Eur. Phys. J. Special Topics **225**, 83–96 (2016)

DOI: 10.1140/epjst/e2016-02613-0

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Fractional order stochastic dynamical systems with distributed delayed control and Poisson jumps

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Received 31 July 2015 / Received in final form 11 January 2016
Published online 29 February 2016

Abstract. In this paper, we study the controllability results for nonlinear fractional order stochastic dynamical systems with distributed delayed control and Poisson jumps in finite dimensional space. New set of sufficient conditions are derived based on Schauder's fixed point theorem and the controllability Grammian matrix is defined by Mittag-Leffler matrix function. Finally, a numerical example has been given to validate the efficiency of the proposed theoretical results.

1 Introduction

Fractional-order calculus is an area of mathematics that deals with derivatives and integrals from non-integer order. Although fractional calculus dates from the 17th century, fractional differential systems have recently been identified an usual field in diffusion, turbulence, electromagnetism, signal processing, and quantum evolution of complex systems. There are two essential differences between fractional-order derivation and integer-order derivation. Firstly, the fractional-order derivative is concerned with the whole time domain for a mechanical or physical process, while the integer-order derivative indicates a variation or certain attribute at particular time. Secondly, the fractional-order derivative is related to the whole space for a physical process, while the integer-order derivative describes the local properties of a certain position (see e.g. [1–6]). An advantage of fractional-order models in comparison with classical integer-order models is that fractional-order systems have infinite memory. Taking into account of this fact, it is easy to see that the incorporation of a memory term into a neural network model is an extremely important improvement. Nowadays, the dynamics analysis of fractional-order artificial neural networks has become a very promising research topic, and has received some attention. The systematic presentation of the applications of fractional differential equations could be seen in the books [7,8] also one can refer the monographs [9–12].

Recently there has been an increasing interest in dynamical systems involving time delays with applications ranging from biology and population dynamics to physics and

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engineering, and from economics to medicine. So what makes time-delayed systems important? The answer is simple: time delays are intrinsic in many real systems, and therefore must be properly accounted for when developing models. Delay is a common feature of many real processes, and with a growing demand for more precise predictions, control and performance there is a greater need for models to behave as close to real systems as possible. Neural networks provide a perfect real-life example where the time delay is an intrinsic part of the system and also one of the key factors that determines the dynamics. In this particular case, time delay occurs in the interaction between neurons, and is induced by the finite switching speed of amplifiers and the communication time of neurons. Recent studies on synchronization in coupled systems with a time delay in the interactions have shown that delays can induce oscillations but they can also enhance synchrony between coupled elements. It is well known that time delays cause different types of oscillations, but it can also be shown that they cause an amplitude death (i.e. no oscillations in a coupled system while each subsystem oscillates when isolated) within certain regions of the parameter space (see e.g. [13–16]).

As one of the fundamental concepts in mathematical control theory, controllability plays an important role both in deterministic and stochastic control theory. Also it plays a vital role in the analysis and design of control systems. The approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert space has been studied in [17]. Controllability of stochastic systems with distributed delays in control has been investigated in [18]. Moreover, in [19] relative controllability of fractional dynamical systems with distributed delays in control has been discussed.

Systematic study of controllability was started at the beginning of sixties in XX century and theory of controllability is based on the mathematical description of the dynamical system. Many dynamical systems are such that the control does not affect the complete state of the dynamical system but only a part of it. On the other hand, very often in real industrial processes it is possible to observe only a certain part of the complete state of the dynamical system. Therefore, it is very important to determine whether or not control of the complete state of the dynamical system is possible. Roughly speaking, controllability generally means, that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. For stochastic control systems both linear and nonlinear the situation is less satisfactory. In recent years the extensions of the deterministic controllability concepts to stochastic control systems have been recently discussed only in a rather few number of publications [20,21]. In many cases, deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. The controllability for neutral stochastic functional differential inclusions with infinite delay in abstract space has been studied in [22]. Stochastic controllability of linear systems with delayed control has received much research attentions [21]. The major drawback of studying the fractional differential equations with jumps is the fractional derivative D^q it self contain the term dt^q. A great deal of attention has been given over the past two decades to the analysis of controllability of linear as well as nonlinear stochastic systems. Convergence of numerical solution to the stochastic delay differential equations with jumps and complete controllability of stochastic evolution equations with jumps has been studied in [23,24]. The neutral stochastic functional differential equation with infinite delay and Poisson jumps in an abstract space has been discussed in [25].

Based on the facts, relative controllability of nonlinear systems with distributed delays in control has attracted increasing attention for [18,19,26,27]. However, to the best of authors knowledge, there are no relevant reports on the study of relative controllability results for fractional order stochastic dynamical systems with distributed delays in the control term and Poisson jumps in the finite dimensional space. Motivated by the above, the goal of this paper is to describe a relative controllability results for fractional order stochastic dynamical systems with distributed delayed control and Poisson jumps. Sufficient conditions for controllability results are obtained by using Schauder's fixed point theorem with a Grammian matrix defined by Mittag-Leffler matrix function.

The paper is organized as follows: In Sect. 2, some well known fractional operators and special functions, along with a set of properties are defined. The solution representation for linear fractional differential equation is also discussed. In Sect. 3, the nonlinear fractional stochastic dynamical systems with distributed delayed control and Poisson jumps is proposed also the controllability results are derived by using the Schauder's fixed point theorem. Numerical example is illustrated in Sect. 4 to show the effectiveness of the derived results. Finally, conclusion is drawn in Sect. 5.

2 Preliminaries

Let $q > 0,\ p > 0$ with $n-1 < q < n,\ n-1 < p < n$, and $n \in \mathbb{N}$. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. The following notations, definitions and properties are well known, for a suitable function $f \in L_1(\mathbb{R}_+), \mathbb{R}_+ = [0, \infty)$. The function f follows the assumptions given in the references [9,12].

Definition 1. *Riemann-Liouville fractional operators:*

$$(I^q_{0+}f)(x) = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} f(t)dt$$
$$(D^q_{0+}f)(x) = D^n(I^{n-q}_{a+}f)(x),$$

and the Laplace transform of the Riemann-Liouville fractional integral is given by

$$\mathcal{L}\{I^q_t f(t)\} = \frac{1}{\lambda^q}\hat{f}(\lambda),$$

where

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt.$$

The Riemann-Liouville fractional derivative of order $0 < q < 1$ for the function f can be defined as

$$D^q_t f(t) = \frac{d}{dt} I^{1-q}_t f(t).$$

Definition 2. *Mittag-Leffler Function:*
A two parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{q,p}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+p)}, \quad q,p > 0, \quad z \in \mathbb{C}.$$

The most interesting properties of the Mittag-Leffler function are associated with their Laplace integral

$$\int_0^\infty e^{-st} t^{p-1} E_{q,p}(\pm a t^q)dt = \frac{s^{q-p}}{(s^q \mp a)}.$$

That is

$$\mathcal{L}\{t^{p-1}E_{q,p}(\pm at^q)\}(s) = \frac{s^{q-p}}{(s^q \mp a)}.$$

Solution representation:

The linear fractional differential equation

$$\begin{aligned} d[I_t^{1-q}(x(t)-x_0)] &= [Ax(t)+f(t)]\,dt, \quad t \in [0,T] := J, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $0 < q < 1$, I_t^{1-q} is the $(1-q)-$ order Riemann-Liouville fractional integral operator $x \in \mathbb{R}^n$ and A is an $n \times n$ matrix. Now applying the Riemann-Liouville integral operator on both sides, we get

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}Ax(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds.$$

Taking Laplace transformation on both sides, we have

$$\widehat{x}(s) = \frac{1}{s}x_0 + \frac{1}{s^q}A\widehat{x}(s) + \frac{1}{s^q}\widehat{f}(s).$$

Taking inverse Laplace transformation on both sides, we get the solution of the fractional differential Eq. (1) as

$$x(t) = E_q(At^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s)ds.$$

Let us consider the linear fractional deterministic system with distributed delays in the control which is represented in the following form

$$\begin{aligned} d[I_t^{1-q}(x(t)-x_0)] &= \left[Ax(t) + \int_{-h}^0 d_\tau B(t,\tau)u(t+\tau)\right]dt, \quad t \in J, \\ x(0) &= x_0, \end{aligned} \tag{2}$$

where $0 < q < 1$, I_t^{1-q} is the $(1-q)-$ order Riemann-Liouville fractional integral operator $x \in \mathbb{R}^n$, and the second integral term is in the Lebesgue Stieltjes sense with respect to τ. Let $h > 0$ be given. For function $u : [-h,T] \longrightarrow R^m$ and $t \in J$, we use the symbol u_t to denote the function on $[-h,0]$, defined by $u_t(s) = u(t+s)$ for $s \in [-h,0)$. A is a $n \times n$ matrix, $B(t,\tau)$ is an $n \times m$ dimensional matrix continuous in τ for fixed t and is of bounded variation in t on $[-h,0]$ for each $t \in J$ and continuous from left in t on the interval $(-h,0)$.

The solution of the system (2) is given by the following expression [28,29]

$$x(t) = E_q(At^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\left[\int_{-h}^0 d_\tau B(s,\tau)u(s+\tau)\right]ds.$$

Definition 3. *The set $y(t) = \{x(t), u_t\}$ is the complete state of the system (2) at time t.*

Definition 4. *System (2) is said to be globally relatively controllable on J if for every complete state $y(0)$ and every vector $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on J such that the corresponding trajectory of the system (2) satisfies $x(T) = x_1$.*

In this paper, we adopt the following notations that will be used throughout this paper.

Let $(\Omega, \mathcal{F}, P)$ denotes the complete probability space with a probability measure P on Ω and $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^*$ be an $n-$dimensional Wiener process defined on the probability space. Let $\{\mathcal{F}_t | t \in J\}$ is the filtration generated by $\{w(s) : 0 \leq s \leq t\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n \times \mathbb{R}^m)$ is the Hilbert space of all $\mathcal{F}_t-$ measurable square integrable random variables with values in $\mathbb{R}^n \times \mathbb{R}^m$. Let $L_2^{\mathcal{F}}(J, \mathbb{R}^n)$ is the Hilbert space of all square-integrable and $\mathcal{F}_t$-measurable processes with values in $\mathbb{R}^n$. Let $C_n(J) = C(J, L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n))$ be the Banach space of continuous maps from J into $L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$. Define $\mathcal{B} = C_n(J) \times C_m(J)$ be the Banach space of continuous $L_2(\Omega, \mathcal{F}, \mathbb{R}^n \times \mathbb{R}^m)$ valued $\mathcal{F}_t-$ adapted square integrable functions $(z(t), v(t))$ with norm $\|(z, v)\|^2 = \|z\|^2 + \|v\|^2$, where $\|z\|^2 = \sup_{t \in J} \mathbb{E}\|z(t)\|^2$, $\mathbb{E}(\cdot)$ denotes the mathematical expectation operator of stochastic process with respect to the given probability measure P and $\mathcal{U}_{ad} := L_2^{\mathcal{F}}(J, \mathbb{R}^m)$ is the set of admissible controls. Let $\{N(dt, d\eta), t, \eta \in J\}$ is a centered Poisson random measure with parameter $\Pi(d\eta)dt$ and $\widetilde{N}(dt, d\eta) = N(dt, d\eta) - \Pi(d\eta)dt$ is a compensated Poisson random measure which is independent of $w(t)$ and satisfied $\int_{-\infty}^{+\infty} \Pi(d\eta) < \infty$. Let us assume the following assumptions for further discussion.

- **(H1)** Let $h > 0$ be given. For functions $u : [-h, T] \longrightarrow \mathbb{R}^m$ and $t \in J$, we use the symbol u_t to denote the function on $[-h, 0]$, defined by $u_t(s) = u(t + s)$ for $s \in [-h, 0)$.
- **(H2)** $B(t, \tau)$ is an $n \times m$ dimensional matrix continuous in t for fixed τ and is of bounded variation in τ on $[-h, 0]$ for each $t \in J$ and continuous from left in τ on the interval $(-h, 0)$.
- **(H3)** The functions f, σ, and g are continuous and satisfies the usual linear growth condition, that is, there exists a constants $L > 0, M > 0, N > 0$ and $\Pi(d\eta)$ is a parameter

  (i) $\|f(t, x, u)\|^2 \leq L(1 + \|x\|^2 + \|u\|^2)$
  (ii) $\|\sigma(t, x, u)\|^2 \leq M(1 + \|x\|^2 + \|u\|^2)$
  (iii) $\int_{-\infty}^{+\infty} \|g(t, x, u, \eta)\|^2 \Pi(d\eta) \leq N(1 + \|x\|^2 + \|u\|^2)$

  for all $t \in J$, and all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$.
- **(H4)** The linear fractional stochastic system with distributed delays in the control and Poisson jumps is controllable on J.

3 Controllability results for fractional order system with distributed delays

Using the well known result of unsymmetric Fubini theorem [30] and change of order of integration in the second term, we have

$$
\begin{aligned}
x(t) = {} & E_q(At^q)x_0 \\
& + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0} (t - (s - \tau))^{q-1} E_{q,q}(A(t - (s - \tau))^q) B(s - \tau, \tau) u_0(s) ds \right] \\
& + \int_{0}^{t} \left[\int_{-h}^{0} (t - (s - \tau))^{q-1} E_{q,q}(A(t - (s - \tau))^q) d_\tau B_t(s - \tau, \tau) \right] u(s) ds
\end{aligned}
$$

where

$$
B_t(s, \tau) = \begin{cases} B(s, \tau), & s \leq t \\ 0, & s > t \end{cases}
$$

and dB_τ denotes the integration of Lebesgue Stieltjes sense with respect to the variable τ in the function $B(t,\tau)$. For brevity, let us introduce the following notations

$$G(t,s) := \int_{-h}^{0} (t-(s-\tau))^{q-1} E_{q,q}(A(t-(s-\tau))^q) d_\tau B_t(s-\tau,\tau),$$

$$N(t) := \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0} (t-(s-\tau))^{q-1} E_{q,q}(A(t-(s-\tau))^q) B(s-\tau,\tau) u_0(s) ds \right].$$

Theorem 1. *The linear control system (2) is relatively controllable on J if and only if the controllability Grammian matrix*

$$W(0,T) = \int_0^T G(T,s) G^*(T,s) \mathbb{E}\{\cdot|\mathcal{F}_s\} ds$$

is positive definite, for some $T > 0$, G^* is the transpose of G. □

For convenience, let us introduce the following constants

$$\varphi = \sup_{t\in J} \|E_{q,q}(A(t-s)^q)\|^2, \ \upsilon = \sup_{t\in J} \|E_q(A(t-s)^q)\|^2,$$

$$a_1 = \max\{6\|G^*(T,t)\|^2 T^2, 1\}$$

$$e = 4\times 6 \|G^*(T,t)\|^2 \|W^{-1}\|^2 \Big[\|x_1\|^2 + \|E_q(AT^q)x_0\|^2 + \left\| \int_{-h}^{0} dB_\tau \left(\int_{\tau}^{0} (t-(s-\tau))^{q-1} E_{q,q}(A(t-(s-\tau))^q) B(s-\tau,\tau) u_0(s) ds \right) \right\|^2 \Big],$$

$$\varsigma = 4\times 6 \Big[\|E_q(AT^q)x_0\|^2 + \left\| \int_{-h}^{0} dB_\tau \left(\int_{\tau}^{0} (t-(s-\tau))^{q-1} E_{q,q}(A(t-(s-\tau))^q) B(s-\tau,\tau) u_0(s) ds \right) \right\|^2 \Big],$$

$$\kappa = 4\times 6 \|G^*(T,t)\|^2 \|W^{-1}\|^2 \left(T^2 \upsilon L + \frac{T^{2q+1}}{q^2} \varphi M_\sigma M + \frac{T^{2q}}{q^2} \varphi N \right) \times \left(1 + \sup_{t\in J} \mathbb{E}\|z(t)\|^2 + \sup_{t\in J} \mathbb{E}\|v(t)\|^2 \right),$$

$$\nu = 4\times 6 \left(T^2 \upsilon L + \frac{T^{2q+1}}{q^2} \varphi M_\sigma M + \frac{T^{2q}}{q^2} \varphi N \right) \left(1 + \sup_{t\in J} \mathbb{E}\|z(t)\|^2 + \sup_{t\in J} \mathbb{E}\|v(t)\|^2 \right).$$

Consider the nonlinear fractional order stochastic dynamical system with distributed delayed control and Poisson jumps represented by the fractional differential equation of the form

$$\begin{aligned} d[I_t^{1-q}(x(t)-x_0)] &= \Big[Ax(t) + \int_{-h}^{0} d_\tau B(t,\tau) u(t+\tau) + I_t^{1-q} f(t,x(t),u(t)) \\ &\quad + \int_0^t \sigma(s,x(s),u(s)) dw(s) \Big] dt \\ &\quad + \int_{-\infty}^{+\infty} g(t,x(t),u(t),\eta) \widetilde{N}(dt,d\eta), \quad t\in J, \\ x(0) &= x_0, \end{aligned} \tag{3}$$

where A and B are defined as above and $f : J\times\mathbb{R}^n\times\mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\sigma : J\times\mathbb{R}^n\times\mathbb{R}^m \longrightarrow \mathbb{R}^{n\times n}$ and $g : J\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}_+ \longrightarrow \mathbb{R}^n$ are continuous functions. Now, for each $(z,v)\in\mathcal{B}$, consider the linear fractional order stochastic dynamical system with distributed delayed control and Poisson jumps

$$\begin{aligned} d[I_t^{1-q}(x(t)-x_0)] &= \Big[Ax(t)+\int_{-h}^{0} d_\tau B(t,\tau)u(t+\tau)+I_t^{1-q}f(t,z(t),v(t)) \\ &\quad +\int_0^t \sigma(s,z(s),v(s))dw(s)\Big]dt \\ &\quad +\int_{-\infty}^{+\infty} g(t,z(t),v(t),\eta)\widetilde{N}(dt,d\eta), \quad t\in J, \\ x(0) &= x_0. \end{aligned} \tag{4}$$

Then the solution of the system (4) can be expressed in the following form (see [28,29])

$$\begin{aligned} x(t) &= E_q(At^q)x_0+\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\left[\int_{-h}^{0} d_\tau B(s,\tau)u(s+\tau)\right]ds \\ &\quad +\int_0^t E_q(A(t-s)^q)f(s,z(s),v(s))ds \\ &\quad +\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\left[\int_0^s \sigma(\theta,z(\theta),v(\theta))dw(\theta)\right]ds \\ &\quad +\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\int_{-\infty}^{+\infty} g(s,z(s),v(s),\eta)\widetilde{N}(ds,d\eta). \end{aligned}$$

Using the well known result of unsymmetric Fubini theorem [30] and change of order of integration in the second term of the following equation, we have

$$\begin{aligned} x(t) &= E_q(At^q)x_0 \\ &\quad +\int_{-h}^{0} dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] \\ &\quad +\int_0^t\left[\int_{-h}^{0}(t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau)\right]u(s)ds \\ &\quad +\int_0^t E_q(A(t-s)^q)f(s,z(s),v(s))ds \\ &\quad +\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\left[\int_0^s \sigma(\theta,z(\theta),v(\theta))dw(\theta)\right]ds \\ &\quad +\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\int_{-\infty}^{+\infty} g(s,z(s),v(s),\eta)\widetilde{N}(ds,d\eta). \end{aligned} \tag{5}$$

Theorem 2. *Suppose that the hypothesis (H1)–(H4) are satisfied then the nonlinear system (3) is globally relatively controllable on J. Provided that the following hold* $d_1 = \max\{e, \varsigma\}, r_0 = \max\{\kappa, \nu\}$.

Proof. Define the operator $\Psi : \mathcal{B} \longrightarrow \mathcal{B}$ by $\Psi(z, v) = (x, u)$, with

$$\begin{aligned} u(t) = G^*(T,t)\mathbb{E}\Big\{W^{-1}\Big(&x_1 - E_q(AT^q)x_0 - N(T) \\ &- \int_0^T E_q(A(T-s)^q)f(s,z(s),v(s))ds \\ &- \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)\left[\int_0^s \sigma(\theta,z(\theta),v(\theta))dw(\theta)\right]ds \\ &- \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)\int_{-\infty}^{+\infty} g(s,z(s),v(s),\eta)\widetilde{N}(ds,d\eta)\Big)\Big|\mathcal{F}_t\Big\} \end{aligned}$$

and $x(t)$ is defined in (5), W^{-1} inverse of W. Then, it is easy to establish the following estimates

$$\begin{aligned} \mathbb{E}\|u(t)\|^2 = \mathbb{E}\Big\|G^*(T,t)W^{-1}\Big(&x_1 - E_q(AT^q)x_0 - \int_{-h}^0 dB_\tau\Big[\int_\tau^0 (T-(s-\tau))^{q-1} \\ &\times E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\Big] \\ &- \int_0^T E_q(A(T-s)^q)f(s,z(s),v(s))ds \\ &- \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)\left[\int_0^s \sigma(\theta,z(\theta),v(\theta))dw(\theta)\right]ds \\ &- \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)\int_{-\infty}^{+\infty} g(s,z(s),v(s),\eta)\widetilde{N}(ds,d\eta)\Big)\Big\|^2 \\ \le 6\|G^*(T,t)\|^2\|W^{-1}\|^2\Big[&\|x_1\|^2 + \|E_q(AT^q)x_0\|^2 \\ &+ \Big\|\int_{-h}^0 dB_\tau\Big(\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\Big)\Big\|^2 \\ &+ \Big(T^2\upsilon L + \frac{T^{2q+1}}{q^2}\varphi M_\sigma M + \frac{T^{2q}}{q^2}\varphi N\Big)\Big(1 + \sup_{t\in J}\mathbb{E}\|z(t)\|^2 + \sup_{t\in J}\mathbb{E}\|v(t)\|^2\Big)\Big] \\ \le \frac{e}{4a_1} &+ \frac{\kappa}{4a_1} \\ \le \frac{1}{4a_1}&[d_1 + r_0] := r_1, \end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2 &\leq 6\mathbb{E}\|E_q(At^q)x_0\|^2 + 6\mathbb{E}\Big\|\int_{-h}^{0} dB_\tau\Big[\int_{\tau}^{0}(t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)\\
&\quad\times B(s-\tau,\tau)u_0(s)ds\Big]\Big\|^2 + 6\mathbb{E}\Big\|\int_0^t\Big[\int_{-h}^{0}(t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)\\
&\quad\times d_\tau B_t(s-\tau,\tau)\Big]u(s)ds\Big\|^2 + 6\mathbb{E}\Big\|\int_0^t E_q(A(t-s)^q)f(s,z(s),v(s))ds\Big\|^2\\
&\quad+6\mathbb{E}\Big\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\Big[\int_0^s \sigma(\theta,z(\theta),v(\theta))dw(\theta)\Big]ds\Big\|^2\\
&\quad+6\mathbb{E}\Big\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\int_{-\infty}^{+\infty} g(s,z(s),v(s),\eta)\widetilde{N}(ds,d\eta)\Big\|^2\\
&\leq 6\Big[\|E_q(AT^q)x_0\|^2 + T^2\|G^*(T,t)\|^2\Big[\frac{1}{4a_1}(d_1+r_0)\Big]\\
&\quad+\Big\|\int_{-h}^{0} dB_\tau\Big(\int_{\tau}^{0}(t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\Big)\Big\|^2\\
&\quad+\Big(T^2 vL + \frac{T^{2q+1}}{q^2}\varphi M_\sigma M + \frac{T^{2q}}{q^2}\varphi N\Big)\Big(1+\sup_{t\in J}\mathbb{E}\|z(t)\|^2 + \sup_{t\in J}\mathbb{E}\|v(t)\|^2\Big)\Big]\\
&\leq \frac{\varsigma}{4} + \frac{1}{4}(d_1+r_0) + \frac{\nu}{4}\\
&\leq \frac{1}{2}\left[d_1+r_0\right] := r_2.
\end{aligned}$$

Therefore, $\mathbb{E}\|u(t)\|^2 \leq r_1$, for all $t \in J$, which gives $\mathbb{E}\|x\|^2 \leq r_2$. Thus, we have proved that, if $\mathcal{B}(r') = \{(z,v) \in \mathcal{B} : \mathbb{E}\|z\|^2 \leq r' \text{ and } \mathbb{E}\|v\|^2 \leq r'\}$, then Ψ maps $\mathcal{B}(r')$ into itself.

Now let us take $t_1, t_2 \in J$ with $t_1 < t_2$, and for all $(x,u) \in \mathcal{B}(r)$, we need to show that $\Psi[\mathcal{B}(r)]$ is equicontinuous for all $r > 0$. To prove the result, compute

$$\begin{aligned}
&\mathbb{E}\|u(t_1)-u(t_2)\|^2\\
&= \mathbb{E}\Big\|G^*(T,t_1)W^{-1}\Big(x_1 - E_q(AT^q)x_0 - \int_{-h}^{0} dB_\tau\Big[\int_{\tau}^{0}(T-(s-\tau))^{q-1}\\
&\quad\times E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\Big] - \int_0^T E_q(A(T-s)^q)f(s,x(s),u(s))ds\\
&\quad-\int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)\Big[\int_0^s \sigma(\theta,x(\theta),u(\theta))dw(\theta)\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)\int_{-\infty}^{+\infty} g(s,x(s),u(s),\eta)\widetilde{N}(ds,d\eta)\Big)\\
&-G^*(T,t_2)W^{-1}\Big(x_1 - E_q(AT^q)x_0 - \int_{-h}^0 dB_\tau\Big[\int_\tau^0 (T-(s-\tau))^{q-1}\\
&\times E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\Big] - \int_0^T E_q(A(T-s)^q)f(s,x(s),u(s))ds\\
&-\int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)\left[\int_0^s \sigma(\theta,x(\theta),u(\theta))dw(\theta)\right]ds\\
&-\int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)\int_{-\infty}^{+\infty} g(s,x(s),u(s),\eta)\widetilde{N}(ds,d\eta)\Big)\Big\|^2\\
&\le 6\|G^*(T,t_1)-G^*(T,t_2)\|^2\|W^{-1}\|^2\Big[\|x_1\|^2 + \|E_q(AT^q)x_0\|^2\\
&+\Big\|\int_{-h}^0 dB_\tau\Big[\int_\tau^0 (T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\Big]\Big\|^2\\
&+\Big(T^2vL + \frac{T^{2q+1}}{q^2}\varphi M_\sigma M + \frac{T^{2q}}{q^2}\varphi N\Big)\Big(1+\sup_{t\in J}\mathbb{E}\|x(t)\|^2 + \sup_{t\in J}\mathbb{E}\|u(t)\|^2\Big)\Big] \qquad (6)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\|x(t_1)-x(t_2)\|^2 &= \mathbb{E}\Big\|E_q(At_1^q)x_0\\
&+\int_{-h}^0 dB_\tau\Big[\int_\tau^0 (t_1-(s-\tau))^{q-1}E_{q,q}(A(t_1-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\Big]\\
&+\int_0^{t_1}\left[\int_{-h}^0 (t_1-(s-\tau))^{q-1}E_{q,q}(A(t_1-(s-\tau))^q)d_\tau B_{t_1}(s-\tau,\tau)\right]u(s)ds\\
&+\int_0^{t_1} E_q(A(t_1-s)^q)f(s,x(s),u(s))ds\\
&+\int_0^{t_1}(t_1-s)^{q-1}E_{q,q}(A(t_1-s)^q)\left[\int_0^s \sigma(\theta,x(\theta),u(\theta))dw(\theta)\right]ds\\
&+\int_0^{t_1}(t_1-s)^{q-1}E_{q,q}(A(t_1-s)^q)\int_{-\infty}^{+\infty} g(s,x(s),u(s),\eta)\widetilde{N}(ds,d\eta)\\
&-E_q(At_2^q)x_0 - \int_{-h}^0 dB_\tau\Big[\int_\tau^0 (t_2-(s-\tau))^{q-1}E_{q,q}(A(t_2-(s-\tau))^q)\\
&\times B(s-\tau,\tau)u_0(s)ds\Big]\\
&-\int_0^{t_2}\Big[\int_{-h}^0 (t_2-(s-\tau))^{q-1}E_{q,q}(A(t_2-(s-\tau))^q)d_\tau B_{t_2}(s-\tau,\tau)\Big]u(s)ds\\
&-\int_0^{t_2} E_q(A(t_2-s)^q)f(s,x(s),u(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^{t_2}(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)\left[\int_0^s \sigma(\theta,x(\theta),u(\theta))dw(\theta)\right]ds\\
&-\int_0^{t_2}(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)\int_{-\infty}^{+\infty} g(s,x(s),u(s),\eta)\widetilde{N}(ds,d\eta)\Big\|^2\\
\le\ & 10\Big\{\|E_q(At_1^q)x_0-E_q(At_2^q)x_0\|^2+\Big\|\int_{-h}^0 dB_\tau\Big[\int_\tau^0[R(t_1,s)-R(t_2,s)]B(s-\tau,\tau)\\
&\times u_0(s)ds\Big]\Big\|^2+(t_2-t_1)\int_{t_1}^{t_2}\|G(t_2,s)\|^2\mathbb{E}\|u(s)\|^2ds+t_1\int_0^{t_1}\|G(t_1,s)-G(t_2,s)\|^2\\
&\times\mathbb{E}\|u(s)\|^2ds+(t_2-t_1)\int_{t_1}^{t_2}\|E_q(A(t_2-s)^q)\|^2L(1+\mathbb{E}\|x(s)\|^2+\mathbb{E}\|u(s)\|^2)ds\\
&+t_1\int_0^{t_1}\|E_q(A(t_1-s)^q)-E_q(A(t_2-s)^q)\|^2L(1+\mathbb{E}\|x(s)\|^2+\mathbb{E}\|u(s)\|^2)ds\\
&+(t_2-t_1)\int_{t_1}^{t_2}(t_2-s)^{2(q-1)}\|E_{q,q}(A(t_2-s)^q)\|^2TM_\sigma M(1+\mathbb{E}\|x(s)\|^2+\mathbb{E}\|u(s)\|^2)ds\\
&+t_1\int_0^{t_1}\|(t_1-s)^{(q-1)}E_{q,q}(A(t_1-s)^q)-(t_2-s)^{(q-1)}E_{q,q}(A(t_2-s)^q)\|^2\\
&\times TM_\sigma M(1+\mathbb{E}\|x(s)\|^2+\mathbb{E}\|u(s)\|^2)ds\\
&+(t_2-t_1)\int_{t_1}^{t_2}(t_2-s)^{2(q-1)}\|E_{q,q}(A(t_2-s)^q)\|^2N(1+\mathbb{E}\|x(s)\|^2+\mathbb{E}\|u(s)\|^2)ds\\
&+t_1\int_0^{t_1}\|(t_1-s)^{(q-1)}E_{q,q}(A(t_1-s)^q)-(t_2-s)^{(q-1)}E_{q,q}(A(t_2-s)^q)\|^2\\
&\times N(1+\mathbb{E}\|x(s)\|^2+\mathbb{E}\|u(s)\|^2)ds\Big\} \qquad (7)
\end{aligned}
$$

where
$R(t_1,s)-R(t_2,s)=(t_1-(s-\tau))^{q-1}E_{q,q}(A(t_1-(s-\tau))^q)-(t_2-(s-\tau))^{q-1}E_{q,q}(A(t_2-(s-\tau))^q)$. Moreover, for all $(x,u)\in\mathcal{B}(r)$,

$$
\begin{aligned}
\mathbb{E}\|u(t)\|^2\le\ & 6\|G^*(T,t)\|^2\|W^{-1}\|^2\Big(\|x_1\|^2+\|E_q(AT^q)x_0\|^2\\
&+\Big\|\int_{-h}^0 dB_\tau\Big[\int_\tau^0(T-(s-\tau))^{q-1}\\
&\times E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\Big]\Big\|^2+T\int_0^T\|E_q(A(T-s)^q)\|^2\\
&\times\mathbb{E}\|f(s,x(s),u(s))\|^2ds+TM_\sigma\int_0^T(T-s)^{2(q-1)}\\
&\times\left[\int_0^s\mathbb{E}\left\|\sigma(\theta,x(\theta),u(\theta))\right\|^2d\theta\right]\|E_{q,q}(A(T-s)^q)\|^2ds\\
&+T\int_0^T(T-s)^{2(q-1)}\|E_{q,q}(A(T-s)^q)\|^2\\
&\times\int_{-\infty}^{+\infty}\mathbb{E}\|g(s,x(s),u(s),\eta)\|^2\widetilde{N}(ds,d\eta)\Big).
\end{aligned}
$$

Thus the right sides of the Eqs. (6) and (7) are independent of $(x,u)\in\mathcal{B}(r)$ and tend to zero as $t_1\longrightarrow t_2$. Hence $\Psi[\mathcal{B}(r)]$ is equicontinuous for all $r>0$ and by the regularity assumptions on f, σ and g are continuous, and hence it is completely continuous by the application of Arzela-Ascoli's theorem. Since $\mathcal{B}(r')$ is closed, bounded and convex, the Schauder fixed point theorem guarantees that Ψ has a fixed point $(z, v) \in \mathcal{B}(r)$ such that $\Psi(z, v) = (z, v) \equiv (x, u)$. Hence $x(t)$ is the solution of the system (3), and it is easy to verify that $x(T) = x_1$. Further the control function $u(t) \in \mathcal{U}_{ad}$ steers the system (3) from initial complete state $y(0)$ to x_1 on J. Hence the system (3) is globally relatively controllable on J. □

Remark 1. System (3) is said to be null controllable on J if for every complete state $y(0)$ and every vector $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on J such that the corresponding trajectory of the system (3) satisfies $x(T) = 0$. Any vector $\omega \in \mathbb{R}^n$ in $n-$ dimensional vector space is said to reachable if there exists an admissible initial point $x_0 \in \mathbb{R}^n$, admissible control input $u(t) \in \mathbb{R}^m$ and $T > 0$ such that the solution of system (3) satisfies $x(T, x_0) = \omega$. System (3) is said to be approximately controllable on J if given an arbitrary $\epsilon > 0$ it is possible to steer from the point x_0 to within a distance ϵ from all points in the state space $\mathbb{R}^n$ at time T. The proposed results can also be extended to study the problem of null controllability, reachability and approximate controllability of the nonlinear fractional order stochastic dynamical system with Poisson jumps.

4 Example

Example 1. Consider the following nonlinear fractional order stochastic dynamical system with distributed delayed control and Poisson jumps

$$\begin{aligned} d\Big[I_t^{1-q}(x(t) - x_0)\Big] &= \Big[Ax(t) + \int_{-1}^{0} d_\tau B(t,\tau)u(t+\tau) \\ &\quad + I_t^{1-q} f(t, x(t)) + \int_0^t \sigma(s, x(s))dw(s)\Big]\, dt \\ &\quad + \int_{-\infty}^{+\infty} g(t, x(t), \eta)\widetilde{N}(dt, d\eta), \\ x(0) &= x_0 \end{aligned}$$

for $t \in J$ and $0 < q < 1$. In the matrix form, we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B(t,\tau) = \begin{pmatrix} e^\tau \cos t & e^\tau \sin t \\ -e^\tau \sin t & e^\tau \cos t \end{pmatrix},\ f(t, x(t)) = \begin{pmatrix} \frac{10x_1(t)}{1+x_1^2(t)+x_2^2(t)} \\ \frac{x_2(t)}{1+x_2^2(t)+t} \end{pmatrix},$$

$$\sigma(t, x(t)) = \begin{pmatrix} \frac{5tx_1(t)e^{-t}}{1+t} \\ \frac{(1+t)x_2(t)e^{-t}}{1+t} \end{pmatrix} \text{and } g(t, x(t), \eta) = \begin{pmatrix} \frac{2tx_1(t)e^{-t}}{\eta} \\ \frac{-x_2(t)e^{-t}}{\eta} \end{pmatrix}.$$

The Mittag-Leffler matrix function of the systems is given by (see [28])

$$E_q(At^q) = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j t^{2jq}}{\Gamma[1+2jq]} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{(2j+1)q}}{\Gamma[1+(2j+1)q]} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j t^{(2j+1)q}}{\Gamma[1+(2j+1)q]} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{2jq}}{\Gamma[1+2jq]} \end{pmatrix}.$$

Further

$$E_{q,q}(A(T-(s-\tau))^q) = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{2jq}}{\Gamma[(1+2j)q]} & \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(2j+1)q}}{\Gamma[(j+1)2q]} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(2j+1)q}}{\Gamma[(j+1)2q]} & \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{2jq}}{\Gamma[(1+2j)q]} \end{pmatrix},$$

and

$$(T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) = \begin{pmatrix} \cos_q(t) & \sin_q(t) \\ -\sin_q(t) & \cos_q(t) \end{pmatrix},$$

where $\cos_q(t)$ and $\sin_q(t)$ are given by

$$\cos_q(t) = \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(2j+1)q-1}}{\Gamma[(1+2j)q]},$$

$$\sin_q(t) = \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(j+1)2q-1}}{\Gamma[(j+1)2q]}.$$

Also,
$$G(T,s) = \int_{-1}^{0} (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) d_\tau B_T(s-\tau,\tau),$$
$$= \begin{pmatrix} p(s) & q(s) \\ -q(s) & p(s) \end{pmatrix},$$
$$p(s) = \int_{-1}^{0} e^\tau [\cos_q(T-(s-\tau))\cos(s-\tau) - \sin_q(T-(s-\tau))\sin(s-\tau)]\, d\tau,$$
$$q(s) = \int_{-1}^{0} e^\tau [\sin_q(T-(s-\tau))\cos(s-\tau) - \cos_q(T-(s-\tau))\sin(s-\tau)]\, d\tau.$$

By simple matrix calculations, one can see that the controllability matrix

$$W(0,T) = \int_0^T G(T,s) G^*(T,s) ds,$$
$$= \int_0^T [p^2(s) + q^2(s)] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ds,$$

is positive definite for any $T > 0$. Further it is easy to verity the hypothesis (H1)-(H4) hold and it is easy to show that for all $x \in \mathbb{R}^2$, $\|f(t,x(t))\|^2 \leq 100\|x\|^2$, $\|\sigma(t,x(t))\|^2 \leq \frac{100t^2\|x\|^2 e^{-2t}}{(1+t)^2}$ and $\|g(t,x(t),\eta)\| \leq \frac{16t^2\|x\|^2 e^{-2t}}{\eta^2}$. Hence all the assumptions in Theorem 2 are satisfied. Thus the fractional order system is globally relatively controllable on J.

5 Conclusion and future work

This paper has promoted the controllability results for fractional order stochastic dynamical systems with distributed delayed control and Poisson jumps. The results have been obtained based on suitable fixed point theorem. Finally, a numerical example has been given to validate the efficiency of the proposed theoretical results. Impulsive differential equations, that is, differential equations involving impulse effect, appear as a natural description of observed evolution phenomena of several real world problems. Inspired by the applications of fractional order system and impulsive differential equations, solving the fractional order stochastic dynamical systems with distributed delayed control and impulsive deserves our future concern.

The work of authors are supported by Council of Scientific and Industrial Research, Extramural Research Division, Pusa, New Delhi, India under the grant No. 25/(0217)/13/EMR-II.

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