Eur. Phys. J. Plus (2024) 139:810
https://doi.org/10.1140/epjp/s13360-024-05532-1

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

The blow up of the thin film equations

Ming Liu[a], **Xian Gao Liu**[b]

General Education College, Hunan University of Information Technology, Changsha, People's Republic of China

Received: 10 April 2024 / Accepted: 3 August 2024

Abstract In 1993 Hocherman and Rosenau Hocherman and Rosenau (Phys D 67: 113-125, 1993) conjectured that long-wave unstable Cahn–Hilliard-type interface models develop finite time singularities when the nonlinearity in the destabilizing term grows faster at larger amplitudes than the nonlinearity in the stabilizing term. In 1998 Bertozzi and Puch (Comm. Pure App. Math. 51: 625–661, 1998) revised the conjecture for a class equations, often used to model thin films in a lubrication context, and proved the global boundedness part of this conjecture. We shall prove the blow up part of this revised conjecture in the power law case in the paper.

1 Introduction

In this paper we study the general fourth-order parabolic equation

$$u_t = \nabla \cdot (A(u)\nabla(f(u) - \Delta u)), \tag{1.1}$$

on a bounded (smooth) domain Ω in $\mathbf{R}^N$. Equations of this form arise quite common from the study of pattern formation in material science, fluid dynamics and population dynamics whenever interfaces are involved.

A well-known one is the Cahn-Hilliard equation which describes the separation of phases in binary fluids(to see [1, 2]). In this model the unknown u is the concentration of one fluid, $A(u) \equiv 1$, and $f(u) - \Delta u$ is the chemical potential.

Another model under intensive investigation recently is the thin film type equation (see the survey Oron-Davis-Bankoff [3]) where u in (1.1) ($N \le 2$) stands for the height of a thin film on a flat surface. Here the mobility A and external force f are certain functions defined for $u \ge 0$ where A is positive for positive u and vanishes at 0.

The well-posedness of (1.1) is usually studied under the Neumann and "no-flux" conditions

$$\nu \cdot \nabla u = \nu \cdot \nabla(f(u) - \Delta u) = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where ν is the unit outer normal at the boundary. In some works the periodic boundary conditions are imposed.

When the functions A and f, and the initial data are sufficiently regular, there is a unique solution for (1.1) under (1.2) in some maximal time interval $(0, T)$, $T > 0$. In many cases, T is finite and the solution demonstrates singular behavior as time approaches T.

There are at least two types of such singular behavior. First, it is blow-up; more precisely, the solution becomes unbounded at T. This happens, for instance, for some solutions to the Cahn-Hilliard equation when f is a cubic polynomial with negative coefficients (Elliott-Zheng [4]). On the other hand, rupture may develop even the solution remains bounded. Starting with a positive initial function, the thin film equation ($A(u) = u^n$, $1 < n < 3$ and $f = 0$) admits a solution which touches down in finite time. Subsequently the equation losses parabolicity and the fourth derivative of the solution becomes infinite ([5]). Rupture could also happen due to the presence of driving forces. For the thin film equations with destabilizing forces, blow-up and rupture could happen simultaneously ([5]).

To proceed further, we note that two basic properties of (1.1) under (1.2). First, we have the conservation law

$$\int_\Omega u(x,t)dx = \int_\Omega u(x,0)dx. \tag{1.3}$$

Second, there is a free energy associated to (1.1), namely,

[a] e-mail: 1403755737@qq.com
[b] e-mail: xgliu@fudan.edu.cn (corresponding author)

Published online: 10 September 2024

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega F(u),$$

where F is the primitive function of f with $F(0) = 0$. We have the energy dissipation relation

$$\mathcal{E}(u(\cdot,t)) + \int_0^t \int_\Omega A(u)|\nabla(-f(u) + \Delta u)|^2 dxdt = \mathcal{E}(u(\cdot,0)). \quad (1.4)$$

In particular, the energy is a Liapunov function for (1.1).

With its strong physical background, (1.1) constitutes an important class of fourth order parabolic equations that deserves a systematic study.

When A is a positive constant and f is the gradient of a double-well potential, the energy is bounded from below, and long time solvability is rather easy to establish. Subsequently most works are concerned with issues like asymptotic behavior, spinodal decomposition, and classification of the steady states. When A depends on the concentration, very few results are known. From the expression for the energy, we see that it is bounded from below when the primitive of f is sublinear. In this case one expects that the equation behaves nicely. On the other hand, the energy becomes unbounded when the primitve grows in a superlinear way, so the solution may behave badly in this case. Taking $f = -u^p$, one may compare (1.1) with the semilinear heat equation

$$u_t = \Delta u + u^p, \quad p > 1, \quad (1.5)$$

which is the negative L^2-gradient flow of the same energy. The classical result of Fujita [6] tells us that under the Dirichlet condition some (nonnegative) solutions of (1.5) blow up in finite time. In view of this, one may guess that some solutions of (1.1) blow up in finite or infinite time in the superlinear case while all solutions exist globally in the sublinear case. In fact, such assertion is made precise by Hocherman and Rosenau in [7]. In this paper the authors propose a conjecture on the blow-up and long time existence of solutions for a large class of general fourth order parabolic equations. When applied to (1.1), it states as,

Conjecture 1.1 *Hocherman-Rosenau conjecture*

Consider (1.1) ($N = 1$) under (1.2) where $A(z) > 0$, $z \in \boldsymbol{R}$, and $f'(z) < 0$ for $z \geq M$, for some M. Then

$$\lim_{z\to\infty}\left|\frac{f(z)}{z}\right| = \begin{cases} \infty, & u \text{ blows up in finite time,} \\ \text{finite}, & \text{marginally linear case,} \\ 0, & \text{the system has globally stable solution(s).} \end{cases} \quad (1.6)$$

All results previously known support this conjecture. For instances, when A is a positive constant, [4] showed that when $N \leq 3$ and f behaves like $-z^3$ for large values of $|z|$, all solutions of (1.1) with sufficiently negative energy blow up in finite time. Their results were further extended in $N = 1$ by Novick-Cohen [NC] to more general f. In Bernoff-Bertozzi [8], the Childress-Spiegel equation ($A = 1$, $f = -z - z^2$) was shown to have finite time blow-up with a self-similar singularity. The blow-up rate of the solution was further studied in Bertozzi-Pugh [9] and Chou [10]. Very few results are known when A is non-constant. In Yin [11] global solutions were obtained when A is positive and F is bounded from below.

In particular, for one dimension the thin film type equation, for simplicity, we consider power law case, i.e., the mobility $A(u) = u^n$, $n > 0$, and $f'(u)u^n = -u^m$ in (1.1), that is

$$f(u) = -q^{-1}u^q, \quad q = m - n + 1.$$

For thin liquid films, the fourth-order term of (1.1) comes from surface tension between the liquid and air and also incorporates any slippage at the liquid/solid interface [12].

Only nonnegative solution are interested. Using the conservation law (1.3) and Gagliardo-Nirenberg interpolation inequality it can be shown that the energy is bounded from below if $q < 3$. This motivated Bertozzi and Pugh [13] to propose the following modification of Hocherman-Rosenau's conjecture for thin film type equation:

Conjecture 1.2 *Modified Hocherman-Rosenau conjecture*

Setting as above, assuming that $f' \leq 0$, consider (1.1) under the periodic boundary conditions. Then

$$\lim_{z\to\infty}\frac{|f'(z)|}{z^2} = \begin{cases} \infty & \text{supercritical: blow-up possible,} \\ \text{finite} & \text{critical case,} \\ 0 & \text{subcritical: solutions are globally bounded.} \end{cases}$$

Moreover, finite time blow-up is possible when $\lim_{z\to\infty}\sqrt{A(z)}f'(z) = -\infty$.

One may consult [13] for a heuristic argument leading to this conjecture. In the same paper they solved the conjecture in the subcritical case. Next, in [14] and [15] the conjecture for the special case $A(z) = z$ and $f(z) = -z^q$, $q \geq 3$ was studied when the periodic condition is replaced by considering initial data with compact support. They showed that every weak solution of the Cauchy problem blows up in finite time whenever its initial energy is negative.

Let us recall the definition of a weak period nonnegative solution:

Definition 1.1 A nonnegative function u, P period function, in

$$L^\infty(0, T; H^1([0, P])) \cap L^2(0, T; H^2([0, P]))$$

a weak solution of (1.1) if for all $\phi \in \mathcal{F}$,

$$\int_0^T \int_0^P u\phi_t = \int_0^T \int_0^P A' u_x \phi_x(-f(u) + u_{xx}) + \int_0^T \int_0^P A\varphi_{xx}(-f(u) + u_{xx}), \tag{1.7}$$

where

$$\mathcal{F} = \left\{\varphi : \varphi_t \in L^2([0, P] \times (0, T)),\ \varphi \in L^2(0, T; H^2(\Omega)),\ \varphi = 0 \text{ near } t = 0, \text{ and } T\right\}.$$

There is a nonnegative period weak solution to see Bernis-Friedman [16], Bertozzi-Pugh [9, 13].

We note that the local existence of the positive solution implies that the solution is smooth, as $A(u) \geq a_0(T) > 0$, (1.1) is parabolic equation, and by using the Gagliardo-Nirenberg inequality, the $L^\infty(0, T; H^1(\Omega))$- and $L^2(0, T; H^3(\Omega))$-norms of u can be estimated by constants depending on $\|u\|_{L^\infty}$, $\mathcal{E}(u_0)$, T etc., in the case $N = 1, 2$. Moreover the energy dissipation relation (1.4) holds.

Let us denote by $T_{\max}$ the maximal existence time of the solution u,

$$T_{max} = \sup\{T > 0, u(x, t)\, exist\ on\, [0, T]\},$$

the solution on $[0, T_{max})$ is called the maximal solution. In particular, if $T_{max} = \infty$, we set $T_\infty = T_{max}$.

In the present article we consider the blow-up part of the Conjecture 1.2 in the case of power law. We rewrite the following the thin film type equation, power law case,

$$h_t = -(h^n h_{xxx})_x - (h^m h_x)_x \tag{1.8}$$

with initial date $h|_{t=0} = h_0 > 0$ and periodic condition $h(x + P, t) = h(x, t)$. We have the conservation law

$$\int h(x, t)dx = \int h_0(x)dx = c_0.$$

The local existence of the positive period solution for $h_0 > 0$ of (1.8) was proved by Bertozzi and Pigh, to see Theorm 3.3 in [13].

We recall the steady state solution h_{ss} of (1.8) if $h_{ss} = h$ satisfies

$$h_{xxx} - h^{m-n} h_x = 0, \tag{1.9}$$

and

$$\int h_{ss} = \int h_0 = c_0.$$

Note that

$$q = m - n + 1.$$

The energy

$$\mathcal{E}(h) = \int_0^P \left(\frac{1}{2}h_x^2 - \frac{h^{q+1}}{q(q+1)}\right)dx,$$

and the energy is decreasing:

$$\frac{d\mathcal{E}(h)}{dt} = -\int_0^P h^n(h_{xxx} + h^{m-n}h_x)^2 dx < 0,$$

moreover

$$\mathcal{E}(h_0) - \mathcal{E}(h)(t) = \int_0^t \int_0^P h^n(h_{xxx} + h^{m-n}h_x)^2. \tag{1.10}$$

For any given $c_0 > 0$, let

$$\gamma \equiv \inf\left\{\mathcal{E}(w) : w \in H^1[0, P],\ \int w = c_0,\ \text{ and } w \text{ is a positive steady state of (1.8)}\right\}.$$

It is known (Laugesen-Pugh [17]) that the energy of every positive periodic steady state h is greater than or equal to the energy of its average. We define, $\overline{h} = \frac{1}{P}\int_0^P hdx = \frac{1}{P}c_0$, then

$$\gamma(P, q, c_0) \equiv \mathcal{E}(\overline{h}) = -\frac{c_0^{q+1}}{q(q+1)|P|^q} < 0.$$

We shall prove the Conjecture 1.2 in the case of super-critical. Our main theorem is

Theorem 1.1 *Assume $q > 3$. Then the solution of (1.8) blows up at T_∞ or at $T_{max} < \infty$, for suitable initial date $h_0 > 0$, with $\int h_0 = c_0$ and $h_0 \in H^1$.*

Remark 1.1 (1) In general, the blows up occur in $\lim_{t \uparrow T_{max}} \|h_{xx}\|_{L^1}(t) = \infty$, for example, the solution h of the Modified Kuramoto-Sivashinsky equation, to see [8, 13].

(2) For thin film equation, the blows up occur in $\{(x, t) : h = 0\}$, because the equation (1.8) is degenerate parabolic equation, to see [5].

(3) We choose initial data h_0 such that

$$\mathcal{E}(h_0) < \mathcal{E}(\overline{h_0}) = \gamma,$$

then solution of (1.8) blows up.

2 The proof of theorem 1.1

First we note that the energy is decreasing:

$$\frac{d\mathcal{E}(h)}{dt} = -\int_0^P h^n (h_{xxx} + h^{m-n} h_x)^2 dx < 0,$$

moreover

$$\mathcal{E}(h_0) - \mathcal{E}(h)(t) = \int_0^t \int_0^P h^n (h_{xxx} + h^{m-n} h_x)^2. \tag{2.1}$$

We prove the theorem by contrary. If the maximal solution of (1.8) does not blow up in finite time. Further suppose that it is globally uniformly bounded, or equivalently, there is some $M > 0$ such that

$$||h||_{L^\infty}(t) \le M \quad \text{for } t \in (0, \infty),$$

then the fact that $\mathcal{E}(h)(t) \le \mathcal{E}(h_0)$ and $||h||_{L^\infty}(t) \le M$ implies

$$||h||_{H^1}(t) \le C(\mathcal{E}(h_0), M, P).$$

So by (2.1) we have

$$\int_0^P h^n (h_{xxx} + h^{m-n} h_x)^2 (t_j) \to 0$$

for some series $t_j \to \infty$.

Suppose $h(\cdot, t_j) \to h_{ss}(\cdot)$, a steady state. Then h_{ss} satisfies

$$h_{xxx} + h^{m-n} h_x = 0,$$

and h_{ss} is not equivelent zero as $\int_0^P h_{ss} = \int_0^P h_0 > 0$, and h_{ss} is a nonnegative periodic steady state. Integrating, h_{ss} satisfies

$$h_{xx} + \frac{h^q - D}{q} = 0 \tag{2.2}$$

with $D = \frac{1}{P} \int_0^P h^q > 0$.

On the other hand, since $h_{ss} \in L^\infty$ and $||h_{ss}||_{H^1} \le C$, we know that $h_{ss} \in C^{1/2}$, moreover by (2.2), $h_{ss} \in C^{2,1/2}$, by elliptic equation regularity. So that there are two cases: (i) h_{ss} is positive periodic function; (ii) h_{ss} is a droplet with zero contact angle.

Case (i): If h_{ss} is a positive periodic nonconstant steady state, then by Theorem 6 of [17] we have

$$\mathcal{E}(h_{ss}) > \mathcal{E}(\overline{h_{ss}}).$$

where $\overline{h_{ss}} = \frac{1}{P} \int_0^P h_{ss}$, and

$$\mathcal{E}(\overline{h_{ss}}) = -P \frac{\overline{h_{ss}}^{q+1}}{q(q+1)} = -\frac{P\overline{h_0}^{q+1}}{q(q+1)} = \gamma(P, q, c_0).$$

Case (ii): If h_{ss} is a droplet with zero contact angle, then h_{ss} satisfies (2.2). Since $D > 0$ for $q \ge 3$, we can rescale the solution h_{ss} as follows

$$K(x) = D^{-1/q} h_{ss}(D^{\frac{1}{2q} - \frac{1}{2}} x)$$

and K satisfies

$$K_{xx} + \frac{K^q - 1}{q} = 0.$$

We fix the solution by requiring that K attains its minimum height 0 at $x = 0$, $K'(0) = 0$. Then we have

$$\frac{1}{2}K_x^2 + \frac{1}{q}[\frac{K^{q+1}}{q+1} - K] = 0. \tag{2.3}$$

Define

$$H(K) = \frac{1}{q}[\frac{K^{q+1}}{q+1} - K].$$

We have

$$\mathcal{E}(K) = \frac{1}{q}\int_0^P K - \frac{2}{q(q+1)}\int_0^P K^{q+1}.$$

Since the maximum value β of K must satisfy

$$2H(\beta) = K'(0)^2 = 0$$

we have $\beta = (1+q)^{\frac{1}{q}}$. From $\frac{dx}{dK} = \frac{1}{\sqrt{-2H(K)}}$ we obtain

$$\begin{aligned} q\mathcal{E}(K) &= 2\left[\frac{-2}{q+1}\int_0^\beta K^{q+1}\frac{dx}{dK}dK + \int_0^\beta K\frac{dx}{dK}dK\right] \\ &= \sqrt{2q}\left[\frac{-2}{q+1}\int_0^\beta \frac{K^{q+1}dK}{\sqrt{K - \frac{K^{q+1}}{q+1}}} + \int_0^\beta \frac{KdK}{\sqrt{K - \frac{K^{q+1}}{q+1}}}\right] \\ &= \sqrt{\frac{2}{q}}(1+q)^{3/(2q)}\int_0^1 \frac{x^{\frac{3}{2q}}(\frac{1}{x} - 2)}{\sqrt{1-x}}. \end{aligned}$$

Define

$$I_q = \int_0^1 \frac{x^{\frac{3}{2q}}(\frac{1}{x} - 2)}{\sqrt{1-x}}.$$

Then

$$I_3 = \int_0^1 \frac{x^{1/2}(\frac{1}{x} - 2)}{\sqrt{1-x}} = 0.$$

We shall prove that $I_q \geq I_3 = 0$ for $q \geq 3$ in following.

In deed,

$$\begin{aligned} \frac{dI_q}{dq} =& \frac{3}{2q^2}\int_0^1 \frac{x^{\frac{3}{2q}}|\ln x|(\frac{1}{x} - 2)}{\sqrt{1-x}}dx \\ =& \frac{3}{2q^2}\left[\int_0^{1/2} \frac{x^{\frac{3}{2q}}|\ln x|(\frac{1}{x} - 2)}{\sqrt{1-x}}dx - \int_{1/2}^1 \frac{x^{\frac{3}{2q}}|\ln x|(2 - \frac{1}{x})}{\sqrt{1-x}}dx\right] \\ =& \frac{3}{2q^2}[A - B]. \end{aligned}$$

For $x \in (0, \frac{1}{2}]$ we first have

$$-\ln x \geq 2(1-x)^2.$$

So

$$\begin{aligned} A &= \int_0^{1/2} \frac{x^{\frac{3}{2q}}|\ln x|(\frac{1}{x} - 2)}{\sqrt{1-x}}dx \\ &\geq \int_0^{1/2} x^{\frac{3}{2q}}(\frac{1}{x} - 2)2(1-x)^{3/2}dx \end{aligned}$$

$$\geq \frac{1}{\sqrt{2}}\int_0^{1/2} x^{3/(2q)}\left(\frac{1}{x}-2\right)dx$$
$$= \frac{1}{\sqrt{2}}\frac{4q^2}{3(2q+3)}\left(\frac{1}{2}\right)^{3/(2q)}.$$

For B we use the inequality

$$\ln x \leq 2(1-x) \quad \text{for } x \in [\frac{1}{2}, 1].$$

$$B = \int_{1/2}^{1} \frac{x^{\frac{3}{2q}}|\ln x|(2-\frac{1}{x})}{\sqrt{1-x}}dx$$
$$\leq 2\int_{1/2}^{1} x^{3/(2q)}\sqrt{1-x}(2-\frac{1}{x})dx$$
$$\leq \sqrt{2}\int_{1/2}^{1} x^{3/(2q)}(2-\frac{1}{x})dx$$
$$= \sqrt{2}\left[\frac{2q(3-2q)}{3(3+2q)} + \frac{4q^2}{3(3+2q)}(\frac{1}{2})^{3/(2q)}\right].$$

Thus

$$A - B \geq \frac{1}{\sqrt{2}}\frac{4q^2}{3(2q+3)}\left(\frac{1}{2}\right)^{3/(2q)} - \sqrt{2}\left[\frac{2q(3-2q)}{3(3+2q)} + \frac{4q^2}{3(3+2q)}\left(\frac{1}{2}\right)^{3/(2q)}\right]$$
$$= \frac{2q}{3(2q+3)}\left[\left(\sqrt{2}-(\sqrt{2}-\frac{1}{\sqrt{2}})\left(\frac{1}{2}\right)^{3/(2q)}\right)2q - 3\sqrt{2}\right]$$
$$> \frac{2q}{3(2q+3)}\left[\left(\sqrt{2}-(\sqrt{2}-\frac{1}{\sqrt{2}})\right)2q - 3\sqrt{2}\right]$$
$$= \frac{2q}{3(2q+3)}\sqrt{2}(q-3)$$
$$\geq 0.$$

Turing back to the energy $\mathcal{E}(h_{ss})$, we get

$$\mathcal{E}(K) = D^{-(\frac{1}{2}+\frac{3}{2q})}\mathcal{E}(h_{ss}).$$

So

$$\mathcal{E}(h_{ss}) \geq 0, \quad \text{for } q \geq 3.$$

If we can choose the initial date $h_0 > 0$ such that

$$\mathcal{E}(h_0) < \mathcal{E}(\overline{h_0}) < 0,$$

(see following Lemma 2.1) then we get a contrary.
We have

Lemma 2.1 *There exist two positive constants l_0 and α_0 large enough such that $h_0(x) = l_0\exp(-(x-P/2)\alpha_0)$ satisfies $\mathcal{E}(h_0) < \mathcal{E}(\overline{h_0})$.*

Proof Suppose $h_0(x) = l\exp(-\alpha(x-P/2))$. Then

$$\int_0^P \frac{1}{2}h_{0x}^2 = \frac{l^2\alpha}{4}(e^{\alpha P} - e^{-\alpha P}),$$
$$\int_0^P h_0^{q+1} = \frac{l^{q+1}}{\alpha(q+1)}(e^{(q+1)P\alpha/2} - e^{-(q+1)\alpha/2}).$$
$$\overline{h_0} = \left(\frac{l(e^{\alpha P/2} - e^{-\alpha P/2})}{\alpha P}\right)^{q+1}.$$

Then

$$\mathcal{E}(h_0) = \frac{\alpha l^2}{4}(e^{P\alpha} - e^{-P\alpha}) - \frac{l^{q+1}}{q(q+1)^2\alpha}(e^{(q+1)P\alpha/2} - e^{-(q+1)P\alpha/2}),$$

$$\mathcal{E}(\overline{h_0}) = -\frac{P}{q(q+1)}\left(\frac{l}{P}\right)^{q+1}\left(\frac{e^{P\alpha/2} - e^{P\alpha/2}}{\alpha}\right)^{q+1}.$$

Define

$$A(\alpha) = \frac{1}{q(q+1)P^q}\left(\frac{e^{P\alpha/2} - e^{-P\alpha/2}}{\alpha}\right)^{q+1},$$
$$B(\alpha) = \frac{1}{q(q+1)}\frac{e^{(q+1)P\alpha/2} - e^{-(q+1)P\alpha/2}}{\alpha}.$$

Then

$$\lim_{\alpha\to\infty}\frac{A(\alpha)}{B(\alpha)} = 0.$$

So there exists a $\alpha_0 > 0$ such that $A(\alpha_0) < \frac{1}{2}B(\alpha_0)$. Now for fixed α_0,

$$\lim_{l\to\infty}\frac{\mathcal{E}(\overline{h_0})}{\mathcal{E}(h_0)} = \frac{A(\alpha_0)}{B(\alpha_0)} > 2.$$

Thus we can take $l_0 > \alpha_0$ such that for $h_0(x) = l_0\exp(-\alpha_0(x - P/2))$

$$\mathcal{E}(h_0) < \mathcal{E}(\overline{h_0}).$$

The proof is finished. □

Acknowledgements Thanks to the reviewers for his/her very good revision suggestions, which made this paper have such a good expression. This work was supported partly by NSFC grant 11971113. The second author thanks Professor K. S. Chou for his help in the research of the issue.

Data Availability Statement No Data associated in the manuscript.

References

1. J.W. Cahn, J. Hilliard, Free energy of a nonuniform system I Interfacial free energy. J. Chem. Phys. **28**, 258–267 (1958)
2. E. Beretta, M. Bertsch, R.D. Passo, Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. Arch. Rational Mech. Anal. **129**, 175–200 (1995)
3. A. Oron, S.H. Davis, S.G. Bankoff, Long-scale evolution of thin liquid films. Rev. Mod. Phy. **69**, 931–980 (1997)
4. C.M. Elliott, S. Zheng, On the Cahn-Hilliard equation. Arch. Rational Mech. Anal. **96**, 339–357 (1986)
5. Bertozzi, A.L., Brenner, M.P., Dupont, T.F., and Kadanoff L.P., Singularities and similarities in interface flow, In "Trends and Perspectives in Applied Mathematics", L. Sirovich, ed., Applied Mathematical Sciences, No. 100, Springer Verlay, New York, 1994, 155-208
6. H. Fujita, On some nonexistence and nonuniqueness. Nonlinear Funct. Anal.: Proc. **18**, 105 (1970)
7. T. Hocherman, P. Rosenau, On KS-type equations describing the evolution and rupture of a liquid interface. Physica D **67**, 113–125 (1993)
8. A.J. Bernoff, A.L. Bertozzi, Singularities in a modified Kuramoto-Sivashinsky equation describing interface motion for phase transition. Phys. D **85**, 375–404 (1995)
9. A.L. Bertozzi, M. Pugh, The lubrication approximation for thin viscous films, regularity and long time behavior of weak solution. Comm. Pure Appl. Math. **49**, 85–123 (1996)
10. K.S. Chou, On a modified Kuramoto-Sivashinsky equation. Diff. Integral Equations **15**, 863–874 (2002)
11. J. Yin, On the Cahn-Hilliard equation with nonlinear principal part. J. Partial Diff. Equations **7**, 77–96 (1994)
12. H.P. Greenspan, On the motion of a small viscous droplet that wets a surface. J. Fluid Mech. **84**, 125–143 (1978)
13. A.L. Bertozzi, M. Pugh, Long-wave instabilities and saturation in thin film equations. Comm. Pure App. Math. **51**, 625–661 (1998)
14. A.L. Bertozzi, M. Pugh, Finite-time blow-up of solutions of some long wave unstable thin film equations. Indiana U. Math. J. **49**, 1323–1366 (2000)
15. C.M. Elliott, H. Garcke, On the Cahn-Hilliard equation with degenerate mobility. SIAM J. Math. Anal. **27**, 404–423 (1996)
16. F. Bernis, A. Friedman, Highter order nonlinear degenerate paeabolic equations. J. Dfferrntial equations **83**, 179–206 (1990)
17. R.S. Laugesen, M.C. Pugh, Energy levels of steady states for thin film type equations. J. Differential Equations **182**, 377–415 (2002)