



Hopf bifurcation and its control in a 3D autonomous system

Liangqiang Zhou^{1,2,a}  and Albert Kabbah^{1,2,b}

¹ Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, Jiangsu, China

² Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles(NUAA), MIIT, Nanjing 210016, Jiangsu, China

Received 29 November 2021 / Accepted 17 February 2022 / Published online 1 March 2022

© The Author(s), under exclusive licence to EDP Sciences, Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract The first section of this paper discusses the stability and Hopf bifurcation for a new dynamical system using stability theory, the center manifold as well as normal form theory. To verify the analytical results, numerical simulations are performed. The second section focuses on controlling the Hopf bifurcation with a robust controller capable of handling a wide range of parameter values. By fine tuning the control parameters, the controller ensures that Hopf bifurcation occurred at P_0 . Furthermore, we postpone the Hopf bifurcation at P_+ by adjusting the control parameters.

1 Introduction

There are several successful industrial uses for nonlinear control systems. Researchers and designers in aviation control, robotics, process control, and biomedical engineering are investigating nonlinear control approaches [1–4]. Cai and Yuan [5] proposed a hybrid control strategy with state feedback and parameter control. The results dedicated that hybrid control could change the critical value for Hopf bifurcation. Yang and Wei [6] also proposed a nonlinear control scheme, which allowed a much parameter regain and could be extended to degenerate Hopf bifurcation. A hybrid control strategy for controlling the Hopf bifurcation in a modified hyperchaotic Lü system was proposed by Tang [7]. They were able to delay the Hopf bifurcation by selecting an appropriate parameter control, thereby extending the system's stability range.

As evidenced by the literature, more work on Hopf bifurcation control has been done [8–15]. Du et al. [16] studied Hopf bifurcation with center manifold theorem and normal form. A washout controller was designed to control bifurcation at the equilibrium E_0 by adjusting the controller parameters. Kim and Chang [17] proposed a new chaotic system and examined its compound structures. This paper seeks to design a controller to further delay the bifurcation at the equilibrium point P_0 and also to extend the stability range of the system [17].

2 Stability Analysis

Consider a governing system as follows

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = xz + by, \\ \dot{z} = -x^2 - cz, \end{cases} \quad (1)$$

where x, y, z are state variables, a, b and c are system parameters.

System (1) has a chaotic attractor for $a = 2, b = 1, c = 0.6$ as shown in Fig. 1.

The system has three equilibrium points, i.e., $P_0(0, 0, 0)$, $P_+(\sqrt{bc}, \sqrt{bc}, -b)$, and $P_-(\sqrt{bc}, -\sqrt{bc}, -b)$ if $bc > 0$. Obviously, the equilibrium point is stable when $a, c > 0$ and $b < 0$ and unstable when $a < 0$ or $b > 0$ or $c < 0$.

The Hopf bifurcation at the equilibrium point P_0 , on the other hand, cannot occur. Dias et al. [18, 19] looked into the existence of invariant set of topological circles to find a good set of parameters for the equilibrium P_+ .

3 Bifurcation method

This section describes how to compute the first and the second Lyapunov coefficients, as represented by l_1 and l_2 , respectively.

Consider the following differential equation

$$\dot{x} = f(x, \xi) \quad (2)$$

^a e-mail: zlgrex@sina.com (corresponding author)

^b e-mail: kalbertt2@gmail.com

where $x \in R^4$ are the state variables, and $\xi \in R^n$ are control parameters. It is also assumed that $f \in C^1[R^4 \times R^n]$. If system (2) has the equilibrium point $x = x_0$ at $\xi = \xi_0$, and $x = x - x_0$, then

$$F(x) = f(x, \xi_0) \tag{3}$$

$F(x)$ is smooth and can be expressed in the Taylor series expansion as follows

$$F(x) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(\|x\|^4), \tag{4}$$

where $A = f_x(0, \xi_0)$, $i = 1, 2, 3, 4$.

$$B_i(x, y) = \sum_{j,k=1}^4 \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k} \Big|_{\xi=0} x_j y_k,$$

$$C_i(x, y, z) = \sum_{j,k,l=1}^4 \frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} \Big|_{\xi=0} x_j y_k y_l. \tag{5}$$

Assume A has a pair of purely imaginary eigenvalues, denoting as $\lambda_{2,3} = \pm i\omega_0$, ($\omega_0 > 0$) at (x_0, ξ_0) , where these eigenvalues admit no other eigenvalues with $Re\lambda = 0$, $p, q \in \mathbb{C}^n$ are vectors satisfying

$$Aq = i\omega_0 q, A^T p = -i\omega_0 p, \langle p, q \rangle = \sum_{i=1}^n \bar{p}_i p_i = 1. \tag{6}$$

For any vector in the eigenspace $y \in T^c$, one can describe it as $y = \omega q + \bar{\omega} \bar{q}$, where $\omega = \langle p, y \rangle \in \mathbb{C}$. The center manifold associated with $\lambda_{2,3} = \pm i\omega_0$ is parametrized by ω and $\bar{\omega}$, employing an immersion with the form $x = H(\omega, \bar{\omega})$, where $H : \mathbb{C}^2 \rightarrow R^4$ can be expanded as follows

$$H(\omega, \bar{\omega}) = \omega q + \bar{\omega} \bar{q} + \sum_{2 \leq j+k \leq 5} \frac{1}{j!k!} h_{jk} \omega^j \bar{\omega}^k + O(\|\omega\|^6), \tag{7}$$

where $h_{jk} \in \mathbb{C}^4$ and $h_{jk} = \overline{h_{jk}}$. Substituting the derivative of $H(\omega, \bar{\omega})$ into (4) yields

$$H_\omega \omega' + H_{\bar{\omega}} \bar{\omega}' = F(H(\omega, \bar{\omega})). \tag{8}$$

The vector h_{jk} can be obtained according to (9). Thus, on the center manifold, we have

$$\omega' = i\omega_0 \omega + \frac{1}{2} G_{32} \omega \|\omega\|^4 + O(\|\omega\|^6), \tag{9}$$

where $G_{jk} \in \mathbb{C}^2$. The first Lyapunov coefficient

$$l_1 = \frac{1}{2} Re G_{21}, \tag{10}$$

where $G_{21} = \langle p, H_{21} \rangle$, $H_{21} = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11})$, $h_{20} = (2i\omega_0 I_4)^{-1} B(q, q)$, $h_{11} = -A^{-1} B(q, \bar{q})$,

and I_4 are unit 4×4 matrices. The second Lyapunov coefficient

$$l_2 = \frac{1}{12} Re G_{32}, \tag{11}$$

where

$$G_{32} = \langle p, H_{32} \rangle,$$

$$H_{32} = 6B(h_{11}, h_{21}) + B(\bar{h}_{20}, h_{30}) + 3B(\bar{h}_{21}, \bar{h}_{20}) + 3B(q, h_{22}) + 2B(\bar{q}, h_{31}) + 6C(q, h_{11}, h_{11}) + 3C(q, \bar{h}_{20}, h_{20}) + 3C(q, q, \bar{h}_{20}) + 6C(q, \bar{q}, h_{21}) + 6C(\bar{q}, h_{20}, h_{11}) + C(\bar{q}, \bar{q}, h_{30}) + D(q, q, q, \bar{h}_{20}) + 6D(q, q, \bar{q}, h_{11}) + 3D(q, \bar{q}, \bar{q}, h_{20}) + E(q, q, q, \bar{q}, \bar{q}) - 6G_{21} h_{21}. \tag{12}$$

By solving the following $n + 1$ -dimensional equation

$$\begin{pmatrix} i\omega_0 I_4 & q \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} h_{21} \\ 0 \end{pmatrix} \tag{13}$$

with the condition $\langle p, h_{21} \rangle = 0$, the vector h_{21} can be obtained.

$$h_{30} = (3i\omega_0 I_4 - A)^{-1} [3B(q, h_{20} + C(q, q, q))],$$

$$h_{31} = (2i\omega_0 I_4 - A)^{-1} [3B(q, h_{21}) + B(\bar{q}, h_{30}) + 3B(h_{20}, h_{11}) + 3C(q, q, h_{11}) + 3C(q, \bar{q}, h_{20}) + D(q, q, q, \bar{q}) - 3G_{21} h_{20}],$$

$$h_{22} = -A^{-1} [D(q, q, q, \bar{q}) + 4C(q, \bar{q}, h_{11}) + 2B(q, \bar{h}_{21}) + 2B(\bar{q}, h_{21}) + B(\bar{h}, h_{20})]. \tag{14}$$

4 Hopf bifurcation at $P_0(0,0,0)$

Now consider the stability of $P_0(0, 0, 0)$ in the controlled system as illustrated below:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = xz + by + u \\ \dot{z} = -x^2 - cz \\ \dot{v} = y - mv \end{cases}, \tag{15}$$

where $u = k_1(y - mv) + k_2(y - mv)^2$, k_1, k_2 are the gains and m is the constant time coefficient which satisfies $m > 0$ and $\mu(0) = 0$.

The Jacobian matrix of the system (15) evaluated at $P_0(0, 0, 0)$ is

$$P_0 = \begin{pmatrix} -a & a & 0 & 0 \\ 0 & b + k_1 & 0 & -mk_1 \\ 0 & 0 & -c & 0 \\ 0 & 1 & 0 & -m \end{pmatrix}. \tag{16}$$

Characteristic polynomial evaluated at the equilibrium point $P_0(0, 0, 0)$ is

$$P(\lambda) = (\lambda + c)[\lambda^3 + (abm - k_1)\lambda^2 + (am - ak_1 - abm)\lambda + a^2bm]. \tag{17}$$

If $c > 0$, $ab + m - k_1$, $a^2bm < 0$, and

$$b < b_0 = \frac{m^2 + k_1^2 - 2mk_1 + am - ak_1}{a^2 + 2am - ak_1}, \tag{18}$$

The equilibrium point P_0 is stable. If $b > b_0$, the equilibrium point P_0 is unstable. Equation (18) will be used to study the Hopf bifurcation at P_0 which occurs at system (15).

The multi-linear symmetric functions of the smooth function f can be written as

$$\begin{aligned} B(x, y) &= (0, -a(x_1y_3 + x_3y_1), x_1y_2 + x_2y_1, 0)^T \\ C(x, y, z) &= (0, 6k_2(x_2, y_2, z_2 - m^3x_4y_4z_4m^2x_2y_4z_4 \\ &\quad + m^2x_4y_2z_4 + m^2x_4y_4z_2 \\ &\quad - mx_2y_2z_4 - mx_4y_2z_2), 0, 0)^T. \end{aligned} \tag{19}$$

The eigenvalues of matrix A are

$$\begin{aligned} \lambda_{1,2} &= \pm i\omega_0 = \pm i\sqrt{am - ak_1 - a^2b}, \\ \lambda_3 &= -c, \text{ and } \lambda_4 = -(ab + m - k_1). \end{aligned} \tag{20}$$

From (8), we have

$$\begin{aligned} q &= \begin{pmatrix} \frac{a(am + \omega_0^2) + a\omega_0(a - m)i}{a^2 + \omega_0^2} \\ m + \omega_0i \\ 0 \\ 1 \end{pmatrix}, \\ p &= (\mathbb{k}_1 + \mathbb{k}_2i, \mathbb{k}_3 + \mathbb{k}_4i, 0, \mathbb{k}_5i)^T, \end{aligned} \tag{21}$$

where

$$\begin{aligned} \sigma_1 &= (a^2 + \omega_0^2)(m^2 - \omega_0^2 + mk_1) \\ &\quad - (2k_1m + \omega_0^2) \\ &\quad \times (am + \omega_0^2) - \omega_0^2(m - k_1)(a - m), \\ \sigma_2 &= 2m\omega_0^2(a^2 + \omega_0^2) - \omega_0(m - k_1)(am + \omega_0^2) \\ &\quad - \omega_0(a - m)(2k_1m + \omega_0^2), \\ \mathbb{k}_1 &= \frac{(a^2 + \omega_0^2)[\sigma_2\omega_0(m - k_1) - \sigma_1(2k_1m - \omega_0^2)]}{a(\sigma_1^2 + \sigma_2^2)}, \\ \mathbb{k}_2 &= \frac{(a^2 + \omega_0^2)[- \sigma_2(2k_1m - \omega_0^2) - \sigma_1\omega_0(m - k_1)]}{a(\sigma_1^2 + \sigma_2^2)}, \\ \mathbb{k}_3 &= \frac{(a^2 + \omega_0^2)(\sigma_1m + \sigma_2\omega_0)}{a(\sigma_1^2 + \sigma_2^2)}, \\ \mathbb{k}_4 &= \frac{(a^2 + \omega_0^2)(\sigma_1\omega_0 - \sigma_2m)}{a(\sigma_1^2 + \sigma_2^2)}, \end{aligned}$$

$$\mathbb{k}_5 = \frac{mk_1(a^2 + \omega_0^2)(\sigma_1 + \sigma_2)}{a(\sigma_1^2 + \sigma_2^2)},$$

$$B(q, q) = \begin{pmatrix} 0 \\ \frac{2a[(a\omega_0^2 - am^2 + 2m\omega_0^2) + (m^2\omega_0 + 2am\omega_0 - \omega_0^2)i]}{a^2 + \omega_0^2} \\ 0 \\ 0 \end{pmatrix},$$

$$B(q, \bar{q}) = \begin{pmatrix} 0 \\ \frac{2a(m^2 + \omega_0^2)}{a^2 + \omega_0^2} \\ 0 \\ 0 \end{pmatrix},$$

$$h_{11} = \begin{pmatrix} 0 \\ \frac{-2a(m^2 + \omega_0^2)}{b(a^2 + \omega_0^2)} \\ \frac{-2a(m^2 + \omega_0^2)}{b(a^2 + \omega_0^2)} \\ 0 \\ \frac{-2a(m^2 + \omega_0^2)}{bm(a^2 + \omega_0^2)} \end{pmatrix},$$

$$h_{20} = (2i\omega_0I_3 - A)^{-1}B(q, q) = (\hbar_1, \hbar_2, 0, \hbar_3)^T,$$

$$\begin{aligned} \sigma_3 &= (ci - 2\omega_0)(2am\omega_0 - 8\omega_0^2 - 2ak_1\omega_0 \\ &\quad - 4k_1\omega_0^2i - 2a^2b\omega_0 \\ &\quad + 4m\omega_0^2i + 4am\omega_0^2i + a^2bmi), \\ \sigma_4 &= a(-2a\omega_0 + mi)(8\omega_0^3 - 4k_1\omega_0^2i \\ &\quad + 2ak_1\omega_0 - 2am\omega_0 \\ &\quad + 2a^2b\omega_0 + 4m\omega_0^2i + 4am\omega_0^2i + a^2bmi), \\ \sigma_5 &= a^2b^2m^2 + 4a^2b^2\omega_0^2 + 8a^2bk_1\omega_0^2 - 8a^2bk_1m\omega_0^2 \\ &\quad + 8a^2bm^2\omega_0^2 + 32a^2b\omega_0^2 \\ &\quad + 4a^2k_1^2\omega_0^2 - 8a^2k_1m\omega_0^2 + 4a^2m^2\omega_0^2 + 16a^2\omega_0^2 \\ &\quad + 16k_1^2\omega_0^4 - 32k_1m\omega_0^4 + 16m^2\omega_0^4 + 64\omega_0^6, \\ \hbar_1 &= (2a\sigma_4[(a\omega_0^2 - am^2 - 2m\omega_0^2) \\ &\quad \times + (m_2\omega_0 + 2am\omega_0 - \omega_0^3)i])(\sigma_5(a^2\omega_0^2))^{-1}, \\ \hbar_2 &= (2(a + \omega_0(c + 2\omega_0i))(m + 2\omega_0i) \\ &\quad \times [(a\omega_0^2 - am^2 - 2m\omega_0^2) \\ &\quad + (m_2\omega_0 + 2am\omega_0 - \omega_0^3)i]) \times (\sigma_3(a^2\omega_0^2))^{-1}, \\ \hbar_3 &= (2(c + 2\omega_0i)(c + 2\omega_0i) \\ &\quad \times [(a\omega_0^2 - am^2 - 2m\omega_0^2) + (m_2\omega_0 \\ &\quad + 2am\omega_0 - \omega_0^3)i]) \\ &\quad \times (\sigma_3(a^2\omega_0^2))^{-1}. \end{aligned} \tag{22}$$

One also has

$$\begin{aligned} B(q, h_{11}) &= (0, 0, ((-2a(m^2 + \omega_0^2) \\ &\quad \times [a\omega_0^2 + m2a^2 + m\omega_0^2 \\ &\quad + \omega_0(2a^2 + \omega_0^2 - am)i]) \\ &\quad \times (b(a^2 + \omega_0^2)^2)^{-1}), 0)^T, \\ B(\bar{q}, h_{20}) &= (0, 0, ((a\hbar_2[(am + \omega_0^2) - \omega_0(a - m)i] \\ &\quad + \hbar_1(a_2 + \omega_0^2)(m - \omega_0i)) \\ &\quad \times (a^2 + \omega_0^2)^{-1}), 0)^T, \\ C(q, q, \bar{q}) &= (0, \hbar_4 + \hbar_5i, 0, 0)^T, \\ H_{21} &= (0, \hbar_4 + \hbar_5i, \hbar_6 + \hbar_7i, 0)^T, \end{aligned}$$

$$G_{21} = (\hbar_4 k_3 - \hbar_5 k_3 + \hbar_4 k_4)i, \tag{23}$$

where

$$\begin{aligned} \hbar_4 &= 6mk_2[(m + 1)(\omega_0^2 + 3m^2) - (3m^2 + 2\omega_0^2)], \\ \hbar_5 &= 6k_2\omega_0^3, \\ \hbar_6 &= ab(am + \omega_0^2) + \hbar_1 bm(a^2 + \omega_0^2) \\ &\quad - 4a(m^2 + \omega_0^2)[a(a^2m + \omega_0^2) + m(a^2 + \omega_0^2)], \\ \hbar_7 &= -ab\omega_0(a - m) - \hbar_1 b\omega_0(a^2 + \omega_0^2) - 4a\omega_0(m^2 + \omega_0^2) \\ &\quad \times [a(am + \omega_0^2) + m(a^2 + \omega_0^2)]. \end{aligned} \tag{24}$$

It is now only necessary to confirm the Hopf bifurcation’s transversal condition. Consider differential equations (1) with the parameter b as a dependent variable. At the crucial parameter $b = b_0$, the real part of the complex eigenvalues confirms

$$\xi'(b_0) = \text{Re} \left\langle p, \frac{dA}{db} \Big|_{b=b_0, q} \right\rangle \tag{25}$$

The Hopf point’s transversal condition holds as $\xi'(b_0) \neq 0$.

Theorem 1 *For the differential equation (15), the first Lyapunov coefficient for the point P_0 is as follows:*

$$l_1 = \frac{1}{2}(\hbar_4 k_3 - \hbar_5 k_4) \tag{26}$$

If l_1 is greater than zero and the Hopf point transversal condition holds, then a transversal Hopf point at P_0 exists in equation (15).

Letting $m = 0.5$, the controller is

$$u = k_1(y - mv) + k_2(y - mv)^2. \tag{27}$$

If $0 < k_1 < 1$, a transversal Hopf point at P_0 exists in equation (15) when $a = 2, c = 4$. Furthermore, if $k_1 > 0$, there exist a stable periodic orbit close to P_0 . The dynamical evolution as b increases is as follows:

1. When $0.2 < b < 1.1$, the system is stable,
2. When $1.1 < b < 1.4$, the system is chaotic with one periodic window in the chaotic band.

Now, if we fix $a = 2, b = 1$, and vary c , the dynamic route of the system is summarized as follows:

1. If $3.5 < c < 20$, the system becomes stable,
2. If $20 < b < 40$, Chaos exists for this system with some periodic windows in the chaotic band,
3. If $40 < c < 80$, a long period-doubling bifurcation window exists.

According to the analysis performed here, the Hopf bifurcation at P_0 cannot occur when $b \in [1.1, 1.4]$. We devise a control law with which the system (15)

exhibits a Hopf bifurcation when $b = 0.8215$, illustrated in Figs. 2, 3, 4.

Let $a = 2, c = 0.6, k_1 = 0.3$ and $k_2 = 0.1$; then, the Hopf bifurcation value can be calculated as $b_0 = 0.8215$. As shown in Figs. 2 and 3, the equilibrium is stable when $b = 0.6132 < b_0$ and unstable when $b = 0.9321 > b_0$. A transversal Hopf point at P_0 occurs when $b_0 = 0.8215$ as shown in Fig. 4. By computation, one can obtain $l_1 = -2.59570$. As a result, the stable periodic solution that bifurcates from P_0 is supercritical.

5 Hopf bifurcation at P_+

We investigate the stability of P_+ in system (15) in this section. The Jacobian matrix at P_+ is

$$P_+ = \begin{pmatrix} -a & a & 0 & 0 \\ -b & b + k_1 & \sqrt{bc} & -mk_1 \\ -2\sqrt{bc} & 0 & -c & 0 \\ 0 & 1 & 0 & -m \end{pmatrix}. \tag{28}$$

Characteristic polynomial of the matrix P_+ is

$$p(\lambda) = \lambda^4 + Q\lambda^3 + R\lambda^2 + S\lambda + T, \tag{29}$$

where

$$\begin{aligned} Q &= a + m + c - k_1, \\ R &= abc + mc - ck_1 + am + ac - ak_1, \\ S &= a^2bc - ack_1 + mac + mabc + abc, \\ T &= 2a^2bcm. \end{aligned} \tag{30}$$

If $Q > 0, R > 0, S > 0, T > 0$ and

$$S(S - QR) + Q^2T = 0, \tag{31}$$

system (15) has transversal Hopf points at P_+ .

Dias et al.[11], found out that the equilibrium point P_+ is stable.

The controller in this section is designed as

$$u = k_1(y - 0.5v) + k_2(y - 0.5v)^2, \tag{32}$$

to set off Hopf bifurcation at P_+ .

If system (15) experiences Hopf bifurcation at the equilibrium P_+ , then condition (34) is satisfied. The multilinear symmetric functions corresponding to f are

$$\begin{aligned} B(x, y) &= (0, -x_1y_3 - x_3y_1, x_1y_2 + x_2y_1, 0)^T, \\ C(x, y, z) &= (0, 6k_2(x_2y_2z_2 - m^3x_4y_4z_4 \\ &\quad + m^2x_2y_4z_4 + m^2x_4y_2z_4 + m^2x_4y_4z_2 \\ &\quad - mx_2y_2z_4 - mx_4y_2z_4), 0, 0)^T. \end{aligned} \tag{33}$$

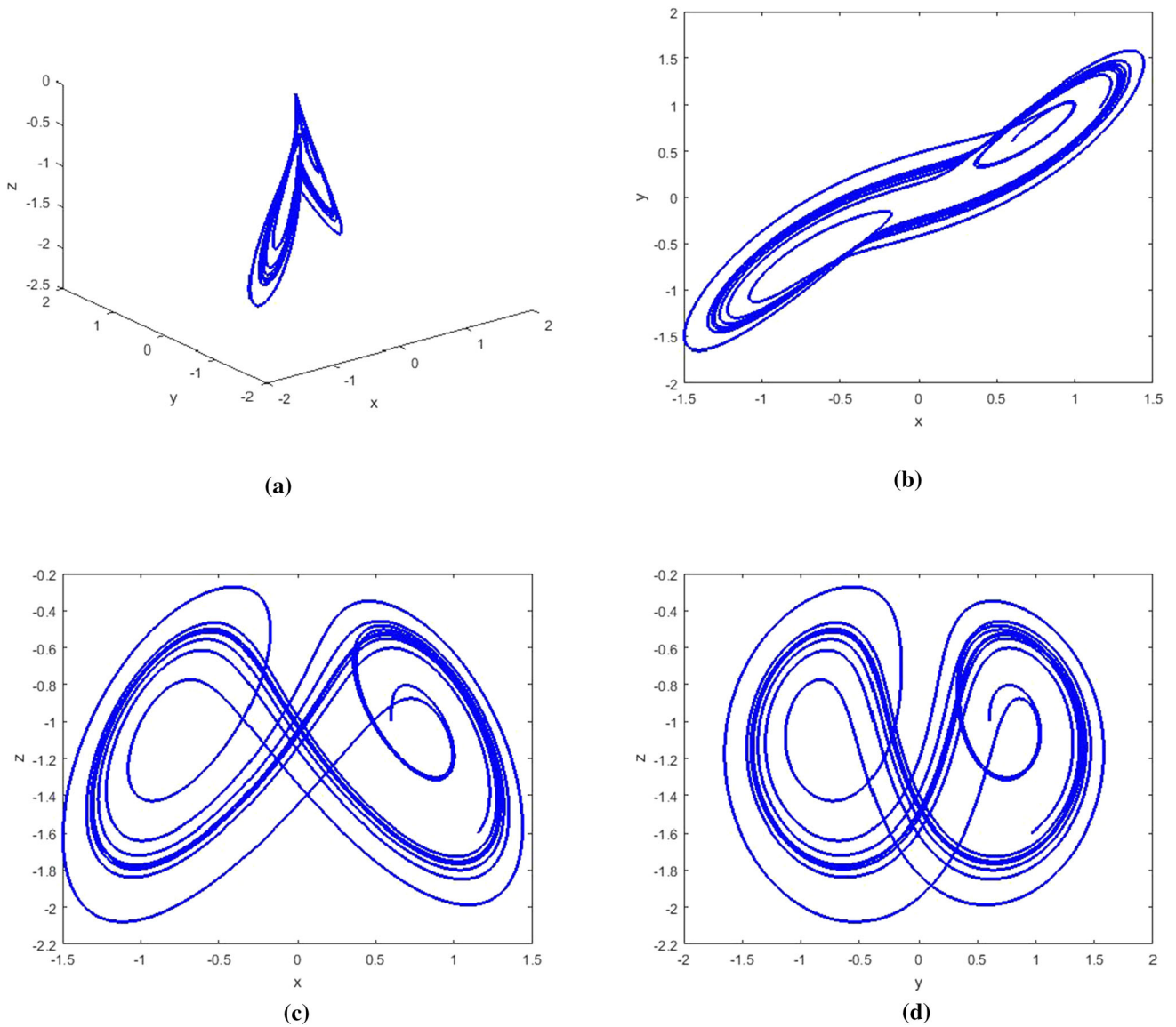


Fig. 1 Trajectory of the chaotic attractor in various projections

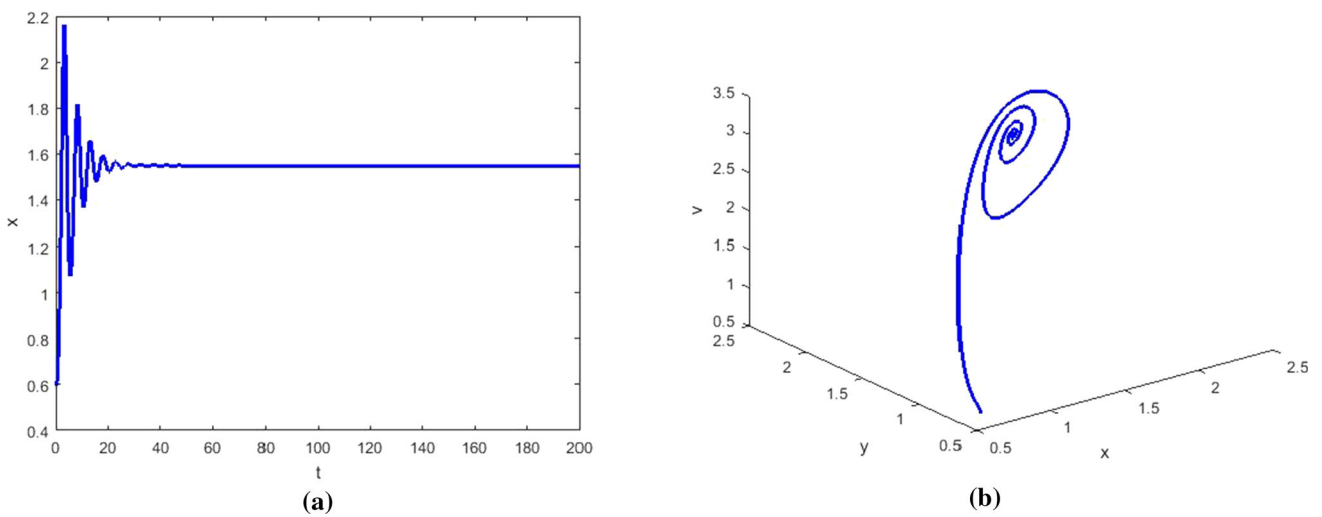


Fig. 2 Time history and phase trajectory of system (15) with $a = 2, b = 0.6, c = 4, m = 0.5, k_1 = 0.3, k_2 = 0.1$

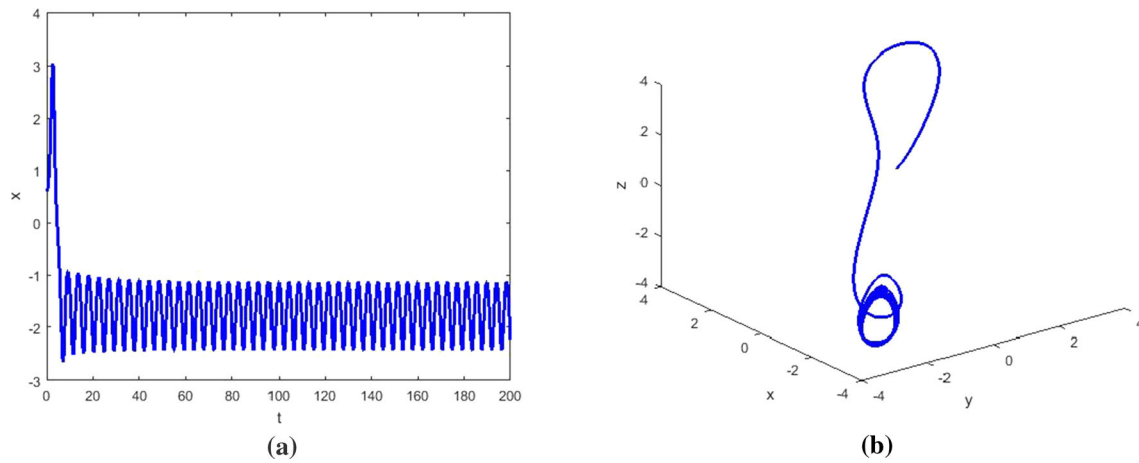


Fig. 3 Time history and trajectory of system (15) with $a = 2, b = 0.9132, c = 4, m = 0.5, k_1 = 0.3, k_2 = 0.1$

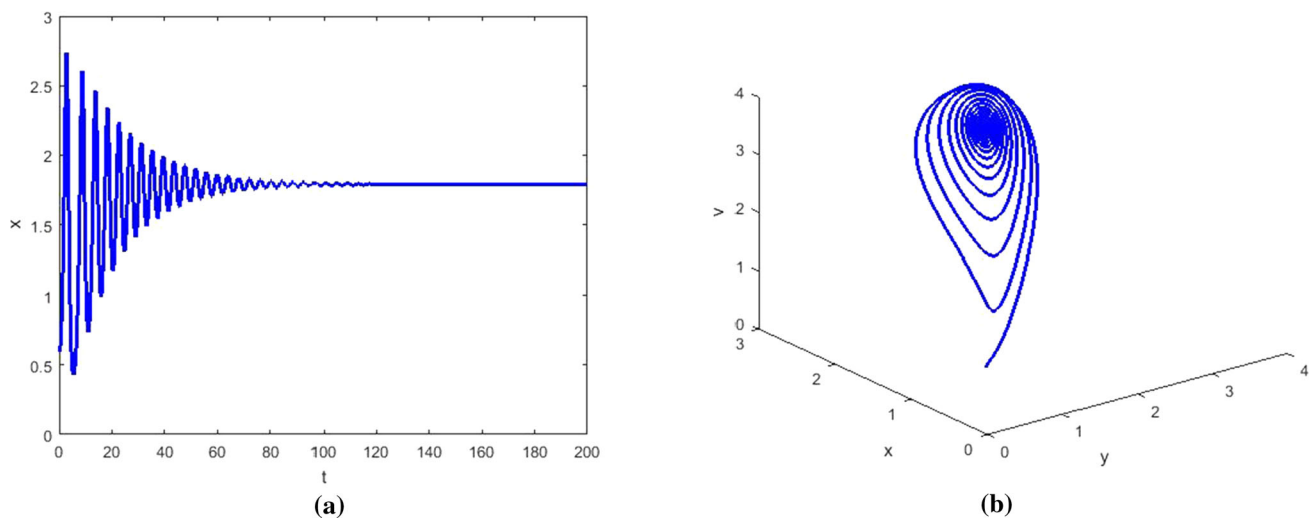


Fig. 4 Time history and trajectory of system (15) with $a = 2, b = 0.8215, c = 4, m = 0.5, k_1 = 0.3, k_2 = 0.1$

The eigenvalues of A are

$$\lambda_{1,2} = \pm 1.2346i, \lambda_3 = -0.4201, \lambda_4 = -1.2156 \quad (34)$$

From (7), we have

$$q = \begin{pmatrix} 0.5106 - 0.1734i \\ 0.3035 + 0.5671i \\ 0.9015 + 1.0013i \end{pmatrix},$$

$$p = \begin{pmatrix} 0.0145 - 0.3015i \\ 0.18075 + 0.2301i \\ 0.167 + 0.3642i \\ 0.2601 + 0.0115i \end{pmatrix},$$

$$B(q, q) = \begin{pmatrix} 0 \\ 8.7215 + 2.4112i \\ -0.2657 - 0.1502i \\ 0 \end{pmatrix},$$

$$B(q, \bar{q}) = \begin{pmatrix} 0 \\ 2.6356 \\ -0.6265 \\ 0 \end{pmatrix},$$

$$h_{11} = \begin{pmatrix} 0.4556 \\ 0.5667 \\ 3.0854 \\ 2.8456 \end{pmatrix},$$

$$h_{20} = \begin{pmatrix} -0.0225 + 0.3675i \\ -0.8720 + 0.3056i \\ 0.1034 + 0.5983i \\ -0.0859 + 0.7146i \end{pmatrix},$$

$$B(q, h_{11}) = \begin{pmatrix} 0 \\ -0.3147 + 0.7594i \\ -0.3421 + 0.6921i \\ 0 \end{pmatrix},$$

$$B(\bar{q}, h_{20}) = \begin{pmatrix} 0 \\ -0.1673 + 0.3117i \\ -0.6115 + 0.5954i \\ 0 \end{pmatrix},$$

$$C(q, q, \bar{q}) = \begin{pmatrix} 0 \\ 3.2i \\ 0 \\ 0 \end{pmatrix},$$

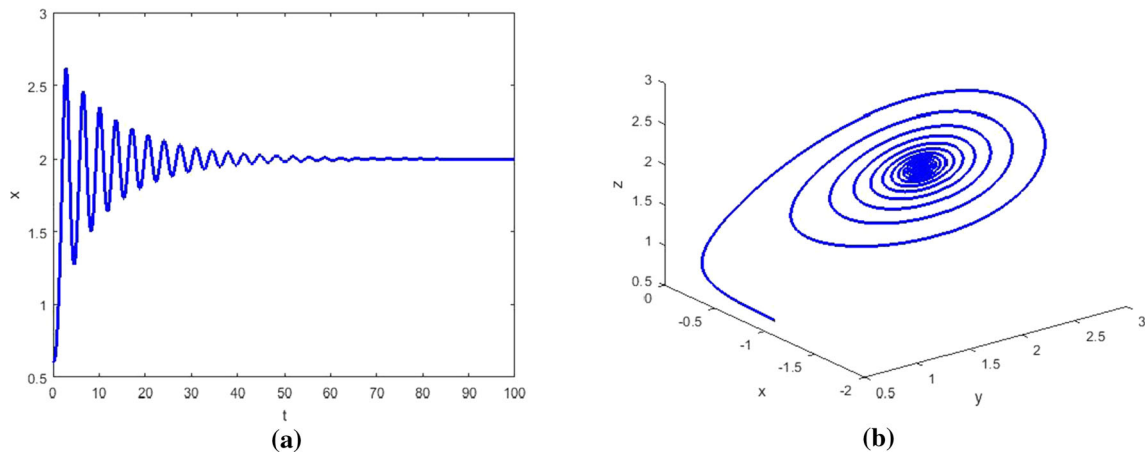


Fig. 5 Time history and trajectory of system (1) with $a = 2, b = 1, c = 4$

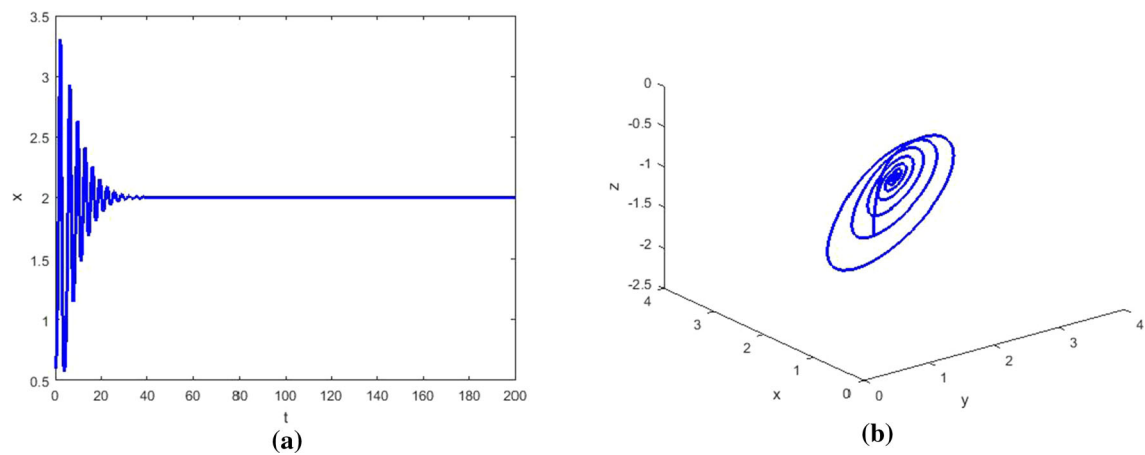


Fig. 6 Time history and trajectory of system (15) with $a = 2, b = 1, c = 4, m = 0.5, k_1 = 0.3, k_2 = 0.1$

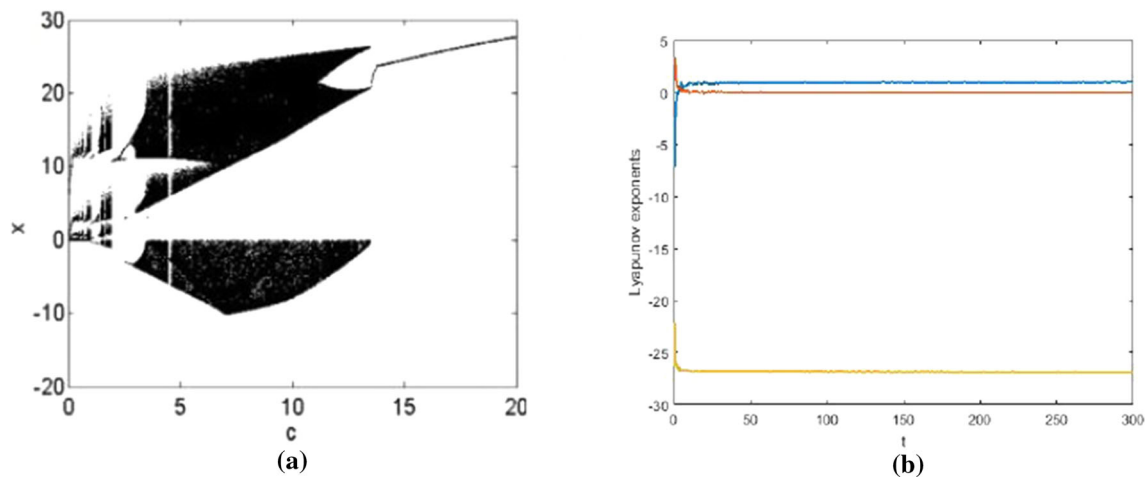


Fig. 7 Bifurcation diagram of x in terms of parameter c (b) Lyapunov exponents spectrum

$$\begin{aligned}
 H_{21} &= \begin{pmatrix} 0 \\ 3.2 + 2.1157i \\ -0.7649 + 1.3954 \\ 0 \end{pmatrix}, \\
 G_{21} &= -3.9815 + 0.8516i.
 \end{aligned}
 \tag{35}$$

Considering equation (15) depended on the parameter c , at $c = c_0$ verifies

$$\xi'(c_0) = Re \left\langle p, \frac{dA}{db} \Big|_{c=c_0, q} \right\rangle = 0.2875 > 0 \tag{36}$$

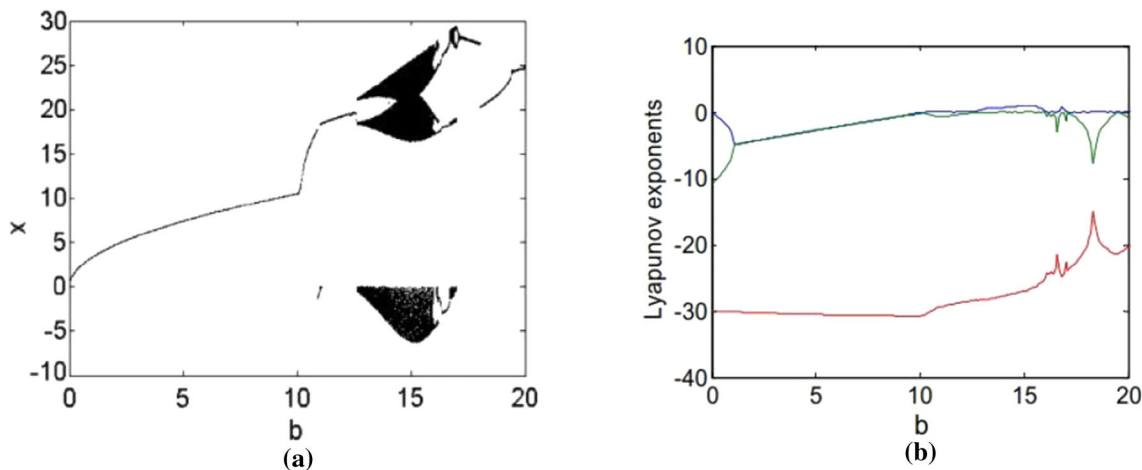


Fig. 8 Bifurcation diagram of x in terms of parameter b (b) Lyapunov exponents spectrum

Thus, the transversal condition holds.

Theorem 2 Considering equation (15), the first Lyapunov coefficient related to P_+ is

$$l_1 = -2.5957k_2 - 0.259 \tag{37}$$

If $k_2 \neq -0.259$, then, $l_1 \neq 0$ which proves that (15) has transversal Hopf points at P_+ for $a = 2, b = 1$ and $c = 0.6$. The k_2 determines the sign of the first Lyapunov coefficient. The first Lyapunov coefficient vanishes when $k_2 = -0.259$.

Given that

$$\begin{aligned} a = 2, b = 1, c = 0.6134, m = 0.5121, k_1 \\ = 0.3423, k_2 = -0.259 \end{aligned} \tag{38}$$

Consider system (15) for (a, b, c, k_1, k_2) , after tedious calculations, we have

$$\begin{aligned} C(q, q, \bar{q}) &= \begin{pmatrix} 0 \\ -0.2583 + 2.0014i \\ 0 \\ 0 \end{pmatrix}, \\ H_{21} &= \begin{pmatrix} 0 \\ -0.9645 + 2.8745i \\ -0.9476 + 3.0015i \\ 0 \end{pmatrix}, \\ G_{21} &= 2.1159i, \\ h_{30} &= \begin{pmatrix} 1.0012 + 0.3753i \\ -0.7854 + 1.8975i \\ -0.1320 + 0.4596i \\ -0.4713 + 0.2005i \end{pmatrix}, \\ h_{31} &= \begin{pmatrix} -3.7732 + 0.2593i \\ -0.96754 - 0.8513i \\ -2.0031 + 0.7642i \\ -0.7639 - 0.0167i \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} h_{21} &= \begin{pmatrix} -5.0034 + 0.5934i \\ -5.2632 - 6.0158i \\ -6.7456 + 1.6632i \\ -2.9763 + 1.9654i \end{pmatrix}, \\ h_{22} &= \begin{pmatrix} -3.0016 + 4.7421i \\ -1.8534 + 6.0139i \\ -7.3375 - 25.6614i \\ 6.2754 - 11.0220i \end{pmatrix}, \\ G_{32} &= -2.52391 - 8.4632i, \\ l_2 &= \frac{1}{12} \text{Re}G_{32} = -0.2103258333. \end{aligned} \tag{39}$$

Theorem 3 Consider $(a, b, c, m, k_1, k_2) \in \Phi$ for equation (15), the second Lyapunov coefficient related to P_+ is

$$l_2 = -0.2103258333. \tag{40}$$

As $l_2 \neq 0$, system (15) has a transversal Hopf point of codimension 2 at P_+ .

Letting $m = 0.5, k_1 = 0.3, k_2 = 0.1$, the controller is designed as

$$u = 0.3(y - 0.5v) + 0.1(y - 0.5v)^2 \tag{41}$$

In an uncontrolled system, P_+ is locally stable. While comparing the controlled system (15) to the uncontrolled system (1), we can see that a transversal Hopf point exists in system (15) at P_+ under the parameter region, shown in Figs. 5 and 6. We obtained distinct dynamical phenomena for system (1) and (15) if we fix parameters $a = 2$ and $b = 1$ and vary c in the interval $[3, 4]$. The bifurcation diagram in relation to parameter a and c , is shown in Figs. 7 and 8, respectively.

As we increase c , system (1) goes through the following dynamical paths: There is a period-two orbit window when $c \in [0, 2.9]$. When $c \in (0, 3.9]$, the system becomes chaotic. When $c = 4$, a Hopf bifurcation for the uncontrolled system occurs at the equilibrium P_+ . In Figure 9, we design a controller to make that system (15) undergoes a Hopf

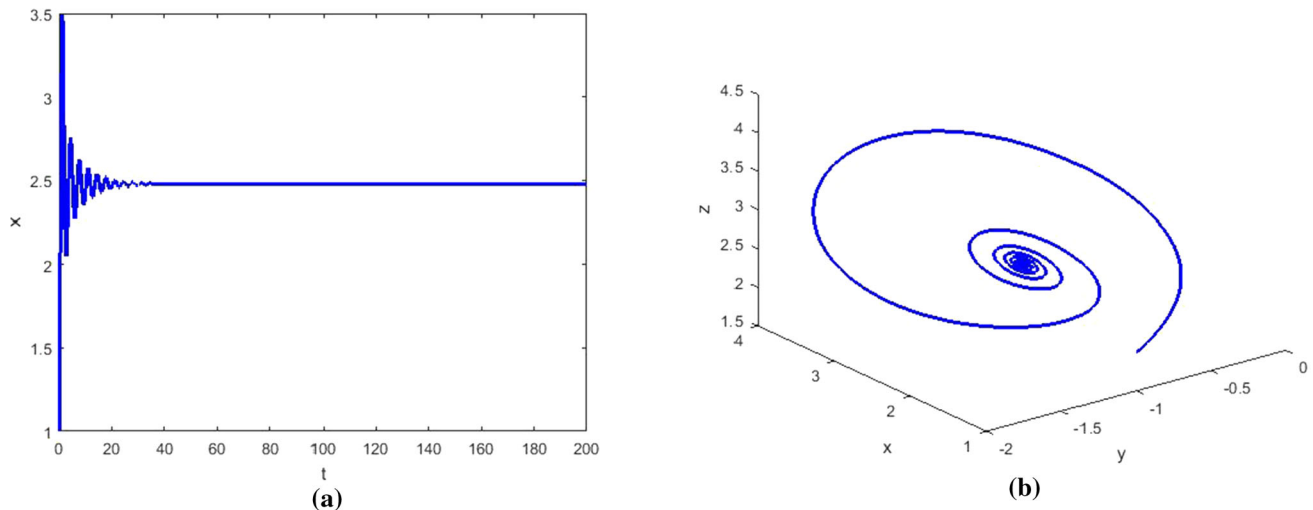


Fig. 9 Time history and trajectory of system (15) with $a = 2$, $b = 1$, $c = 6.1324$, $m = 0.5$, $k_1 = 0.3$, $k_2 = 0.1$

bifurcation when $c = 6.1324$. The analysis performed here shows that Hopf bifurcation is delayed. By calculation, $l_1 = -2.59570$ can be obtained. As a result, the periodic solution that bifurcates from P_+ is supercritical as well as stable.

6 Conclusions

The control of Hopf bifurcation for a new system is investigated in this paper. Through the use of appropriate controls, the controller is built with the desired location and properties. Numerical simulations demonstrate that supercritical Hopf bifurcation exists. The controller ensures that P_0 undergoes a controllable Hopf bifurcation. Adjusting the controller parameters, the Hopf bifurcation at the equilibrium point P_+ is delayed.

Acknowledgements This research is financially supported by the National Science Foundation of China (Nos. 11772148, 11872201, 12172166).

References

1. B.H.K. Lee, S.J. Price, Y.S. Wong, Nonlinear aeroelastic analysis of airfoils: bifurcation and chaos. *Prog. Aerosp. Sci.* **35**, 205–334 (1999)
2. S.H. Strogatz, *Nonlinear Dynamics, and chaos: with applications to physics, biology, chemistry, and engineering* (CRC Press, Boca Raton, 2018)
3. E.N. Lorenz, Deterministic nonperiodic flow. *J. Atmos. Sci.* **20**, 130–141 (1963)
4. P. Yu, G. Chen, Hopf bifurcation control using nonlinear feedback with polynomial functions. *Int. J. Bifurcation Chaos* **14**, 1683–1704 (2004)
5. P. Cai, Z.Z. Yuan, Hopf bifurcation and chaos control in a new chaotic system via hybrid control strategy. *Chin. J. Phys.* **55**, 64–70 (2017)
6. Z. Wei, I. Moroz, Z. Wang et al., Dynamics at infinity, degenerate Hopf and zero-Hopf bifurcation for Kingni-Jafari system with hidden attractors. *Int. J. Bifurcation Chaos* **26**, 1650125 (2016)
7. J.S. Tang, Ping Cai, Z.B. Li, Controlling Hopf bifurcation of a new modified hyperchaotic Lü system. *Math. Problems Eng.* **2015**, (2015)
8. Q. Yang, Z. Wei, Anti-control of Hopf bifurcation in the new chaotic system with two stable node-foci. *Appl. Math. Comput.* **217**, 422–429 (2010)
9. Z. Wang, W. Sun, Z. Wei et al., Dynamics and delayed feedback control for a 3D jerk system with hidden attractor. *Nonlinear Dyn.* **82**, 577–588 (2015)
10. J. Liu, J. Guan, Z. Feng, Hopf bifurcation analysis of KdV-Burgers-Kuramoto chaotic system with distributed delay feedback. *Int. J. Bifurcation Chaos* **29**, 1950011 (2019)
11. A.K. Tiba, A.F. Araujo, Control strategies for Hopf bifurcation in a chaotic associative memory. *Neurocomputing* **323**, 157–174 (2019)
12. C.J. Xu, Y.S. Wu, Chaos control and bifurcation behavior for a Sprott E system with distributed delay feedback. *Int. J. Autom. Comput.* **12**, 182–191 (2015)
13. D.S. Chen, H.O. Wang, G. Chen, Anti-control of Hopf bifurcations. *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **48**, 661–672 (2001)
14. Y. Liu, S. Pang, D. Chen, An unusual chaotic system and its control. *Math. Comput. Model.* **57**, 2473–2493 (2013)
15. C.K. Tse, Y.M. Lai, H.H.C. Iu, Hopf bifurcation and chaos in a free-running current-controlled Cuk switching regulator. *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **47**, 448–457 (2000)
16. W.J. Du, Y.D. Chu, Y.X. Chang et al., Control of Hopf bifurcation in Autonomous Systems Based on Washout Filter. *J. Appl. Math.* **2013**, 482351 (2013)
17. D. Kim, P.H. Chang, A new butterfly-shaped chaotic attractor. *Results Phys.* **3**, 14–19 (2013)

18. F.S. Dias, L.F. Mello, J.G. Zhang, Nonlinear analysis in a Lorenz-like system. *Nonlinear Anal. Real-World Appl.* **11**, 34913500 (2010)
19. B.D. Hassard, N.D. Kazarinoff, Y. Wan, *Theory and Applications of Hopf Bifurcation* (Cambridge University Press, London, UK, 1981)