Eur. Phys. J. Special Topics **225**, 187–196 (2016)

DOI: 10.1140/epjst/e2016-02608-3

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

On $\Lambda - \phi$ generalized synchronization of chaotic dynamical systems in continuous–time

A. Ouannas[1] and M.M. Al-sawalha[2,a]

[1] Department of Mathematics and Computer Science, Tebessa University, Tebessa, Algeria
[2] Mathematics Department, Faculty of Science, University of Hail, Hail, Kingdom of Saudi Arabia

Received 1 August 2015 / Received in final form 11 January 2016
Published online 29 February 2016

Abstract. In this paper, a new type of chaos synchronization in continuous-time is proposed by combining inverse matrix projective synchronization (IMPS) and generalized synchronization (GS). This new chaos synchronization type allows us to study synchronization between different dimensional continuous-time chaotic systems in different dimensions. Based on stability property of integer-order linear continuous-time dynamical systems and Lyapunov stability theory, effective control schemes are introduced and new synchronization criterions are derived. Numerical simulations are used to validate the theoretical results and to verify the effectiveness of the proposed schemes.

1 Introduction

The nature of our physical world is most commonly described by systems of nonlinear equations. These nonlinear models of real-life problems generally exhibit chaotic behaviors which is possess some special features, such as having bounded trajectories with a positive leading positive Lyapunov exponent of the dynamics of the chaotic system, extreme sensitivity to initial conditions and having noise-like behaviors. It can be applied in the vast areas of in secure communications, chemical reactions, biomedical science, social science, and many other fields. The idea of chaos synchronization is to use the output of the master system to control the slave system so that the output of the response system follows the output of the master system asymptotically. Recently there has been growing interest in the investigation of various kinds of synchronization in chaotic or hyperchaotic systems. This interest is spurred by the possible applications of synchronous chaos particularly in secure communications [1–7]. Recently, the topic of synchronization between different dimensional chaotic systems attract more and more attentions. Until now, a variety of control schemes have been proposed to study the problem of synchronization between chaotic dynamical systems with different dimensions such as modified function projective synchronization [8], generalized matrix projective synchronization [9], generalized synchronization

[a] e-mail: sawalha_moh@yahoo.com

[10–12], inverse generalized synchronization [13], full state hybrid projective synchronization [14], Q-S synchronization [15], increased order synchronization [16,17] and reduced order generalized synchronization [18,19]. Among the aforementioned methods, the most effective synchronization approaches are inverse matrix projective synchronization (IMPS) and generalized synchronization (GS) and have been used widely to achieve the chaos synchronization with different dimensions [20–24]. However in (IMPS), the slave system synchronize with the master system up to scaling constant matrix. on the other hand in (GS) there exist a functional relationship between the states of the master and the slave chaotic systems. However when we combine (IMPS) and (GS), a new generalized-type of chaos synchronization, called $\Lambda - \phi$ generalized synchronization, will appear. To the best of our knowledge most of theoretical results about synchronization of chaos focus on the systems whose models are identical or strictly different systems and systems of different order, especially the systems in biological science and social science. One example is the synchronization that occurs between heart and lung, where one can observe that both circulatory and respiratory systems behave in synchronous way, but their models are essentially different and they have different order. So, the study of synchronization for strictly different dynamical systems and different order dynamical systems is both very important from the perspective of control theory and very necessary from the perspective of practical application.

In this paper, a new type of synchronization for different dimensional chaotic dynamical system is proposed and experimented. The sufficient conditions for achieving $\Lambda - \phi$ generalized synchronization of two chaotic systems are derived based upon the stability theory of linear systems and Lyapunov stability theorem. This paper provides further contribution to the topic of $\Lambda - \phi$ generalized synchronization. This paper introduces a general control schemes with different structures that can be applied to wide classes of chaotic and hyperchaotic systems. The proposed control method is simple, efficient and easy to implement in practical applications. The rest of the present paper is organized as follows. In Sect. 2 the definition of $\Lambda - \phi$ generalized synchronization is introduced. In Sect. 3 different schemes for $\Lambda - \phi$ generalized synchronization are proposed and new synchronization criterion are presented. In Sect. 4 the derived criterions and the proposed schemes are applied to some typical different dimensional chaotic systems. Finally, the paper is concluded in Sect. 5.

2 $\Lambda - \phi$ generalized synchronization

Consider the following coupled chaotic systems

$$\dot{X}(t) = F(X(t)), \tag{1}$$

$$\dot{Y}(t) = G(Y(t)) + U, \tag{2}$$

where $X(t) \in \mathbf{R}^n$, $Y(t) \in \mathbf{R}^m$ are the states of the master system (1) and the slave system (2), respectively, $F : \mathbf{R}^n \to \mathbf{R}^n$, $G : \mathbf{R}^m \to \mathbf{R}^m$ and $U \in \mathbf{R}^m$ is a controller.

Before proceeding to the definition of $\Lambda - \phi$ generalized synchronization for the coupled chaotic systems (1) and (2), the definitions of inverse matrix projective synchronization (IMPS) and generalized synchronization (GS) are provided.

Definition 1. *The n-dimensional master system $X(t)$ and the m-dimensional slave system $Y(t)$ are said to be inverse matrix projective synchronization, if there exists* *a controller* $U = (u_i)_{1\leq i\leq m}$ *and a matrix* $\Lambda \in \mathbf{R}^{n\times m}$*, such that the synchronization error*

$$e(t) = \Lambda Y(t) - X(t), \tag{3}$$

satisfies that $\lim_{t\to+\infty} \|e(t)\| = 0$.

Definition 2. *The* n*-dimensional master system* $X(t)$ *and the* m*-dimensional slave system* $Y(t)$ *are said to be* $\Lambda-\phi$ *generalized synchronization, if there exists a controller* $U = (u_i)_{1\leq i\leq m}$ *and a differentiable vector function* $\phi : \mathbf{R}^n \to \mathbf{R}^m$*, respectively, such that the synchronization error*

$$e(t) = Y(t) - \phi\left(X(t)\right), \tag{4}$$

satisfies that $\lim_{t\to+\infty} \|e(t)\| = 0$.

Remark 1. Generalized synchronization and inverse matrix projective synchronization of chaotic dynamical systems with different dimensions, based on Lyapunov stability theory, have been studied and carried out, for example, see Refs. [10, 20].

Definition 3. *The* n*-dimensional master system* $X(t)$ *and the* m*-dimensional slave system* $Y(t)$ *are said to be* $\Lambda-\phi$ *generalized synchronization, if there exists a controller* $U = (u_i)_{1\leq i\leq m}$*, a constant matrix* Λ*,* $d \times m$*, and a differentiable vector function* $\phi : \mathbf{R}^n \to \mathbf{R}^d$*, respectively, such that the synchronization error*

$$e(t) = \Lambda Y(t) - \phi\left(X(t)\right), \tag{5}$$

satisfies that $\lim_{t\to+\infty} \|e(t)\| = 0$*. Where* d *is called the synchronization dimension.*

Remark 2. When $(\Lambda, \phi(.)) = (I, X(t))$, $(\Lambda, \phi(.)) = (I, -X(t))$, $(\Lambda, \phi(.)) = (\Lambda, X(t))$ and $(\Lambda, \phi(.)) = (I, \phi(X(t)))$ complete synchronization, anti- synchronization, inverse matrix projective synchronization and generalized synchronization will appear, respectively.

3 Different schemes for $\Lambda - \phi$ generalized synchronization

The aim of this section is to address two different schemes of $\Lambda - \phi$ generalized synchronization. We take two kinds of cases: (I) when the synchronization dimension $d = n$ and (II) when the synchronization dimension $d = m$ into consideration.

3.1 Case I: $d = n$

In this case, we assume that the synchronization dimension $d = n$, where $n < m$. Here, we assume that the master and the slave chaotic systems can be considered in the following forms

$$\dot{X}(t) = AX(t) + f_1(X(t)), \tag{6}$$
$$\dot{Y}(t) = g_1(Y(t)) + U, \tag{7}$$

where $X(t) \in \mathbf{R}^n$, $Y(t) \in \mathbf{R}^m$ are the states of the master system (6) and the slave system (7), respectively, A is a $n \times n$ constant matrix, $f_1 : \mathbf{R}^n \to \mathbf{R}^n$ is the nonlinear part of the system (6), $g_1 : \mathbf{R}^m \to \mathbf{R}^m$ and $U = (u_i)_{1\leq i\leq m}$ is a vector controller.

The error system, according to the definition of $\Lambda - \phi$ generalized synchronization, between the master system (6) and the slave system (7) is defined by

$$e(t) = \Lambda Y(t) - \phi(X(t)), \tag{8}$$

where $\phi : \mathbf{R}^n \to \mathbf{R}^n$ is a continuously differentiable function and

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \cdots & \Lambda_{1n} & \cdots & \Lambda_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \Lambda_{n1} & \cdots & \Lambda_{nn} & \cdots & \Lambda_{nm} \end{pmatrix}, \tag{9}$$

is the constant scaling matrix. The error system (8) can be written as

$$\dot{e}(t) = (A - C_1)\, e(t) + \Lambda U + R_1, \tag{10}$$

where $C_1 \in \mathbf{R}^{n\times n}$ is a control matrix to be designed later and

$$R_1 = (C_1 - A)\, e(t) + \Lambda g_1(Y(t)) - \mathbf{D}\phi(X(t)) \times (AX(t) + f_1(X(t)))\,, \tag{11}$$

where $\mathbf{D}\phi(X(t))$ is the Jacobian matrix of the function $\phi : \mathbf{R}^n \to \mathbf{R}^n$, given by

$$\mathbf{D}\phi(X(t)) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{pmatrix}. \tag{12}$$

Theorem 1. $\Lambda - \phi$ *generalized synchronization between the master system (6) and the slave system (7) will occur if the following conditions are satisfied*

(i) The vector controller is constructed as

$$(u_1, ..., u_n)^T = -\tilde{\Lambda}^{-1} R_1, \tag{13}$$

$$u_i = 0, \text{for} : i = n + 1, ..., m, \tag{14}$$

where $\tilde{\Lambda}^{-1}$ *is the inverse of* $\tilde{\Lambda} = \begin{pmatrix} \Lambda_{11} & \cdots & \Lambda_{1n} \\ \vdots & \ddots & \vdots \\ \Lambda_{n1} & \cdots & \Lambda_{nn} \end{pmatrix}$

(ii) All eigenvalues of $A - C_1$ *have negative real part.*

Proof. Using Eq. (14), then the error system (10) can be described as follow:

$$\dot{e}(t) = (A - C_1)\, e(k) + \tilde{\Lambda}\tilde{U} + R_1, \tag{15}$$

where $\tilde{U} = (u_1, \ldots, u_n)^T$. Now, applying the control law described by Eq. (13) to Eq. (15) yields the resulting error dynamics as follows:

$$\dot{e}(t) = (A - C_1)\, e(k). \tag{16}$$

According to the stability theory of the linear systems, if all eigenvalues of $A - C_1$ have negative real parts, This choice will lead to the error states Eq. (16) converge to zero as time t tends to infinity. Therefore, the systems (6) and (7) are globally $\Lambda - \phi$ generalized synchronized in $n-$D.

3.2 Case II: $d = m$

Define the master and the slave systems as follows

$$\dot{X}(t) = f_2(X(t)), \tag{17}$$

and

$$\dot{Y}(t) = BY(t) + g_2(Y(t)) + U, \tag{18}$$

where $X(t) \in \mathbf{R}^n$, $Y(t) \in \mathbf{R}^m$ are the state of the master system (17) and the slave system (18), $f_2 : \mathbf{R}^n \to \mathbf{R}^n$, $B \in \mathbf{R}^{m\times m}$, $g_2 : \mathbf{R}^m \to \mathbf{R}^m$ is a nonlinear function and $U \in \mathbf{R}^m$ is the control law. In this case, the error system, is given by

$$e(t) = \Lambda Y(t) - \phi(X(t)), \tag{19}$$

where $\Lambda \in \mathbf{R}^{m\times m}$ is the scaling matrix and $\phi : \mathbf{R}^n \to \mathbf{R}^m$ is a continuously differentiable function. The error system (19) can be written as

$$\dot{e}(t) = (B - C_2)\, e(t) + \Lambda U + R_2, \tag{20}$$

where

$$R_2 = (C_2 - B)\, e(t) + \Lambda BY(t) + \Lambda g_2(Y(t)) - \mathbf{D}\phi(X(t)) \times f_2(X(t)), \tag{21}$$

and C_2 is an, $m \times m$, unknown control matrix to be determined.

Theorem 2. *$\Lambda - \phi$ generalized synchronization between the master system (17) and the slave system (18) will occur if the following conditions are satisfied*
(I) $U = -\Lambda^{-1}R_2$, where Λ^{-1} is the inverse of the matrix Λ.
(II) $(B - C_2)^T + (B - C_2)$ is a negative definite matrix.

Proof. By using condition (I), the error system (20) can be described as

$$\dot{e}(t) = (B - C_2)\, e(t). \tag{22}$$

Construct the candidate Lyapunov function in the form: $V(e(t)) = e^T(t)e(t)$, then we obtain

$$\begin{aligned} \dot{V}(e(t)) &= \dot{e}^T(t)e(t) + e^T(t)\dot{e}(t) \\ &= e^T(t)(B - C_2)^T e(t) + e^T(t)(B - C_2)e(t) \\ &= e^T(t)\left[(B - C_2)^T + (B - C_2)\right] e(t), \end{aligned}$$

and by using condition (II), we get $\dot{V}(e(t)) < 0$. Thus, from the Lyapunov stability theory, it is immediate that $\lim_{t\to\infty} e_i(t) = 0$, $1 \le i \le m$. Therefore, the systems (17) and (18) are globally $\Lambda - \phi$ generalized synchronized in $m-$D.

4 Numerical examples

In this section, we will consider two examples to illustrate the effectiveness of the synchronization results which proposed in the previous section.

4.1 Example 1: $\Lambda - \phi$ generalized synchronization of Lorenz system and hyperchaotic Cai system in 3-D

Assume that Lorenz system drives the hyperchaotic Cai system. Therefore, we define the master system as follows

$$\begin{aligned}\dot{x}_1 &= \alpha(x_2 - x_1),\\ \dot{x}_2 &= \gamma x_1 - x_2 - x_1 x_3,\\ \dot{x}_3 &= -\beta x_3 + x_1 x_2,\end{aligned} \tag{23}$$

where x_1, x_2 and x_3 are state variables and α, γ and β are positive parameters. Bifurcation studies show that with the parameters $\alpha = 10, \gamma = 28$, system (23) exhibits chaotic behavior when $\beta = \frac{8}{3}$ [25]. The controlled Hyperchaotic Cai system can be described as

$$\begin{aligned}\dot{y}_1 &= a(y_2 - y_1) + u_1,\\ \dot{y}_2 &= by_1 + cy_2 - y_1 y_3 + u_2,\\ \dot{y}_3 &= y_2^2 - dy_4 + u_3,\\ \dot{y}_4 &= -hy_1 + u_4,\end{aligned} \tag{24}$$

where y_1, y_2, y_3 and y_4 are state variables, a, b, c, d and h are positive parameters and $(u_1, u_2, u_3, u_4)^T$ is the vector controller. Hyperchaotic Cai system (i.e., the system (24) with $u_1 = u_2 = u_3 = u_4 = 0$) exhibits hyperchaotic behavior when $a = 27.5, b = 3, c = 19.3, d = 2.9$ and $h = 3.3$ [26]. The linear part A and the nonlinear part f_1 of the Lorenz system (23) are given by

$$A = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}, \quad f_1(x_1, x_2, x_3) = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}.$$

In this case, the scaling matrix $\Lambda = (\Lambda_{ij})_{3\times 4}$ and the function $\phi : \mathbf{R}^3 \to \mathbf{R}^3$ are selected as

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \end{pmatrix}, \quad \phi(x_1, x_2, x_3) = \begin{pmatrix} x_1^2 \\ x_1^2 + x_2^2 \\ x_1^2 + x_2^2 + x_3^2 \end{pmatrix}.$$

Hence, the error system between the master system (23) and the slave system (24) is defined by

$$\begin{aligned}e_1 &= y_1 + y_4 - x_1^2,\\ e_2 &= 2y_2 + y_4 - x_1^2 - x_2^2,\\ e_3 &= 3y_3 + 2y_4 - x_1^2 - x_2^2 - x_3^2.\end{aligned} \tag{25}$$

According to our approach presented in Sect. 3.1, if we choose the control matrix C_1 as

$$C_1 = \begin{pmatrix} 0 & 10 & 0 \\ 28 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

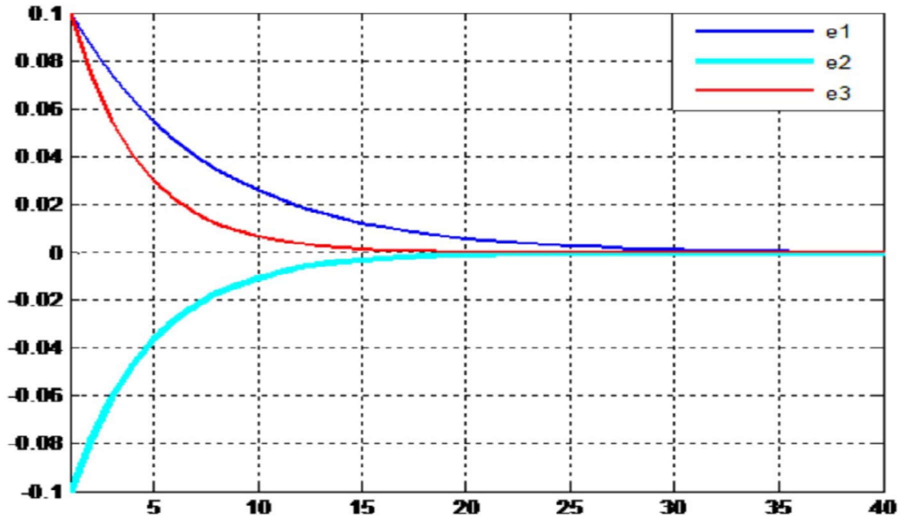

Fig. 1. $\Lambda - \phi$ generalized synchronization errors, e_1, e_2, e_3, of the slave system (24) and the drive system (23) with time t.

then the controllers u_i, $i = 1, 2, 3, 4$ can be constructed as follow:

$$\begin{aligned} u_1 &= 10e_1 + (a+h)\, y_1 - ay_2 + 2\alpha x_1(x_2 - x_1), \\ u_2 &= \frac{1}{2}e_2 + \left(\frac{1}{2}h - b\right) y_1 - cy_2 - y_1y_3 + \alpha x_1(x_2 - x_1) + x_2\left(\gamma x_1 - x_2 - x_1x_3\right), \\ u_3 &= \frac{8}{9}e_3 - y_2^2 + dy_4 + \frac{2}{3}hy_1 - \frac{2}{3}\alpha x_1^2 - \frac{2}{3}x_2^2 - \frac{2}{3}\beta x_3^2 + \frac{2}{3}\left(\alpha + \gamma\right) x_1x_2, \\ u_4 &= 0. \end{aligned} \tag{26}$$

It is easy to know that all eigenvalues of $A - C_1$ have negative real part. Then the conditions of Theorem 1 are satisfied. Therefore, the systems (23) and (24) are globally $\Lambda - \phi$ generalized synchronized in 3-D and the error system can be described as follow:

$$\begin{aligned} \dot{e}_1 &= -10e_1, \\ \dot{e}_2 &= -e_2, \\ \dot{e}_3 &= -\frac{8}{3}e_3. \end{aligned} \tag{27}$$

The evolution of $\Lambda - \phi$ generalized synchronization errors is shown in Fig. 1.

4.2 Example 2: $\Lambda - \phi$ generalized synchronization of Rössler system and hyperchaotic Liu system in 4D

Now, we consider Rössler system as the master system and hyperchaotic Liu system as the slave system. The Rössler system can be described as

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + \alpha x_2, \\ \dot{x}_3 &= \gamma + x_3(x_1 - \beta), \end{aligned} \tag{28}$$

here x_1, x_2 and x_3 are state variables and α, γ and β are positive parameters. Bifurcation studies show that with the parameters $\alpha = 0.2, \gamma = 0.2$, system (28) exhibits chaotic behavior when $\beta = 5.7$ [28]. The controlled hyperchaotic Liu system can be described as

$$\begin{aligned}\dot{y}_1 &= a(y_2 - y_1) + u_1,\\ \dot{y}_2 &= by_1 + y_1y_3 - y_4 + u_2,\\ \dot{y}_3 &= -cy_3 - y_1y_2 + y_4 + u_3,\\ \dot{y}_4 &= dy_1 + y_2 + u_4,\end{aligned} \tag{29}$$

where y_1, y_2, y_3 and y_4 are state variables, a, b, c, d and h are positive parameters and $(u_1, u_2, u_3, u_4)^T$ is the vector controller. Hyperchaotic Liu system (i.e., the system (29) with $u_1 = u_2 = u_3 = u_4 = 0$) exhibits hyperchaotic behavior when $a = 10, b = 35, c = 1.4$ and $d = 5$ [27]. Here, the linear part B and the nonlinear part g_2 of the hyperchaotic Liu system are given by

$$B = \begin{pmatrix} -10 & 10 & 0 & 0\\ 35 & 0 & 0 & -1\\ 0 & 0 & -1.4 & 0\\ 5 & 1 & 0 & 0\end{pmatrix}, \quad g_2(y_1, y_2, y_3, y_4) = \begin{pmatrix} 0\\ y_1y_3\\ -y_1y_2\\ 0\end{pmatrix}.$$

In this case, the scaling matrix $\Lambda = (\Lambda_{ij})_{4\times4}$ and the function $\phi : \mathbf{R}^3 \to \mathbf{R}^4$ are chosen as

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\end{pmatrix}, \quad \phi(x_1, x_2, x_3) = \begin{pmatrix} x_1\\ x_1 + x_2 + x_3\\ x_2 + x_3\\ x_1\end{pmatrix}.$$

Then, the error system between the master system (28) and the slave system (29) is defined by

$$\begin{aligned}e_1 &= y_1 - x_1,\\ e_2 &= 2y_2 - x_1 - x_2 - x_3,\\ e_3 &= y_3 - x_2 - x_3,\\ e_4 &= 3y_4 - x_1.\end{aligned} \tag{30}$$

According to the control scheme proposed in Sect. 3.2, if we select the control matrix C_2 as

$$C_2 = \begin{pmatrix} 0 & 10 & 0 & 0\\ 35 & -1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 5 & 1 & 0 & -1\end{pmatrix},$$

then the controllers u_i, $i = 1, 2, 3, 4$, can be designed as follow:

$$\begin{aligned}u_1 &= -10e_1 - a(y_2 - y_1) - x_2 - x_3,\\ u_2 &= -\frac{1}{2}e_2 - by_1 - y_1y_3 + y_4 + \frac{1}{2}x_1 + \frac{1}{2}(\alpha - 1)x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_3(x_1 - \beta) + \frac{1}{2}\gamma,\\ u_3 &= -1.4e_3 + cy_3 + y_1y_2 - y_4 + x_1 + \alpha x_2 + x_3(x_1 - \beta) + \gamma,\\ u_4 &= -\frac{1}{3}e_4 - dy_1 - y_2 - \frac{1}{3}x_2 - \frac{1}{3}x_3.\end{aligned} \tag{31}$$

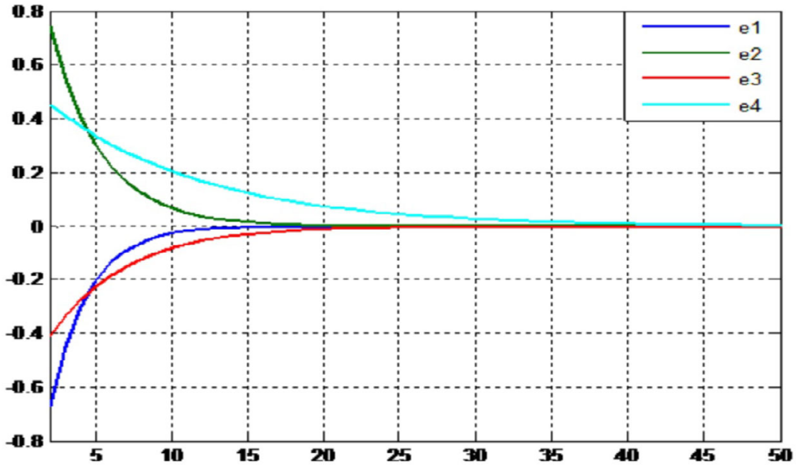

Fig. 2. $\Lambda - \phi$ generalized synchronization errors, e_1, e_2, e_3 and e_4, of the slave system (29) and the master system (28) with time t.

It is easy to show that $(B - C_2)^T + (B - C_2)$ is a negative definite matrix, and the conditions of Theorem 2 are satisfied. Therefore, the systems (28) and (29) are globally $\Lambda - \phi$ generalized synchronized in 4-D and the error system can be written as follow:

$$\begin{aligned} \dot{e}_1 &= -10e_1, \\ \dot{e}_2 &= -e_2, \\ \dot{e}_3 &= -1.4e_3, \\ \dot{e}_4 &= -e_4. \end{aligned} \tag{32}$$

The evolution $\Lambda - \phi$ generalized synchronization errors is shown in Fig. 2.

5 Conclusion

This paper has illustrated a new generalized type of synchronization, called $\Lambda - \phi$ generalized synchronization, between a master system of dimension n and a slave system of dimension m. Some new synchronization criterions were derived and proved theoretically using the stability of linear system and Lyapunov stability theory. Firstly, to observe $\Lambda - \phi$ generalized synchronization behavior with respect to dimension n, the synchronization scheme was proposed based on the control of the linear part of the master system. Secondly, to achieve $\Lambda - \phi$ generalized synchronization with respect to dimension m, the synchronization criterion was obtained via controlling the linear part of the slave system. Finally, numerical examples and simulations results were used to verify the effectiveness of the proposed schemes.

References

1. J. Warminski, Z. Szmit, J. Latalski, Eur. Phys. J. Special Topics **223**, 827 (2014)
2. A.L. Fradkov, B. Andrievsky, M. Ananyevskiy, Eur. Phys. J. Special Topics **223**, 773 (2014)

3. A.L. Fradkov, B. Andrievsky, M. Ananyevskiy, Eur. Phys. J. Special Topics **223**, 729 (2014)
4. L. Marcheggiani, R. Chacn, S. Lenci, Eur. Phys. J. Special Topics **223**, 757 (2014)
5. J. Alvarez, R. Cuesta, D. Rosas, Eur. Phys. J. Special Topics **223**, 687 (2014)
6. K. Wiesenfeld, Eur. Phys. J. Special Topics **223**, 665 (2014)
7. O. Moskalenko, N. Phrolov, A. Koronovskii, A. Hramov, Eur. Phys. J. Special Topics **222**, 729 (2013)
8. H.J. Liu, Z.L. Zhu, H. Yu, Q. Zhu, Discrete Dynam. Nat. Soc. **2013**, 7 (2013)
9. Z. Wu, X. Xu, G. Chen, X. Fu, J. Franklin I. **351**, 4584 (2014)
10. G. Zhang, Z. Liu, Z. Ma, Chaos, Solitons Fractals **32**, 773 (2007)
11. Y. Yu, H.X. Li, J. Yu, Mod. Phys. Lett. B **23**, 2593 (2009)
12. X. He, C. Li, J. Huang, L. Xiao, Optik **126**, 454 (2015)
13. A. Ouannas, Z. Odibat, Int. J. Appl. Comput. Math. (2015)
14. M. Hu, Z. Xu, R. Zhang, A. Hu, Phys. Lett. A, **365**, 315 (2007)
15. A.M.A. El-Sayed, A. Elsaid, H.M. Nour, A. Elsonbaty, J. Appl. Analy. Comp. **4**, 323 (2014)
16. S. Ogunjo, Int. J. Nonlinear Sci. **16**, 105 (2013)
17. M.M. Al-sawalha, M.S.M. Noorani, Chin. Phys. Lett. **28**, 11 (2011)
18. K.S. Ojo, S.T. Ogunjo, A.N. Njah, I.A. Fuwape, Pramana **84**, 1 (2014)
19. M.M. Al-sawalha, M.S.M. Noorani, Commun. Nonlinear. Sci. Numer. Simulat. **15** (2010)
20. A. Ouannas, E.E. Mahmoud, J. Comput. Intell. Electron. Syst. **3**, 188 (2014)
21. K.S. Ojo, A.N. Njah, S.T. Ogunjo, Pramana. J. Phys. **80**, 825 (2013)
22. J. Qiang, Phys. Lett. A **370**, 40 (2007)
23. Z.L. Wang, Nonlinear Dyn. **59**, 455 (2010)
24. X. Li, Int. J. Nonlinear Sci. **8**, 79 (2009)
25. E.N. Lorenz, J. Atmos. Sci. **20**, 130 (1963)
26. H. Wang, G. Cai, J. Inf. Comput. Sci. **4**, 251 (2009)
27. L. Liu, C. Liu, Y. Zhang, Chinese J. Phys. **46**, 386 (2008)
28. O.E. Rössler, Phys. Lett. A **57**, 397 (1976)