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Explosive synchronization in clustered scale-free networks: Revealing the existence of chimera state

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Abstract. The collective dynamics of Kuramoto oscillators with a positive correlation between the incoherent and fully coherent domains in clustered scale-free networks is studied. Emergence of chimera states for the onsets of explosive synchronization transition is observed during an intermediate coupling regime when degree-frequency correlation is established for the hubs with the highest degrees. Diagnostic of the abrupt synchronization is revealed by the intrinsic spectral properties of the network graph Laplacian encoded in the heterogeneous phase space manifold, through extensive analytical investigation, presenting realistic MC simulations of nonlocal interactions in discrete time dynamics evolving on the network.

1 Introduction

A fundamental problem in dynamics of complex systems exhibiting nonlocal coupling is the integration of information processed in different spatiotemporal regimes. Apart from the effect of noise-induced phase synchronization [1] on maintaining the stable linear coherent state, it is shown that spatially modulated delayed feedback [2] applied to a system of globally coupled oscillators, induces coexistence of generic compositions of coherent and incoherent fluctuations of the phase difference among specific domains of coupled oscillators.

The particular inherent mode, exhibited by the system of nonlocaly coupled oscillators integrated in the complex Ginzburg-Landau equation, was recently revealed by Kuramoto and colleagues [3] as a remarkable phenomenon of simultaneous coexistence of coherence and incoherence signatures, which is called a chimera state [4]. Under such cases, it was shown that chimera states can be induced both by the external source and by avalanches of internal exponentially or linearly driven perturbation in form of time delayed feedback stimulation when the group and phase delay and oscillator coupling strength are monitored [5–7].

Recent results have putted in forefront the particular effects which impose the geometrical structure and topological characteristics [8] of the system over the value of the critical coupling, manifesting the explosive synchronization [9].

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Here we study a scale free (SF) network of coupled Kuramoto like oscillators where the global system is influenced by a stochastic switching perturbation which induces noise independently on the internal Markovian dynamics of the system. Such characteristic nonlinear response can be represented via Laplacian coupling schemes [10] which are induced on one dimensional simplicial complex as a dichotomic Markovian dynamics of adjacently coupled oscillators in the vicinity of a continuous homogeneous oscillatory instability [11]. In particular, we have analyzed the effect of coupling strengths of phase oscillators, associated with different edge weights, on emergence of chimera states during the explosive synchronization. It is shown that the synchronization of clusters of nodes exhibits essential correlation with the specific dynamics of the network encoded in topological structure.

Assuming the globally coupled network of oscillators mapped to a simplicial complex, coupled dynamical ODE-s are numerically integrated by applying the fourth-order Runge-Kutta scheme with time-step $dt \leq 0.01$, where the spatial coupling of the 1-dimensional simplicial complexes is obtained using a 5-point stencil method in a 2-D grid [12]. During the simulation, the total number of oscillators was set to $N = 1000$. Characteristic frequencies of individual topological clusters are estimated using FFT of time-series for a continuance 10^5 time intervals. The evolution of the model for a specific set of variables of local order parameter r and corresponding eigenvalues λ_i, is analyzed over 2000 MC realizations of the uniform intrinsic frequency distribution $-1 \leq \omega_i \leq 1$ with random initial conditions. Estimation of order parameters is performed in forward and backward transitions [13] for associated λ, in adiabatic approximation, where increasing of each successive $\Delta\lambda$ is done with step of 0.01.

2 Results and discussion

We consider the coupled oscillator model [14] as a connected digraph $G(V, E)$ defined on a vertex set $V := \{1, \ldots, n\}$ and edge set $E \subset V \times V$, where the nodes are spatially localized into n clusters of a SF network with specific geometry and different topological and frequency configurations, and subjected to Laplacian coupling schemes [10]. In particular, accounting that clusters of the nodes match the subcomplexes which are coupled by exact adjacency relations, we construct a weighted and directed network of N coupled phase oscillators [15]. The phase of each oscillator is given by $\vartheta_i(t)$, where $i = 1, \ldots, N$ evolves in time following the Kuramoto regime [3]:

$$
\begin{aligned}
\dot{\vartheta}_{i,1}(t) &= \begin{pmatrix} f(x_{1,1}, t) \\ \vdots \\ f(x_{n,1}, t) \end{pmatrix} + \alpha_{i,1} \sum_{j=1}^{d_{i,1}} A_{ij} \sin(\vartheta_{j,1} - \vartheta_{i,1}), \\
\dot{\vartheta}_{i,2}(t) &= \begin{pmatrix} f(x_{1,2}, t) \\ \vdots \\ f(x_{n,2}, t) \end{pmatrix} + \alpha_{i,2} \sum_{j=1}^{d_{i,2}} A_{ij} \sin(\vartheta_{j,2} - \vartheta_{i,2}), \\
&\vdots \\
\dot{\vartheta}_{i,n}(t) &= \begin{pmatrix} f(x_{1,n}, t) \\ \vdots \\ f(x_{n,n}, t) \end{pmatrix} + \alpha_{i,n} \sum_{j=1}^{d_{i,n}} A_{ij} \sin(\vartheta_{j,n} - \vartheta_{i,n}),
\end{aligned} \tag{1}
$$

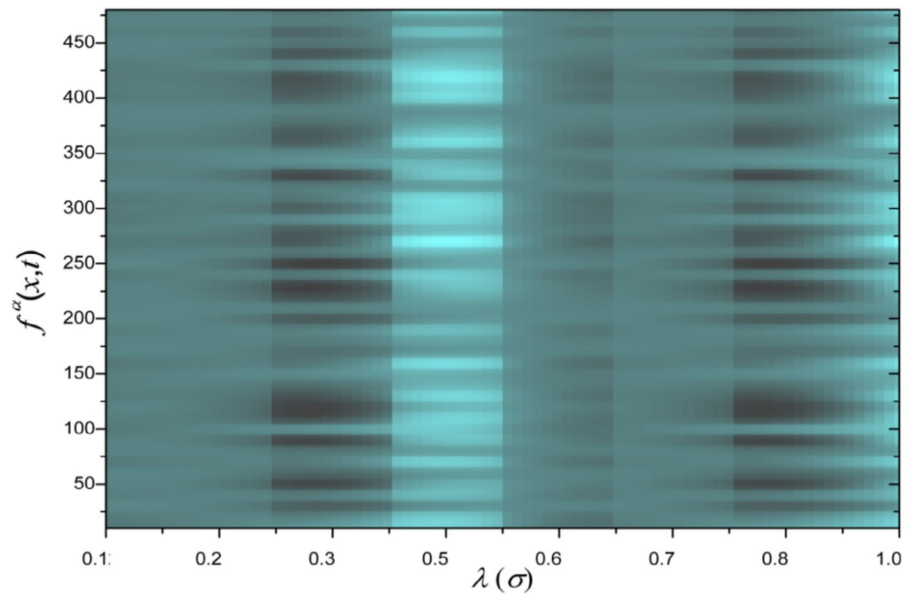

Fig. 1. Amplitude-frequency characteristic obtained by applying oscillations of nearly constant amplitude and variable frequency along weighted edges of a SF network (representing 1000 Kuramoto oscillators) in the linear mode with increasing non normalized coupling strength $\alpha_{i,n}$. Input and output amplitudes of the respective associated frequencies $\omega_{i,n} = f(x_{i,n}, t)$ are set into the relationship with the corresponding eigenvalues in the Laplacian spectrum, showing the coexistence of non-locally coupled coherent and incoherent phase domains.

where $\omega_{i,n} = f(x_{i,n}, t)$ term represents the intrinsic frequency of the i-th oscillator (see Fig. 1) belonging to n-th subcomplex, $\alpha_{i,n}$ is the coupling strength between i-th and n-th collection of nodes, $k_i = \sum_{j=1}^{N} A_{ij}$ is the degree of i-th node, where A_{ij} are the elements of adjacency matrix: $A_{ij} = \begin{cases} 1 & if\ i,j \in E, \\ 0 & if\ i,j \notin E. \end{cases}$

The equations of motion in (1) are essentially unaffected by the permutations of the subscripts i, n in each cluster. Thus, from the fact that all nodes represent elements of a simplicial complex K (encoded in the Laplacian matrix, Fig. 2), the nodes associated with different subcomplexes all receive the same total stimulus from the nodes belonging to neighboring subcomplexes. Under given circumstances, if a different weighs are introduced over a network structure, such connection between the local topology and the coupling strength leads to a phenomenon of explosive synchronization [9] between associated subcomplexes. To every edge it is associated the local order parameter, characterizing the i, n-th oscillator dynamics of the network, $r_i(t)e^{i\varphi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\vartheta_j(t)}$, $0 \leqslant r_i \leqslant 1$, and representing the key factor in modeling the evolution of each oscillator instantaneous phase with time (see Fig. 3).

In particular case, the time domain is introduced considering that $f(x, t)$ from Eq. (1) defines the dynamics of a discrete Witten-Morse function [16]. Note that in general case when the time dependency does not figure, for each N we can consider the Laplacian [17] (where ∂^* is the adjoint of boundary operator ∂ with respect to

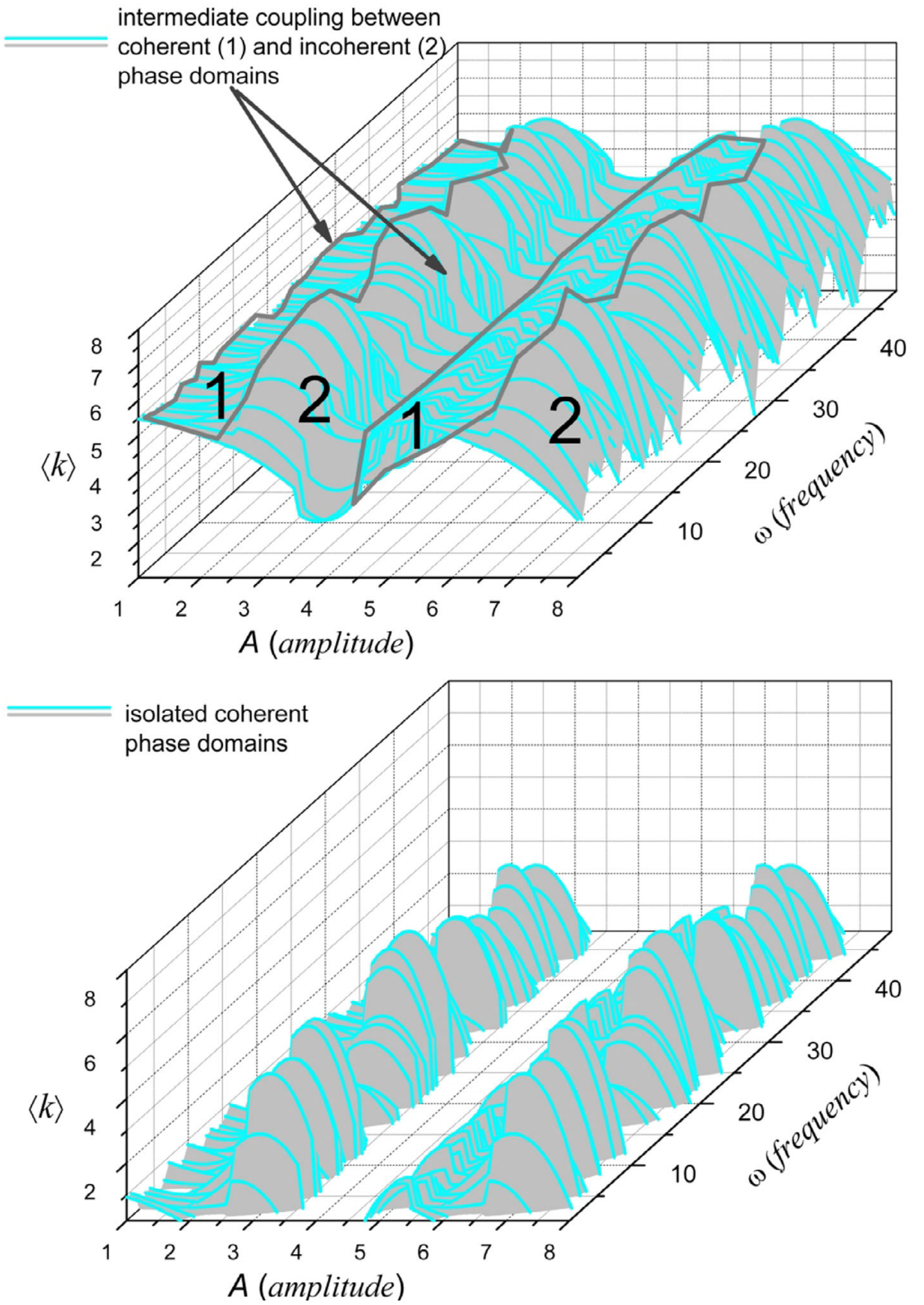

Fig. 2. Top: phase space representation of the Laplacian matrix encoded in heterogeneous SF network of 1000 nodes, exhibiting nonsymmetric coupling between the hubs and peripheral nodes. Two pairs of essentially different (inhomogeneous) non-locally coupled structures are created in the phase panel of SF network: two coherent and two incoherent parameter domains under intermediate coupling regime $\alpha_{i,n} = 0.1 - 0.5$ as a function of average degree $\langle k \rangle$. Bottom: normalized Laplacian matrix encoded in SF network of 1000 nodes, where input coupling strengths of each distinct node are summed to unity. Increasing of the coupling strength to $\alpha_{i,n} = 0.7 - 1.0$ (strong coupling regime) increases the synchronization and strongly confines the coupled structures preventing at the same time the simultaneous existence of coherent and incoherent phase domains.

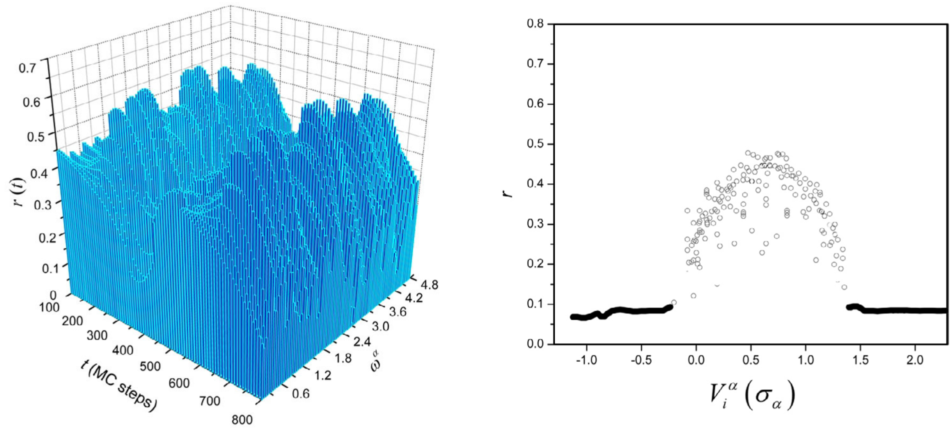

Fig. 3. Right: snapshot of the local order parameter $r(t)$ dependency on coupling induced oscillator frequencies ω^α, for the 1000 nodes distribution function $\rho(k;\vartheta,t)$, with parameters: $t = 800$ MC steps, $k = 7$, referring to coexistences of synchronous and asynchronous phase domains, obtained as a solution of discrete Witten-Morse function. Introduced non-local coupling is $\alpha_{i,n} = 0.2$. Left: local order parameter vs N-component eigenvectors V^α, of the Laplacian $\sum_{j=1}^{N} L_{ij}(\sigma_\alpha)V_j^\alpha = \lambda_i(\sigma_\alpha)V_i^\alpha$, for SF network of $N = 1000$ nodes, with degree distribution $P(k) \sim k^{-3}$. The existence of a chimera state is marked by point symbols (open circles) on the parameter space diagram. The coupling strength α is mapped to eigenspace and consequently increased for associated λ_i starting from λ_0 by amounts $\Delta\lambda = 0.01$, in order to obtain a scaled order parameter r for $\lambda_i = \lambda_0, \lambda_0 + \Delta\lambda, \ldots, \lambda_0 + n\Delta\lambda$. Prior to each $\Delta\lambda$ step, the system was integrated for 10^5 time steps to approach near stationary values.

inner products on the chain C_N spaces):

$$\Delta_N \equiv \partial_{n+1}\partial_{n+1}^* + \partial_n^*\partial_n : C_N \to C_N. \tag{2}$$

Definition: A function $f(x,t) : K \to \mathbb{R}$, defined on a simplicial complex K, represents a *discrete Morse function* if for all simplices $\sigma \in K$ holds:

i) If σ is an irregular face of $u^{(i+1)}$ then $f(u) > f(\sigma)$ and $\#\{u^{(i+1)} > \sigma | f(u) \leqslant f(\sigma)\} \leqslant 1$.
ii) If $v^{(i-1)}$ is an irregular face of σ then $f(v) < f(\sigma)$ and $\#\{v^{(i-1)} < \sigma | f(v) \geqslant f(\sigma)\} \leqslant 1$.

Considering that the network topology and time scales are mapped on dynamical manifold M, where the sets of i-dimensional simplexes σ^i are organized into finite simplicial complex K, a discrete Morse function on M behaves as a function on K.

A (*discrete*) *Witten-Morse function* in fact represents the operation of identifying of a single real number to each simplex on simplicial complex K, representing the Laplacian: $\Delta_{\alpha N}(t) \equiv \partial_i(t)\partial_i^*(t) + \partial_i^*(t)\partial_i(t) : C_\alpha(K,\mathbb{R}) \to C_\alpha(K,\mathbb{R})$, where $\partial_i^*(t)$ is the adjoint of $\partial_i(t)$ with respect to the inner product on $C_\alpha(K,\mathbb{R})$ chains such that the boundary operations over simplices σ are given by the following relations:

$$\begin{aligned} \partial_i(t)\sigma &= \sum_{v^{(i-1)}<\sigma} \pm e^{t(f(v)-f(\sigma))}v, \\ \partial_i^*(t)\sigma &= \sum_{u^{(i+1)}>\sigma} \pm e^{t(f(\sigma)-f(u))}u, \end{aligned} \tag{3}$$

where $\sigma, v, u \in K$, and if $\sigma < u$, then σ is a *face* of u. For each oriented i-cell, $e^{tf}(\sigma) = e^{tf(\sigma)}\sigma$, the induced Laplace operators are given by $\partial_n(t) = e^{tf(v)}\partial_n e^{-tf(\sigma)}$, where the boundary operator $\partial_n(t)$ is a map from the simplices of dimension n to their faces. Hence, Eq. (2) is obtained from the fact that the graph Laplacian $L = D - A$ coincides with a one dimensional simplicial complex (here encoded in SF network), then the adjacency matrix is given by:

$$\begin{aligned} A(x,t) &= \sum_{i=1}^{N} k_{ii} - \partial_1(t)(\sigma^{(i)})\partial_1^*(t)(\sigma^{(i)}) \\ &= \sum_{i=1}^{N} k_{ii} - \left(\sum_{v^{(i-1)}<\sigma} \sum_{\sigma_\alpha^i > v} \pm e^{t(2f(v)-f(\sigma)-f(\sigma_\alpha))} \right) \sigma_\alpha, \end{aligned} \tag{4}$$

$\forall x = x(i,j)$ which denotes the coordinate of each $i = 1, \ldots, N$ oscillator (node), where $D = \sum_{i=1}^{N} k_{ii}$ is the degree matrix.

Without loss of generality the level of phase coherence in the non-locally coupled network of oscillators $\dot{\vartheta}_N(t)$ can be estimated via the global order parameter [9]: $R(t)e^{i\varphi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\vartheta_j(t)}, 0 \leqslant R(t) \leqslant 1$, where the network written in compact form is represented by the Laplacian coupling scheme:

$$\begin{aligned} \dot{\vartheta}_N(t) = &\begin{pmatrix} f(x_{1,1},t) & f(x_{1,2},t) & & f(x_{1,n},t) \\ \vdots & \vdots & \cdots & \vdots \\ f(x_{n,1},t) & f(x_{n,2},t) & & f(x_{n,n},t) \end{pmatrix} + \alpha_{i,n} \sum_{i=1}^{N} k_{ii} \\ &- \left(\sum_{v^{(i-1)}<\sigma} \sum_{\sigma_\alpha^i > v} \pm e^{t(2f(v)-f(\sigma)-f(\sigma_\alpha))} \right) \sigma_\alpha \sum_{j=1}^{d_{i,n}} \sin(\vartheta_{j,n} - \vartheta_{i,n}). \end{aligned} \tag{5}$$

In particular, the normal coordinates $\dot{\vartheta}_i$ in Eq. (1) split into the subsets encoded into the Laplacian spectra, displaying the specific phase dynamics. They reflect not only the network topology, but also detect system degrees of freedom responsible for, on one side, the explosive synchronization by promoting the onset of partial synchronization, and on the other for the emergence of chimera state as a mark of inhibition of the global synchronization in a clustered SF network. The collective coordinates (Eq. (5)) can be represented in terms of projections onto eigenvectors of the Laplacian, as following:

$$\dot{\theta}(t) = F(\theta) + \alpha (L \otimes I_m) \theta(t), \tag{6}$$

where L is the Laplacian symmetric matrix

$$L = W\mathbf{M}W^*, \tag{7}$$

where $\mathbf{M} = diag(\lambda_0, \lambda_1, \ldots, \lambda_n)$ is the eigenvalue matrix.

Defining a vector field $\mathbf{X} = col(x_1, x_2, \ldots, x_n)$ on a manifold $\mathbf{M}$, Eq. (1) can be expanded in terms of Laplacian eigenmodes

$$\Theta(t) = (W \otimes I_m)\,\mathbf{X} = \sum_{i=1}^{n} \theta_i(t) \otimes \mathbf{x_i}. \tag{8}$$

The spectral gap from Eq. (6) is defined as $g(A) = \lambda_1 - \lambda_2$, while the total contribution of the spectrum is given by $\omega(A) = \lambda_2 - \lambda_n$. The gaps in the spectrum are signature of different time scales between Laplacian eigenmodes which on the other hand detect different topological scales of the system [18]. The ratio $R = (\lambda_1 - \lambda_2)/(\lambda_2 - \lambda_n)$ denotes a global parameter of the spectral distance between the first eigenvalue and the main part of the distribution of eigenvalues normalized by the algebraic connectivity λ_2 in relation to a total spectrum.

Following the increase of algebraic connectivity, λ_2, more and more free oscillators will be inclined to synchronize toward each of the distinct topological clusters, but these clusters are suppressed from merging with each other as a result of the presence of lowest eigenvalue mode $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ with high level of multiplicity which corresponds to the number of discrete components in network (mapped to simplicial complex). At the same time, increasing of the coupling strength α strongly affects the Witten-Morse function which behaves as a weighting function to each simplex on simplicial complex, as

$$W(x,t)k_i = A(x,t)\exp\left[if^\alpha(x,t)\right], \forall x = x(i,j), \tag{9}$$

where $f^\alpha(x,t) \equiv \sum_i \varphi_i V_i^\alpha$, with $\omega^\alpha \equiv \sum_i \varphi_i V_i^\alpha$, is induced oscillator frequency by the non-local coupling, φ_i are coupled oscillators phases, and $A(x,t)$ is substituted from Eq. (4) giving rise to a critical value when no more free oscillators are left and when the nodes from the hubs with high degree of correlation k are diverging in synchrony toward phase chimera state [15,19] establishing asymmetrical connection to peripheral nodes characterized by low degree k, displaying as a result a discontinuous and abrupt dynamics of $R(t)$ as a consequence of the sudden explosive transition to synchronization, see Fig. 4.

By setting a time differential of Eq. (8) and combining Eq. (5),

$$\dot{W} = A(x,t) - (1+iC_2)|A(x,t)|^2 A(x,t) + \alpha(1+iC_1)\left(\bar{A}(x,t) - A(x,t)\right), \tag{10}$$

we obtain the familiar form of nonlocal complex Ginzburg-Landau equation [20], where C_1, C_2, and α denote linear and the nonlinear dispersion constant, and coupling strength, respectively, and $\bar{A}(x,t)$ is the nonlocal mean field variable. In $\alpha \ll 1$ limit Eq. (9) reduces to a manifold of dynamical phase space corresponding to chimera states, which is determined by critical points of Witten-Morse function.

In particular, following the Morse Lemma [21] there are points $(\varphi_1(t), \ldots, \varphi_n(t))$, for $x \in (0, \ldots, 0)$, such that in these coordinates

$$F(\varphi_1(t), \ldots, \varphi_n(t)) = F(x) - {\varphi_1}^2(t) + \sum_{i=2}^{n-1} k{\varphi_i}^2(t), \tag{11}$$

where from beginning at the point x, function F decreases from both sides in the φ_1 direction, while in the transverse directions it increases. In particular case, for the critical edge σ_α of a discrete Morse function $f(x,t)$, (see Eqs. (5), (9)), starting from the edge to either boundary node $f(x,t)$ decreases, while in each transverse direction it increases. As a result, $\sigma_\alpha^{(i)}$ as a critical simplex on induced $f^\alpha(x,t)$, represents the i-dimensional "unstable" α-coupled space at a smooth critical point of index i [21]. Coupled oscillator phases defined at smooth critical points, satisfy the flow equation

$$\tfrac{d}{dt}\varphi_i + \nabla_\varphi f^\alpha(x,t) = 0. \tag{12}$$

The Eq. (12) states that coupled phases $\varphi_i(t)$ of oscillators trajectories diffuse in the direction of $-\frac{\nabla f^\alpha}{|\nabla f^\alpha|}$. For coordinates $x(t_i) \in (0, \ldots, 0)$ by modeling choice from

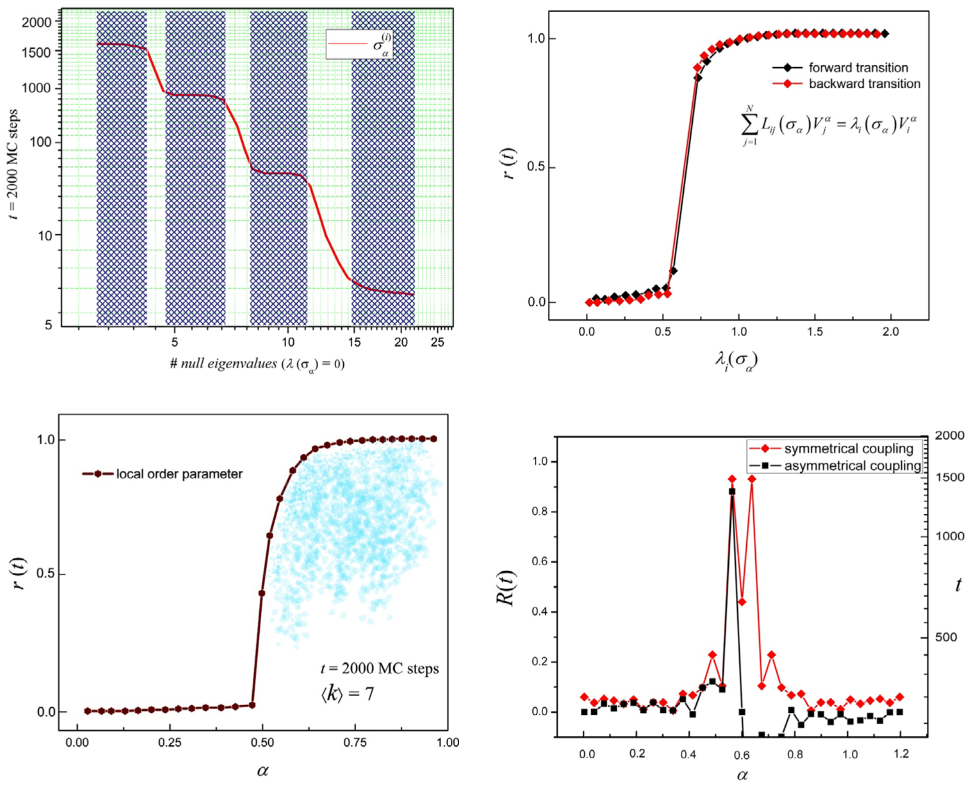

Fig. 4. Top from left to right: stability diagram showing the number of synchronized critical clusters $\sigma_\alpha^{(i)}$ (see text) equivalent to number of null eigenvalues on i-dimensional α coupled space as a function of time for 2000 MC iterations. The blue-pattern areas mark the phase chimeras that are observed in the weak coupling limit. Right: synchronization diagram with forward (black line with diamonds) and backward (red line with diamonds) synchronization transitions as a function of Laplacian eigenvalues of SF network with $N = 1000$ oscillators with introduced non-local coupling $\alpha = 0.45$ and degree $k = 7$. Bottom: the local order parameters of SF network, r, in nonstationary regime as a function of the coupling strength for different α, showing the abrupt transition to synchronization regime. The simulation parameter values are $\langle k \rangle = 7$, $C = 0.95$, see Eq. (14). Right: the evolution of the global order parameter in 2000 MC time steps as a function of various α for symmetrical and asymmetrical non-local coupling, which is exhibited in normalized and heterogeneous SF network with high clustering, respectively.

Eq. (11), and considering that $f^\alpha(x,t) \equiv \omega^\alpha(x(t))$ (see Eq. (9)), solutions satisfy

$$\omega^\alpha(x(t)) + \int_0^T \left| \frac{d\varphi_i}{dt} \right|^2 dt = \omega^\alpha(x_0), \ \forall T > 0. \tag{13}$$

The collective dynamics of the system of non-local coupled oscillators evolves at the total dissipation rate corresponding to $\sum_{i=2}^{n-1} C\dot{\varphi}_i^2$, where $C > 0$ is a total dispersion constant. Putting Eq. (11) and the dissipation together, results in the set of equations from which we can construct geometric flow of coupled oscillator phases at critical points, which corresponds to explosive synchronization, as following

$$\begin{aligned} C\dot{\varphi}_1 &= k\,(\varphi_2 - \varphi_1)\,, \\ C\dot{\varphi}_i &= k\,(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1})\,, \\ C\dot{\varphi}_n &= k\,(\varphi_{n-1} - \varphi_n)\,, \end{aligned} \tag{14}$$

for $i = 2, \ldots, n-1$.

3 Conclusions

In conclusion, we reported on a chimera state existence during cluster explosive transitions to synchronization regime. Using the discrete Witten-Morse function framework for the observation of SF network topology we have established the connection between natural frequencies and local topology of each cluster setting. Particular forms of the distribution of natural frequencies in fact serve as a basis for exact encodings of the local structure dynamics which lead to explosive synchronization. Our findings confirmed the simultaneous existence of both coherent and incoherent phase domains during transition to explosive synchronization in particular examples of (SF) networks, explicitly under intermediate coupling.

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