-1)^j \Psi_j E\left[x_d(k+d-j)\,|Z_k\,\right] \\
&= (A_d + \Psi_1 I)\tilde{x}_d(k+d-1|k) - \Psi_2 E\left[x_d(k+d-2)\,|Z_k\,\right] \\
&\quad + \Psi_3 E\left[x_d(k+d-3)\,|Z_k\,\right] + \ldots + (-1)^{d-1}\Psi_{d-1} E\left[x_d(k+1)|Z_k\right] \\
&\quad - \sum_{j=d}^{k+d} (-1)^j \Psi_j x_d(k+d-j|k) \to \tilde{x}_d(k+d|k) \\
&= (A_d + \Psi_1 I)\tilde{x}_d(k+d-1|k) - \Psi_2 \tilde{x}_d(k+d-2|k) \\
&\quad + \Psi_3 \tilde{x}_d(k+d-3|k) + \ldots + (-1)^{d-1}\Psi_{d-1}\tilde{x}_d(k+1|k) \\
&\quad - \sum_{j=d}^{k+d} (-1)^j \Psi_j x_d(k+d-j|k).
\end{aligned} \tag{31}$$

Therefore we obtained d–step ahead prediction in (31), easily by substituting k with (k-d) in (31) we can find $\tilde{x}_d(k|k-d)$ as follows:

$$\begin{aligned}
\tilde{x}_d(k|k-d) &= (A_d + \Psi_1 I)\tilde{x}_d(k-1|k-d) - \Psi_2 \tilde{x}_d(k-2|k-d) \\
&\quad + \Psi_3 \tilde{x}_d(k-3|k-d) + \ldots + (-1)^{d-1}\Psi_{d-1}\tilde{x}_d(k-d+1|k-d) \\
&\quad - \sum_{j=d}^{k} (-1)^j \Psi_j x_d(k-j|k)
\end{aligned} \tag{32}$$

also from (25) we find:

$$\begin{cases} P_d(k|k-d) = E\left[\left(x(k) - \tilde{x}_d(k\,|k-d)\,\right)\left(x(k) - \tilde{x}_d(k\,|k-d)\,\right)^T\right] \\ P(k\,|k) = E\left[\left(x(k) - \widehat{x}(k\,|k)\right)\left(x(k) - \widehat{x}(k\,|k)\right)^T\right]. \end{cases}$$

It is obvious that $P_d(k|k-d) = P(k\,|k)$, so we only need to calculate $P_d(k|k-d)$, using (32) and assumption 3 we find prediction error as follows:

$$\begin{aligned}
x(k) - \tilde{x}_d(k|k-d) &= x(k) - (A_d + \Psi_1 I)\tilde{x}_d(k-1|k-d) + \Psi_2\tilde{x}_d(k-2|k-d) \\
&\quad - \Psi_3\tilde{x}_d(k-3|k-d) + \ldots + (-1)^{d-1}\Psi_{d-1}\tilde{x}_d \\
&\quad \times(k-d+1|k-d) + \sum_{j=d}^{k}(-1)^j\Psi_j x_d(k-j|k) \\
&= (A_d + \Psi_1 I)x_d(k-1) + w(k-1) - \sum_{j=2}^{k}(-1)^j\Psi_j x_d(k-j) \\
&\quad -(A_d + \Psi_1 I)\tilde{x}_d(k-1|k-d) + \Psi_2\tilde{x}_d(k-2|k-d) \\
&\quad -\Psi_3\tilde{x}_d(k-3|k-d) + \ldots + (-1)^{d-1}\Psi_{d-1}\tilde{x}_d(k-d+1\,|k-d) \\
&\quad +\sum_{j=d}^{k}(-1)^j\Psi_j x_d(k-j|k) \to x(k) - \tilde{x}_d(k|k-d) \\
&= (A_d + \Psi_1 I)\,(x(k-1) - \tilde{x}_d(k-1|k-d)) + w(k-1) \\
&\quad - \Psi_2(x(k-2) - \tilde{x}_d(k-2|k-d)) + \Psi_3(x(k-3) \\
&\quad -\tilde{x}_d(k-3|k-d)) + \ (-1)^{d-1}\Psi_{d-1}(x(k-d+1) \\
&\quad -\tilde{x}_d(k-d+1|k-d)) + \sum_{j=d}^{k}(-1)^j\Psi_j\,(x(k-j) - x_d(k-j|k)).
\end{aligned} \tag{33}$$

The product of each side of (33) with itself and after applying the expectation operator, we find:

$$\begin{aligned}
&P_d(k|k-d) = (A_d + \Psi_1 I)P_d(k-1|k-d)(A_d + \Psi_1 I)^T + \Psi_2 P_d(k-2|k-d)\Psi_2^T \\
&+ \Psi_3 P_d(k-3|k-d)\Psi_3^T + \ldots + \Psi_{d-1}P_d\,(k-d+1\,|k-d)\,\Psi_{d-1}^T \\
&+ \sum_{j=0}^{k-d}\Psi_j P_d(k-d-j|k-d)\Psi_j^T + Q(k-1)
\end{aligned} \tag{34}$$

as regards all the cross-terms are zero, and $P_d(k-d-j|k-d) = P(k-j|k)$ which is the smoothing covariance matrix, obtains from lemma 2, and the terms $P_d(k-1|k-d)$,$P_d(k-2|k-d)$,$\ldots$, $P_d\,(k-d+1\,|k-d)$ have been calculated in the previous steps. So the proof is completed.

6 Simulation and results

We want to estimate the states of a fractional order system with process and measurement noise covariance matrices and time-delay in the measurement equation. The discrete-time fractional order system is expressed as follows:

$$\begin{aligned}
&A_d = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \, C = \begin{bmatrix} 0.1 \; 0.3 \end{bmatrix} \\
&N = \begin{bmatrix} 0.7 \; 1.2 \end{bmatrix} \quad d = 25, T = 1 \\
&E\left[v_k v_k^T\right] = 0.3 \;,\; E\left[w_k w_k^T\right] = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}
\end{aligned}$$

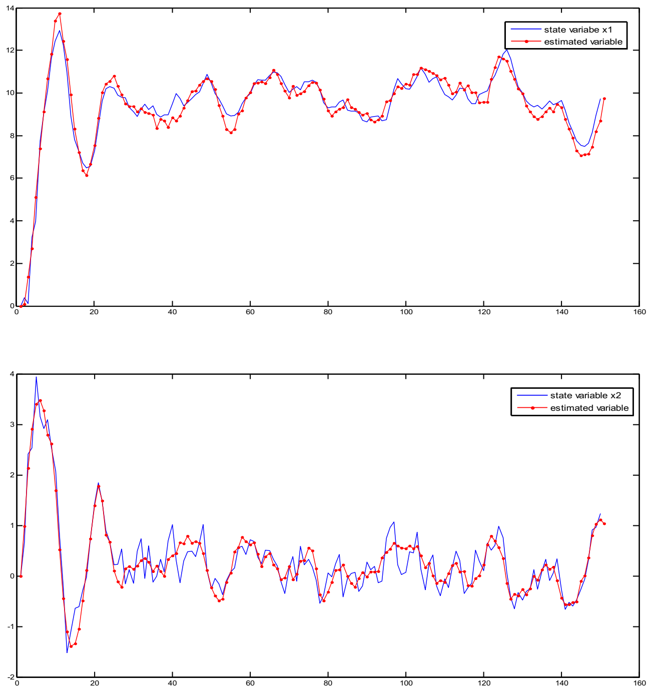

Fig. 3. Real states x1 and x2 and their estimation.

and the initial parameters for fractional Kalman filter are:

$$P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, \; Q = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix} \quad R = [0.3] \,.$$

Using problem 1, the results of fractional Kalman filter state estimation for the above fractional order system are shown in Fig. 3. as can be seen, the states have been estimated accurately.

7 Conclusions

This paper has presented the fractional Kalman filtering for time-delay fractional order systems when a constant time-delay is entered in the observation signal. This filter, for example, can be applied in multi-sensor data fusion when one of the sensors has a delay in data processing unit. Using this filter we can fuse information from different sensors with unequal processing speeds.

References

1. M. Nakagawa, K. Sorimachi, Comm. Comput. Sci. **75**, 1814 (1992)
2. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications* (Academic press, San Diego, 1998)
3. S. Westerlund, Dead matter has memory! Causal consulting (Kalmar, Sweden, 2002)

4. D. Baleanu, J.A.T. Machado, A.C. Luo, *Fractional dynamics and control* (Springer Science & Business Media, 2011)
5. Y. Chen, I. Petráš, D. Xue, in *American Control Conference, ACC'09* (2009), p. 1397
6. A. Dzieliński, D. Sierociuk, in *Computational Intelligence for Modelling, Control and Automation* (2005), p. 804
7. I. Podlubny, IEEE Trans. Automatic Control **44**, 208 (1999)
8. H. Sadeghian, H. Salarieh, A. Alasty, A. Meghdari, Automatica **50**, 282 (2014)
9. D. Xue, C. Zhao, Y. Chen, in *Proceedings of American control conference* (2006), p. 3182
10. A. Dzielinski, D. Sierociuk, in *Fractional Differentiation and its Applications* (2006), p. 511
11. H. Sadeghian, H. Salarieh, A. Alasty, A. Meghdari, Mechatronics **23**, 764 (2013)
12. D. Sierociuk, A. Dzielinski, Int. J. Appl. Math. Comput. Sci. **16**, 129 (2006)
13. D. Sierociuk, I. Tejado, B.M. Vinagre, Signal Proc. **91**, 542 (2011)
14. S. Victor, R. Malti, H. Garnier, A. Oustaloup, Automatica **49**, 926 (2013)
15. S. Agrawal, M. Srivastava, S. Das, Chaos, Solitons & Fractals **45**, 737 (2012)
16. W.M. Ahmad, J. Sprott, Chaos, Solitons & Fractals **16**, 339 (2003)
17. S. Bhalekar, Eur. Phys. J. Special Topics, 1495 (2014)
18. M.-F. Danca, W.K. Tang, Q. Wang, G. Chen, Eur. Phys. J. B **86**, 1 (2013)
19. C. Li, G. Chen, Chaos, Solitons & Fractals **22**, 549 (2004)
20. H. Sadeghian, H. Salarieh, A. Alasty, A. Meghdari, Comput. Math. Appl. **62**, 1482 (2011)
21. M.S. Tavazoei, M. Haeri, S. Jafari, S. Bolouki, M. Siami, IEEE Trans. Ind. Electronics **55**, 4094 (2008)
22. X.-Y. Wang, J.-M. Song, Comm. Nonlinear Sci. Numer. Simul. **14**, 3351 (2009)
23. Y. Wang, K. Sun, S. He, H. Wang, Eur. Phys. J. Special Topics **223**, 1591 (2014)
24. W. Deng, C. Li, Physica A: Stat. Mech. Appl. **353**, 61 (2005)
25. I. Grigorenko, E. Grigorenko, Phys. Rev. Lett. **91**, 34101 (2003)
26. R.E. Kalman, J. Fluids Eng. **82**, 35 (1960)
27. M. Basin, E. Sanchez, R. Martinez-Zuniga, Int. J. Innov. Comput. Info. Cont. **3**, 309 (2007)
28. M. Briggs, R. Vinter, IMA J. Math. Cont. Info. **6**, 167 (1989)
29. H. Zhao, Control Conference (CCC), 33rd Chinese (2014), p. 8947
30. H. Zhao, P. Cui, W. Wang, D. Yang, IEEE Trans. Signal Proc. (2013), p. 316
31. B.D. Anderson, J.B. Moore, *Optimal filtering*: Courier Corporation (2012)
32. T. Kailath, A.H. Sayed, B. Hassibi, *Linear estimation* (Prentice Hall, NJ, 2000)
33. T. Kaczorek, *Selected problems of fractional systems theory* (Springer Science, 2011)
34. C.A. Monje, Y. Chen, B.M. Vinagre, D. Xue, V. Feliu-Batlle, *Fractional-order systems and controls: fundamentals and applications* (Springer Science, 2010)
35. S. Särkkä, Bayesian estimation of time-varying systems: Discrete-time systems, *Aalto University, School of Science and Technology* (2010)
36. P. Ziubinski, D. Sierociuk, in *International Conference on Fractional Differentiation and Its Applications (ICFDA)* (2014), p. 1
37. X. Sun, G. Yan, B. Zhang, in *International Conference on Measurement, Information and Control (ICMIC)* (2013), p. 151