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Synchronisation of fractional-order time delayed chaotic systems with ring connection

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Abstract. In this paper, synchronisation of fractional-order time delayed chaotic systems in ring networks is investigated. Based on Lyapunov stability theory, a new generic synchronisation criterion for N-coupled chaotic systems with time delay is proposed. The synchronisation scheme is applied to N-coupled fractional-order time delayed simplified Lorenz systems, and the Adomian decomposition method (ADM) is developed for solving these chaotic systems. Performance analysis of the synchronisation network is carried out. Numerical experiments demonstrate that synchronisation realises in both state variables and intermediate variables, which verifies the effectiveness of the proposed method.

1 Introduction

Significant progress in fractional-order calculus has been witnessed in the last several decades since its applications in many fields, such as flow dynamics, quantum theory and anomalous diffusion [1–3]. In recent years, many investigations have been devoted to the chaotic behaviours of fractional-order dynamical systems [3–6].

At the same time, synchronisation of fractional-order chaotic systems is a hot topic [7,8]. Because of its importance in information encryption and secure communication, synchronisation of fractional-order network is enjoying growing interest among researchers [9–16]. Many synchronisation schemes for fractional-order chaotic networks are proposed, including generalised synchronisation [10], robust synchronisation [11], adaptive synchronisation [12], linear feedback control method [13] and coupled synchronisation [14]. Among these methods, the coupled synchronisation with ring connection can be implemented most easily [15], and there are many practical applications [16]. Meanwhile, time delay is unavoidable due to signal propagation in communication processes. Thus it is necessary to investigate synchronisation of fractional-order time delayed chaotic ring networks.

Furthermore, to make the fractional-order time delayed network be available for digital application, a suitable numerical algorithm for solution should be chosen. At present, the numerical simulation of fractional-order time delayed chaotic systems is

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mainly carried out using the Adams-Bashforth-Moulton (ABM) algorithm [17]. It is accurate for fractional-order calculus, but the calculation speed is quite slow [18]. A frequency domain method has also been derived to solve the fractional-order time delayed chaotic systems [19]. However, whether this algorithm is reliable in detecting chaos in nonlinear systems has been questioned [20]. By employing the Adomian decomposition method (ADM), the fractional-order chaotic system can generate chaos with much lower order [21] and can be implemented in digital circuit [22]. It is recommended for solving fractional-order time delayed nonlinear systems [23,24], but applying ADM for synchronisation of fractional-order time delayed chaotic network is still a novel approach.

Motivated by the above discussion, in this paper, we focus on network synchronisation of fractional-order time delayed chaotic systems by applying ADM. In Sect. 2, ADM is developed to solve fractional-order time delayed chaotic systems and the synchronisation scheme is presented. In Sect. 3, N-coupled ring connection synchronisation of fractional-order time delayed simplified Lorenz systems is investigated. Finally, we summarize the results and indicate future directions

2 Synchronisation principle of fractional-order time delayed chaotic systems

In this section, ADM is derived to obtain the numerical solution of a fractional-order time delayed chaotic system, and the synchronisation scheme is proposed.

2.1 Preliminaries and Adomian decomposition method

Firstly, definitions of fractional calculus are recalled. Fundamental properties and a useful Lemma about the fractional-order calculus are presented. Solution algorithm based on ADM for time delayed chaotic systems is introduced.

Definition 1 [25]: The Caputo derivative of fractional order q of $x(t)$ is defined as

$$D_{t_0}^{q} x(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_{t_0}^{t} (x-\tau)^{m-q-1} x^{(m)}(\tau) d\tau, m-1 < q < m \\ \frac{d^m}{dt^m} x(t), q = m \end{cases}, \tag{1}$$

where $m \in N$, $x^{(m)}(t)$ is the m^{th}-order derivative, $\Gamma(\cdot)$ is the Gamma function and $t > 0$.

Definition 2 [26]: The Riemann-Liouville fractional integral operator with fractional order q is defined as

$$J_{t_0}^{q} x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (x-\tau)^{q-1} x(\tau) d\tau. \tag{2}$$

When $q = 0$, $J_{t_0}^{0} x(t) = x(t)$. The fundamental properties of the operator are described as follows.

$$J_{t_0}^{q} (t-t_0)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+q)} (t-t_0)^{\gamma+q}, \tag{3}$$

$$J_{t_0}^q C = \frac{C}{\Gamma(q+1)}(t-t_0)^q, \tag{4}$$

$$J_{t_0}^q J_{t_0}^r x(t) = J_{t_0}^{q+r} x(t), \tag{5}$$

where $t \in (t_0, t_1)$, $\gamma > -1$, $r > 0$, and C is a real constant.

Lemma 1: For $q \in (0, 1]$, $D_{t_0}^q |x(t)| = \text{sgn}(x(t)) D_{t_0}^q x(t)$.

Proof: If $x(t) = 0$, then $D_{t_0}^q |x(t)| = 0$. Let $y(t) = |x(t)|$. If $x(t) > 0$, then

$$D_{t_0}^q |x(t)| = \frac{1}{\Gamma(1-q)} \int_{t_0}^t \frac{\dot{y}(s)}{(t-s)^q} ds = \frac{1}{\Gamma(1-q)} \int_{t_0}^t \frac{\dot{x}(s)}{(t-s)^q} ds = D_{t_0}^q x(t). \tag{6}$$

If $x(t) < 0$, then

$$D_{t_0}^q |x(t)| = \frac{1}{\Gamma(1-q)} \int_{t_0}^t \frac{\dot{y}(s)}{(t-s)^q} ds = -\frac{1}{\Gamma(1-q)} \int_{t_0}^t \frac{\dot{x}(s)}{(t-s)^q} ds = -D_{t_0}^q x(t). \tag{7}$$

Therefore, this Lemma is proved.

Here, ADM is derived for solving fractional-order time delayed system. For a given fractional-order time delayed chaotic system with the form of

$$\begin{cases} D_{t_0}^q \mathbf{x}(t) = f(\mathbf{x}(t), \mathbf{x}(t-\tau)) + \mathbf{g}(t) \text{ for } t > 0 \\ \mathbf{x}(t) = \mathbf{H}(t), \text{ for } t \in [-\tau, 0] \end{cases}, \tag{8}$$

where $\mathbf{x}(t)$=[$x_1(t)$, $x_2(t)$, ..., $x_n(t)$] are state variables of the sytem, and $\mathbf{g}(t)$=[$g_1(t)$, $g_2(t)$, ..., $g_n(t)$] are constants for autonomous systems. So it can be divided into three parts as the form

$$D_{t_0}^q \mathbf{x}(t) = L(\mathbf{x}(t), \mathbf{x}(t-\tau)) + N(\mathbf{x}(t), \mathbf{x}(t-\tau)) + \mathbf{g}(t), \tag{9}$$

where $m \in N$, $m - 1 < q \leq m$. $L(x(t), x(t-\tau))$ and $N(x(t), x(t-\tau))$ are the linear and nonlinear terms of the fractional differential equations respectively. We have [24]

$$\mathbf{x} = J_{t_0}^q L(\mathbf{x}, \mathbf{x}_\tau) + J_{t_0}^q N(\mathbf{x}, \mathbf{x}_\tau) + J_{t_0}^q \mathbf{g} + \Phi, \tag{10}$$

where $\Phi = \sum_{k=0}^{m-1} \mathbf{b}_k (t-t_0)^k / k!$, $\mathbf{x}_\tau = \mathbf{x}(t-\tau)$ and $\mathbf{b}_k$ are the initial conditions. By applying the recursive relation

$$\mathbf{x}^0 = \begin{cases} J_{t_0}^q \mathbf{g} + \Phi \text{ for } t > 0 \\ \mathbf{H}(t) \text{ for } t \leq 0 \end{cases}, \tag{11}$$

$$\mathbf{x}^i = \begin{cases} J_{t_0}^q L(\mathbf{x}^{i-1}, \mathbf{x}_\tau^{i-1}) + J_{t_0}^q \mathbf{A}^{i-1}(\mathbf{x}^0, ..., \mathbf{x}^{i-1}, \mathbf{x}_\tau^0, ..., \mathbf{x}_\tau^{i-1}), \text{for, } t > 0 \\ 0, \text{ for, } t \leq 0 \end{cases}, \tag{12}$$

where $i = 1, 2, ..., \infty$. The analytical solution of the fractional-order system is presented as

$$\mathbf{x}(t) = \sum_{i=0}^{\infty} \mathbf{x}^i, \tag{13}$$

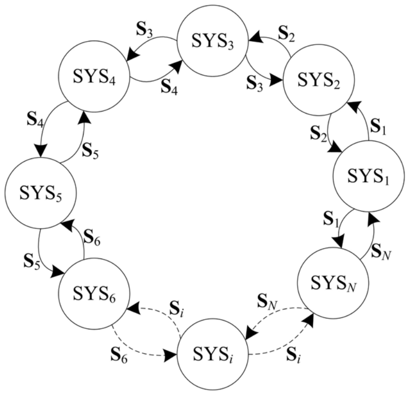

Fig. 1. The topological structure diagram of a ring network with N nodes.

where the nonlinear terms of the fractional-order differential equations $N(\mathbf{x}, \mathbf{x}_\tau)$ is evaluated by [28]

$$N(\mathbf{x}, \mathbf{x}_\tau) = \sum_{n=0}^{\infty} \mathbf{A}^{n-1}(\mathbf{x}^0, \cdots, \mathbf{x}^{n-1}, \mathbf{x}_\tau^0, \cdots, \mathbf{x}_\tau^{n-1}), \tag{14}$$

$$\mathbf{A}^n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{n}\lambda^k \mathbf{x}^k\right)\left(\sum_{k=0}^{n}\lambda^k \mathbf{x}_\tau^k\right)\right]_{\lambda=0}. \tag{15}$$

Thus, solution of the fractional-order time delayed chaotic system based on ADM is obtained.

2.2 Synchronisation scheme

The synchronisation scheme for bidirectional N-coupled fractional-order time delayed chaotic systems with ring connection is described by

$$\begin{cases} D_{t_0}^q \mathbf{x}_1(t) = L(\mathbf{x}_1, \mathbf{x}_1(t-\tau)) + N(\mathbf{x}_1(t), \mathbf{x}_1(t-\tau)) + d(\mathbf{x}_N + \mathbf{x}_2 - 2\mathbf{x}_1) \\ D_{t_0}^q \mathbf{x}_2(t) = L(\mathbf{x}_2, \mathbf{x}_2(t-\tau)) + N(\mathbf{x}_2(t), \mathbf{x}_2(t-\tau)) + d(\mathbf{x}_1 + \mathbf{x}_3 - 2\mathbf{x}_2) \\ \qquad \vdots \\ D_{t_0}^q \mathbf{x}_{N-1}(t) = L(\mathbf{x}_{N-1}, \mathbf{x}_{N-1}(t-\tau)) + N(\mathbf{x}_{N-1}(t), \mathbf{x}_{N-1}(t-\tau)) \\ \qquad + d(\mathbf{x}_{N-2} + \mathbf{x}_N - 2\mathbf{x}_{N-1}) \\ D_{t_0}^q \mathbf{x}_N(t) = L(\mathbf{x}_N, \mathbf{x}_N(t-\tau)) + N(\mathbf{x}_N(t), \mathbf{x}_N(t-\tau)) \\ \qquad + d(\mathbf{x}_{N-1} + \mathbf{x}_1 - 2\mathbf{x}_N) \end{cases}. \tag{16}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_N \in R^n$ are the state vectors of the fractional-order time delayed chaotic systems, and d is the coupled strength. For this network, the structure is shown in Fig. 1, where $\mathbf{S}_i$ represents the data sent by system (i).

Definition 3: Define $\mathbf{e}_i = \mathbf{x}_i - \mathbf{x}_{i+1}$ for $i = 1, 2, ..., N-1$ and $\mathbf{e}_N = \mathbf{x}_N - \mathbf{x}_1$ for $i = N$, when all $||\mathbf{e}_i|| \to 0$ as $t \to \infty$, the network is synchronised. Here $\|\mathbf{e}_i\| = \sum\limits_{j=1}^{p} |e_{ij}| = \sum\limits_{j=1}^{p} \left|x_{ij} - x_{(i+1)j}\right|$ and p is the dimension of the chaotic system.

Theorem 1: For bidirectional N-coupled fractional-order time delayed chaotic systems with ring connection as presented in Eq. (16), the synchronisation can be achieved with any $d > 0$.

Proof: According to Definition 3, we define the Lyapunov function as

$$V(t) = \sum_{i=1}^{N} \|\mathbf{e}_i\|. \tag{17}$$

Obviously, it is also the synchronisation error of the network. Then

$$\begin{aligned} D_{t_0}^q V(t) &= \sum_{i=1}^{N} D_{t_0}^q \|\mathbf{e}_i\| \\ &= \sum_{j=1}^{p} D_{t_0}^q |e_{1j}| + \sum_{j=1}^{p} D_{t_0}^q |e_{2j}| + \cdots + \sum_{j=1}^{p} D_{t_0}^q |e_{Nj}| \end{aligned}. \tag{18}$$

According to **Lemma 1**, we have

$$\begin{aligned} D_{t_0}^q V(t) = &\sum_{j=1}^{p} \mathrm{sgn}(e_{1j})[D_{t_0}^q x_{1j} - D_{t_0}^q x_{2j}] + \sum_{j=1}^{p} \mathrm{sgn}(e_{2j})[D_{t_0}^q x_{2j} - D_{t_0}^q x_{3j}] \\ &+ \cdots + \sum_{j=1}^{p} \mathrm{sgn}(e_{Nj})[D_{t_0}^q x_{Nj} - D_{t_0}^q x_{1j}] \\ \leq &\sum_{j=1}^{p} [D_{t_0}^q x_{1j} - D_{t_0}^q x_{2j}] + \sum_{j=1}^{p} [D_{t_0}^q x_{2j} - D_{t_0}^q x_{3j}] \\ &+ \cdots + \sum_{j=1}^{p} [D_{t_0}^q x_{Nj} - D_{t_0}^q x_{1j}] \\ = &\ 0 \end{aligned}. \tag{19}$$

When $D_{t_0}^q V(t) = 0$, it means $e_{ij} \geq 0$ for $i = 1, 2, ..., N$ and $j = 1, 2, ..., p$. So we have $D_{t_0}^q x_{1j} \geq D_{t_0}^q x_{2j}$, $D_{t_0}^q x_{2j} \geq D_{t_0}^q x_{3j}$, ..., $D_{t_0}^q x_{(N-1)j} \geq D_{t_0}^q x_{Nj}$ and $D_{t_0}^q x_{Nj} \geq D_{t_0}^q x_{1j}$. Thus only when $D_{t_0}^q x_{1j} = D_{t_0}^q x_{2j} = \cdots = D_{t_0}^q x_{Nj}$ for $j = 1, 2, ..., p$, $D_{t_0}^q V(t) = 0$, otherwise, $D_{t_0}^q V(t) < 0$. That is to say, $V(t) \to 0$ as $t \to \infty$. The coupled network is synchronised.

Remark 1: The network can be synchronised for any number of nodes ($N \geq 3$).

Remark 2: According to Theorem 1, the network can be synchronised for any $d > 0$. However, in practice, it is difficult to obtain synchronisation when d is too small. In our testing, the suggested value of d should be larger than 1.

3 Synchronisation simulation of the time delayed simplified Lorenz system with ring connection

In this section, synchronisation of N fractional-order time delayed simplified Lorenz systems with ring connection is investigated.

3.1 Model and numerical solution of the network

The fractional-order simplified Lorenz system is investigated by [5,6]. It is described as

$$\begin{cases} D^q_{t_0} x_1 = 10(x_2 - x_1) \\ D^q_{t_0} x_2 = (24 - 4c)x_1 - x_1 x_3 + c x_2\,, \\ D^q_{t_0} x_3 = x_1 x_2 - 8x_3/3 \end{cases} \tag{20}$$

where x_1, x_2, x_3 are state variables, and c is the system parameter. In order to control the fractional-order simplified Lorenz system, we need to design the state feedback controller $u = (u_1, u_2, u_3)$ [29], then Eq. (20) becomes

$$\begin{cases} D^q_{t_0} x_1 = 10(x_2 - x_1) + u_1 \\ D^q_{t_0} x_2 = (24 - 4c)x_1 - x_1 x_3 + c x_2 + u_2\,. \\ D^q_{t_0} x_3 = x_1 x_2 - 8x_3/3 + u_3 \end{cases} \tag{21}$$

Here, if we set the controllers $u_1 = \kappa x_1(t - \tau)$, $u_2 = 0$, $u_3 = 0$, the following delayed control system is defined as

$$\begin{cases} D^q_{t_0} x_1 = 10(x_2 - x_1) + \kappa x_1(t - \tau) \\ D^q_{t_0} x_2 = (24 - 4c)x_1 - x_1 x_3 + c x_2\,, \\ D^q_{t_0} x_3 = x_1 x_2 - 8x_3/3 \end{cases} \tag{22}$$

where κ is the control parameter and τ is the time delay. Here, $D^q_{t_0}$ is the Caputo differential operator of order $q \in (0, 1]$. The fractional-order time delayed simplified Lorenz system with bidirectional coupling is described by

$$\begin{cases} D^q_{t_0} x_{i1} = 10(x_{i2} - x_{i1}) + \kappa x_{i1}(t - \tau) + d(x_{(i-1)1} + x_{(i+1)1} - 2x_{i1}) \\ D^q_{t_0} x_{i2} = (24 - 4c)x_{i1} - x_{i1} x_{i3} + c x_{i2} + d(x_{(i-1)2} + x_{(i+1)2} - 2x_{i2})\,. \\ D^q_{t_0} x_{i3} = x_{i1} x_{i2} - 8x_{i3}/3 + d(x_{(i-1)3} + x_{(i+1)3} - 2x_{i3}) \end{cases} \tag{23}$$

Next, we present the numerical solution of the network based on the ADM. Consider an uniform grid $\{t_n = nh,\ n = -m,\ -(m\text{-}1),\ ...,\ -1,\ 0,\ 1,\ 2,\ ...,\ N\}$, where $m = \lceil \tau/h \rceil$ and N is the length of the time series. By applying ADM, the solution of system (23) is represented as

$$\begin{cases} x_{i1}(n+1) = \sum_{j=0}^{5} K^j_{i1} h^{jq} / \Gamma(jq+1) \\ x_{i2}(n+1) = \sum_{j=0}^{5} K^j_{i2} h^{jq} / \Gamma(jq+1) \\ x_{i3}(n+1) = \sum_{j=0}^{5} K^j_{i3} h^{jq} / \Gamma(jq+1) \end{cases}. \tag{24}$$

The intermediate variables are calculated by

$$K^0_{i1} = x_{i1}(n), \quad K^0_{i2} = x_{i2}(n), \quad K^0_{i3} = x_{i3}(n), \tag{25}$$

$$\begin{cases} K^1_{i1} = 10(K^0_{i2} - K^0_{i1}) + \kappa K^0_{i1\tau} + d(K^0_{(i-1)1} + K^0_{(i+1)1} - 2K^0_{i1}) \\ K^1_{i2} = (24 - 4c)K^0_{i1} - K^0_{i1} K^0_{i3} + c K^0_{i2} \\ \qquad + d(K^0_{(i-1)2} + K^0_{(i+1)2} - 2K^0_{i2}) \\ K^1_{i3} = K^0_{i1} K^0_{i2} - 8K^0_{i3}/3 + d(K^0_{(i-1)3} + K^0_{(i+1)3} - 2K^0_{i3}) \end{cases}, \tag{26}$$

$$\begin{cases} K_{i1}^2 = 10(K_{i2}^1 - K_{i1}^1) + \kappa K_{i1\tau}^1 + d(K_{(i-1)1}^1 + K_{(i+1)1}^1 - 2K_{i1}^1) \\ K_{i2}^2 = (24-4c)K_{i1}^1 - K_{i1}^0 K_{i3}^1 - K_{i1}^1 K_{i3}^0 + cK_{i2}^1 \\ \qquad +d(K_{(i-1)2}^1 + K_{(i+1)2}^1 - 2K_{i2}^1) \\ K_{i3}^2 = K_{i1}^1 K_{i2}^0 + K_{i1}^0 K_{i2}^1 - 8K_{i3}^1/3 + d(K_{(i-1)3}^1 + K_{(i+1)3}^1 - 2K_{i3}^1) \end{cases}, \tag{27}$$

$$\begin{cases} K_{i1}^3 = 10(K_{i2}^0 - K_{i1}^0) + \kappa K_{i1\tau}^2 + d(K_{(i-1)1}^2 + K_{(i+1)1}^2 - 2K_{i1}^2) \\ K_{i2}^3 = (24-4c)K_{i1}^2 - K_{i1}^0 K_{i3}^2 - K_{i1}^1 K_{i3}^1 \frac{\Gamma(2q+1)}{\Gamma^2(q+1)} - K_{i1}^2 K_{i3}^0 \\ \qquad +cK_{i2}^2 + d(K_{(i-1)2}^2 + K_{(i+1)2}^2 - 2K_{i2}^2) \\ K_{i3}^3 = K_{i1}^0 K_{i2}^2 + K_{i1}^1 K_{i2}^1 \frac{\Gamma(2q+1)}{\Gamma^2(q+1)} + K_{i1}^2 K_{i2}^0 - 8K_{i3}^2/3 \\ \qquad +d(K_{(i-1)3}^2 + K_{(i+1)3}^2 - 2K_{i3}^2) \end{cases}, \tag{28}$$

$$\begin{cases} K_{i1}^4 = 10(K_{i2}^3 - K_{i1}^3) + \kappa K_{i1\tau}^3 + d(K_{(i-1)1}^3 + K_{(i+1)1}^3 - 2K_{i1}^3) \\ K_{i2}^4 = (24-4c)K_{i1}^3 - K_{i1}^0 K_{i3}^3 - (K_{i1}^2 K_{i3}^1 + K_{i1}^1 K_{i3}^2)\frac{\Gamma(3q+1)}{\Gamma(q+1)\Gamma(2q+1)} \\ \qquad -K_{i1}^3 K_{i3}^0 + cK_{i2}^3 + d(K_{(i-1)2}^3 + K_{(i+1)2}^3 - 2K_{i2}^3) \\ K_{i3}^4 = K_{i1}^0 K_{i2}^3 + (K_{i1}^2 K_{i2}^1 + K_{i1}^1 K_{i2}^2)\frac{\Gamma(3q+1)}{\Gamma(q+1)\Gamma(2q+1)} + K_{i1}^3 K_{i2}^0 \\ \qquad +8K_{i3}^3/3 + d(K_{(i-1)3}^3 + K_{(i+1)3}^3 - 2K_{i3}^3) \end{cases}, \tag{29}$$

$$\begin{cases} K_{i1}^5 = 10(K_{i2}^4 - K_{i1}^4) + \kappa K_{i1\tau}^4 + d(K_{(i-1)1}^4 + K_{(i+1)1}^4 - 2K_{i1}^4) \\ K_{i2}^5 = (24-4c)K_{i1}^4 - K_{i1}^0 K_{i3}^4 - (K_{i1}^3 K_{i3}^1 + K_{i1}^1 K_{i3}^3)\frac{\Gamma(4q+1)}{\Gamma(q+1)\Gamma(3q+1)} \\ \qquad -K_{i1}^2 K_{i3}^2 \frac{\Gamma(4q+1)}{\Gamma^2(2q+1)} - K_{i1}^4 K_{i3}^0 + cK_{i2}^4 \\ \qquad +d(K_{(i-1)2}^4 + K_{(i+1)2}^4 - 2K_{i2}^4) \\ K_{i3}^5 = K_{i1}^0 K_{i2}^4 + (K_{i1}^3 K_{i2}^1 + K_{i1}^1 K_{i2}^3)\frac{\Gamma(4q+1)}{\Gamma(q+1)\Gamma(3q+1)} \\ \qquad +K_{i1}^2 K_{i2}^2 \frac{\Gamma(4q+1)}{\Gamma^2(2q+1)} + K_{i1}^4 K_{i2}^0 - 8K_{i3}^4/3 \\ \qquad +d(K_{(i-1)3}^4 + K_{(i+1)3}^4 - 2K_{i3}^4) \end{cases}, \tag{30}$$

where $K_{i1\tau}^j = K_{i1}^j(t-\tau)$. Obviously, when $t_n \le mh$, $t_{n-\tau}$ is located between $(n-1-m)h$ and $(n-m)h$. When $t_n > mh$, $t_{n-\tau}$ is located between $(n-m)h$ and $(n-m+1)h$. By applying linear interpolation, $K_{i1\tau}^j$ is calculated by

$$K_{i1\tau}^j(t_n) = \left(1 - m + \frac{\tau}{h}\right) K_{i1}^j(t_{n-1-m}) + \left(m - \frac{\tau}{h}\right) K_{i1}^j(t_{n-m}), \tag{31}$$

for $t_n \le mh$, and it is calculated by

$$K_{i1\tau}^j(t_n) = \left(1 - m + \frac{\tau}{h}\right) K_{i1}^j(t_{n-m}) + \left(m - \frac{\tau}{h}\right) K_{i1}^j(t_{n-m+1}), \tag{32}$$

for $t_n > mh$.

According to Eqs. (24)–(32), numerical simulation can be carried out. It also shows that ADM provides a good scheme for digital circuit implementation of fractional-order chaotic system synchronisation.

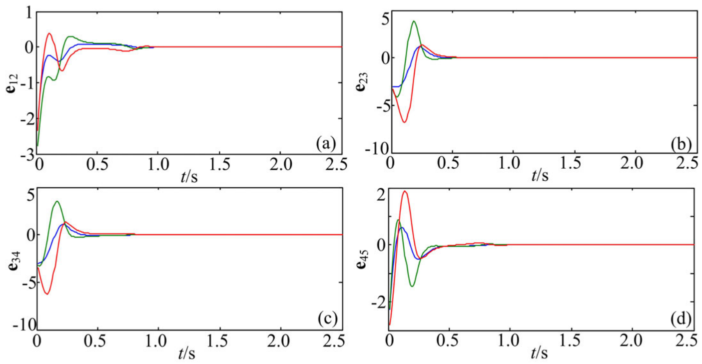

Fig. 2. Synchronisation error curves. (a) $\mathbf{e}_{12}$, (b) $\mathbf{e}_{23}$, (c) $\mathbf{e}_{34}$, (d) $\mathbf{e}_{45}$.

3.2 Performance analysis of the network synchronisation

Here, we study the numerical simulation of the network with $c = -1$, $\tau = 1$, $\kappa = 1$, $q = 0.98$, $d = 5$, where the network contains five systems. The configuration matrix of the network is obtained by

$$G = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{33}$$

The initial values for systems one to five are (1, 2, 3), (4, 5, 6), (7, 8, 9,), (10, 11, 12) and (13, 14, 15), respectively. When $t \leq 0$, $\mathbf{H}_1(t) = 1$, $\mathbf{H}_2(t) = 2$, $\mathbf{H}_3(t) = 3$, $\mathbf{H}_4(t) = 4$ and $\mathbf{H}_5(t) = 5$. Let $d = 5$; the synchronisation results between different systems are shown in Fig. 2. Here the synchronisation error $\mathbf{e}_{ij} = \mathbf{x}_i - \mathbf{x}_j$. It is defined as the error of different state variables between system (i) and system (j). According to Fig. 2, synchronisation is realized between different systems, which means the network is synchronised.

According to the synchronisation scheme shown in Fig. 1, $\mathbf{S}_i$ $(i = 1, 2, ..., N)$ are sent to the target systems to make the network synchronised. In this paper, $\mathbf{S}_i$ contains K_{il}^k (k=0, 1, 2, 3, 4, 5, l=1, 2, 3). Taking Sys_1 as an example, phase diagrams of these intermediate variables are plotted in Fig. 3. It shows that these signals are also chaotic. We define the synchronisation error between different intermediate variables as

$$E_k = \sum_{i=1}^{4}\sum_{j=1}^{3}\left|K_{ij}^k - K_{(i+1)j}^k\right|, \tag{34}$$

where k=0, 1, 2, 3, 4 and 5. When k=0, E_0 is the error of the state variables. The simulation results are illustrated in Fig. 4. Synchronisation is also achieved in these intermediate variables, which means they can also be used in the real application.

The synchronisation setup time with different q, τ and d are obtained as shown in Fig. 5. It indicates in Fig. 5(a) that the synchronisation setup time increases with order q, which means the fractional-order time delayed chaotic systems are more suitable in the practice than integer-order ones. According to Fig. 5(b), the synchronisation setup time decreases with d. Thus we should choose a relatively large coupling

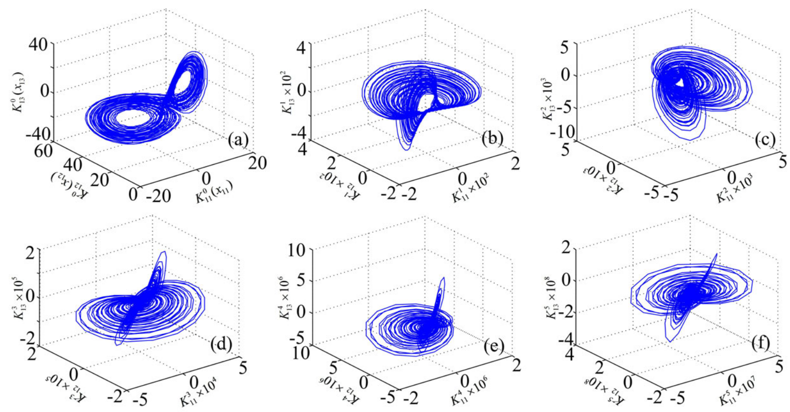

Fig. 3. Phase diagrams of intermediate variables in system one. (a)$K_{11}^0 - K_{12}^0 - K_{13}^0$, (b) $K_{11}^1 - K_{12}^1 - K_{13}^1$, (c) $K_{11}^2 - K_{12}^2 - K_{13}^2$, (d) $K_{11}^3 - K_{12}^3 - K_{13}^3$, (e) $K_{11}^4 - K_{12}^4 - K_{13}^4$, (f) $K_{11}^5 - K_{12}^5 - K_{13}^5$.

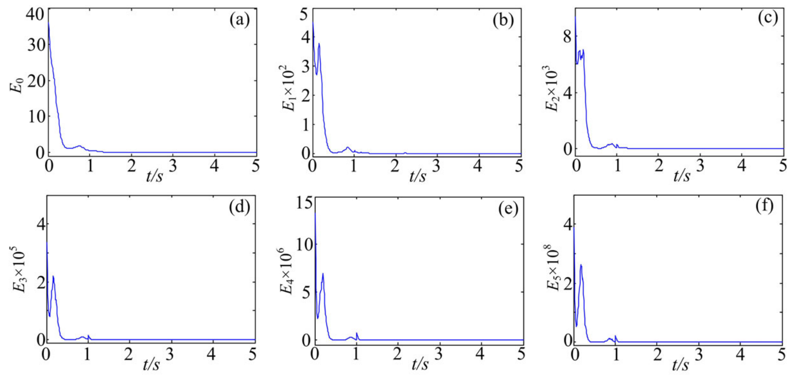

Fig. 4. Synchronisation results of intermediate variables. (a) E_0, (b) E_1, (c) E_2, (d) E_3, (e) E_4, (f) E_5.

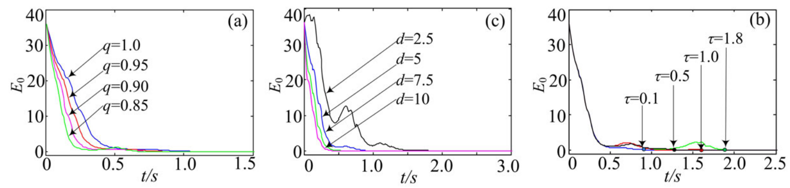

Fig. 5. Synchronisation setup time with different parameter. (a) q, (b) d, (c) τ.

strength d for practical application. Figure 5(c) also shows that the larger the time delay τ is, the larger the synchronisation setup time is.

4 Conclusion

In this paper, synchronisation of the fractional-order time delayed simplified Lorenz systems with ring connection is investigated based on the Adomian decomposition method (ADM). Both theoretical analysis and numerical simulation illustrate that the N-coupled fractional-order time delayed systems can be well synchronised. We also found that the synchronisation setup time decreases with the increase of control parameter d and increases with the increase of derivative order q and time delay τ. Synchronisation is also found in the intermediate variables of different systems. Our further work will focus on FPGA circuit design and application of the network.

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