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Generalized ghost dark energy model in extended modified theory of symmetric teleparallel gravity

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Abstract This paper focuses on developing a generalized ghost dark energy model $f(\mathcal{Q}, \mathcal{T})$ through the correspondence scheme for a noninteracting scenario involving pressureless matter and a power-law scale factor. We analyze the cosmological implications of our model using the equation of state parameter, $\omega_{GGDE} - \omega'_{GGDE}$ and the $r-s$ planes. We also examine stability of the model by considering squared speed of sound parameter. The equation of state parameter shows the phantom era, while the squared speed of sound indicates a stable generalized ghost dark energy model throughout the cosmic evolution. The $\omega_{GGDE} - \omega'_{GGDE}$ plane depicts a freezing region, whereas the $r-s$ plane corresponds to the Chaplygin gas model. We conclude that our results are consistent with the current observational data.

1 Introduction

In the early twentieth century, Albert Einstein introduced the general theory of relativity (GR), which revolutionized our comprehension of the cosmos. It is expanding through a significant accumulation of accurate observations and delving into the concealed realms of the universe in current cosmology. The current concept of cosmic expansion has been reinforced by several observable findings, including large-scale structures, cosmic microwave background radiations and supernovae type Ia [1]. The universe in which we live is experiencing a unique epoch, in which it expands at an accelerated rate. Cosmologists examined this expansion by conducting numerous attempts on distant galaxies. It is assumed that this universe expansion is due to an extraordinary force called dark energy (DE), which works through immense negative pressure. Several models have been proposed to explain phenomena related to DE and the evolution of the universe. There are two main approaches to study DE, i.e., dynamical DE models and modified theories of gravity [2].

In recent years, scholars have proposed various approaches to address these issues, yet they remain enigmatic to this day. To characterize DE, researchers found the equation of state (EoS) parameter beneficial for the model being examined. The Veneziano ghost DE (GDE) model possesses noteworthy physical properties within the cosmos [3]. In flat spacetime, the GDE does not contribute to vacuum energy density but in curved spacetime, it enhances small vacuum energy density in proportion to $\Lambda^3_{QCD}H$, where Λ_{QCD} represents the quantum chromodynamics (QCD) mass scale and H denotes the Hubble parameter. With $\Lambda_{QCD} \sim 100MeV$ and $H \sim 10^{-33}eV$, $\Lambda^3_{QCD}H$ gives the order of observed GDE density. This small value efficiently provides the essential exotic force driving the accelerating universe, thereby resolving the fine-tuning issue [4]. The energy density of GDE is given as

$$\rho_{GDE} = \alpha H,$$

where α is an arbitrary constant.

The GDE model solves various issues effectively, but it encounters stability problems [5]. It suggests that the energy density relies not only on H explicitly but also on higher-order terms of H, leading to what is known as the generalized GDE (GGDE) model. In this model, the vacuum energy associated with the ghost field can be regarded as a dynamic cosmological constant. In reference [6], the author explained that the Veneziano QCD ghost field's contribution to vacuum energy does not precisely follow the H-order. The additional term H^2 is particularly important during the initial phase of the universe evolution, serving as early DE [7]. Instead, a subleading term H^2 arises because the conservation of the energy–momentum tensor holds independently [8]. The GGDE density is given as follows

$$\rho_{GGDE} = \alpha H + \beta H^2, \tag{1}$$

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where β is another arbitrary constant.

Numerous DE models can be explored to study the ongoing accelerated expansion of the universe. Khodam et al. [9] investigated the reconstruction of $f(\mathcal{R}, \mathcal{T})$ gravity (here $\mathcal{R}$ and $\mathcal{T}$ are the Ricci scalar and trace of the energy–momentum tensor, respectively) within the GGDE model, delving into cosmic evolution through the analysis of cosmological parameters. Malekjani [10] explored the cosmographic aspects of GGDE and concluded that this model demonstrates strong consistency with observational data. Ebrahimi et al. [11] studied the interacting GGDE within a non-flat cosmos, revealing that the EoS parameter aligns with the phantom era of the universe. Chattopadhyay [12] reconstructed the QCD ghost $f(T)$ model (T represents the torsion scalar) to examine the cosmic evolution and found both the phantom and quintessence phases of the cosmos. Sharif and Saba [13] explored the cosmography of GGDE in $f(\mathcal{G}, \mathcal{T})$ ($\mathcal{G}$ is the Gauss–Bonnet invariant) gravity. Biswas et al. [14] investigated the evolution trajectories of the EoS parameter, $\omega_D - \omega_D'$ and the state finder planes in GGDE DGP model. Recently, Sharif and Ajmal [15] studied the cosmography of GGDE $f(\mathcal{Q})$ theory.

Several gravitational theories, including GR, are acquiring importance because of their ability to explain the accelerating expansion of the universe. The Levi-Civita connection in GR explains gravitational interactions in Riemannian space based on the assumption of geometry free from torsion and non-metricity. Moreover, it is important to consider that the general affine connection has a broader formulation [16], and GR can be derived within alternative spacetime geometries beyond Riemannian. Teleparallel gravity presents an alternative framework to GR, where gravitational interaction is characterized by torsion, denoted as T [17]. Teleparallel equivalent of GR utilizes the Weitzenböck connection, which entails zero curvature and non-metricity [18]. In the study of a cosmological model within Weyl–Cartan (WC) spacetime, the Weitzenböck connection is investigated with vanishing of the combined curvature and scalar torsion [19].

In a torsion-free space, gravity is modified by non-metricity, defined as $\mathcal{Q}_{\lambda\zeta\xi} = \nabla_\lambda g_{\zeta\xi}$. This theory is called symmetric teleparallel gravity (STG) [20]. Jimenez et al. [21] recently introduced $f(\mathcal{Q})$ theory also named as nonmetric gravity, the scalar function $\mathcal{Q}$ represents the non-metricity. Lazkoz et al. [22] described the constraints on $f(\mathcal{Q})$ gravity by using polynomial functions of the redshift. The energy conditions for two distinct $f(\mathcal{Q})$ gravity models have also been discussed [23]. The behavior of cosmic parameters in the same context was examined by Koussour et al. [24]. Chanda and Paul [25] studied the formation of primordial black holes in this theory.

Xu et al. [26] extended $f(\mathcal{Q})$ gravity to $f(\mathcal{Q}, \mathcal{T})$ gravity by adding trace of the energy–momentum tensor into $f(\mathcal{Q})$ theory. The Lagrangian governing the gravitational field is considered to be a gravitational function of both $\mathcal{Q}$ and $\mathcal{T}$. The field equations of the corresponding theory are derived by varying the gravitational action with respect to both the metric and the connection. The goal of introducing this theory is to analyze its theoretical implications, its consistency with real-world experimental evidence and its applicability to cosmological scenarios. Nájera and Fajardo [27] studied cosmic perturbation theory in $f(\mathcal{Q}, \mathcal{T})$ gravity and found that non-minimal coupling between matter and curvature perturbations has a major influence on the universe evolution. Pati et al. [28] investigated the dynamical features and cosmic evolution in the corresponding gravity. Mandal et al. [29] studied the cosmography of holographic DE $f(\mathcal{Q}, \mathcal{T})$ gravity.

In this paper, we use a correspondence technique to recreate the GGDE $f(\mathcal{Q}, \mathcal{T})$ model in a noninteracting scenario. We investigate cosmic evolution using the EoS parameter and phase planes. The paper is structured as follows. In section **2**, we introduce $f(\mathcal{Q}, \mathcal{T})$ gravity and its corresponding field equations. In section **3**, we explore the reconstruction procedure to formulate the GGDE $f(\mathcal{Q}, \mathcal{T})$ paradigm. In section **4**, we analyze cosmic behavior using the EoS parameter and phase planes. We also examine the stability of the resulting model. Finally, we discuss our findings in the last section.

2 The $f(\mathcal{Q}, \mathcal{T})$ theory

This section presents the basic structure of modified $f(\mathcal{Q}, \mathcal{T})$ theory. In this theory, the framework of spacetime is the torsion-free teleprallel geometry, i.e., $R^\rho_{\eta\gamma\nu} = 0$ and $T^\rho_{\gamma\nu} = 0$. The connection in the WC geometry is expressed as [26]

$$\hat{\Gamma}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} + \mathbb{C}^\lambda_{\alpha\beta} + \mathbb{L}^\lambda_{\alpha\beta}, \tag{2}$$

where $\Gamma^\zeta_{\gamma\alpha}$ is the Levi-Civita connection, $\mathbb{C}^\lambda_{\alpha\beta}$ is the contortion tensor, and $\mathbb{L}^\lambda_{\alpha\beta}$ is the deformation tensor, given by

$$\Gamma^\zeta_{\mu\alpha} = \frac{1}{2} g^{\zeta\sigma}(g_{\sigma\alpha,\mu} + g_{\sigma\mu,\alpha} - g_{\sigma\mu,\alpha}), \tag{3}$$

$$\mathbb{C}^\lambda_{\alpha\beta} = \hat{\Gamma}^\lambda_{[\alpha\beta]} + g^{\lambda\phi} g_{\alpha\kappa} \hat{\Gamma}^\kappa_{[\beta\phi]} + g^{\lambda\phi} g_{\beta\kappa} \hat{\Gamma}^\kappa_{[\alpha\phi]}, \tag{4}$$

and

$$\mathbb{L}^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\phi}(\mathcal{Q}_{\beta\alpha\phi} + \mathcal{Q}_{\alpha\beta\phi} - \mathcal{Q}_{\lambda\alpha\beta}), \tag{5}$$

where

$$\mathcal{Q}_{\lambda\alpha\beta} = \nabla_\lambda g_{\alpha\beta} = -g_{\alpha\beta,\lambda} + g_{\beta\phi}\hat{\Gamma}^\phi_{\alpha\lambda} + g_{\phi\alpha}\hat{\Gamma}^\phi_{\beta\lambda}. \tag{6}$$

The gravitational action can be written as [30]

$$S = \frac{1}{2k}\int g^{\alpha\beta}(\Gamma^{\rho}_{\phi\alpha}\Gamma^{\phi}_{\beta\rho} - \Gamma^{\rho}_{\phi\rho}\Gamma^{\phi}_{\alpha\beta})\sqrt{-g}d^4x. \tag{7}$$

Using the anti-symmetric relation ($\Gamma^{\mu}_{\varsigma\alpha} = -\mathbb{L}^{\mu}_{\varsigma\alpha}$) in Eq.(7), the integral action takes the form

$$S = -\frac{1}{2k}\int g^{\alpha\beta}(\mathbb{L}^{\rho}_{\phi\alpha}\mathbb{L}^{\phi}_{\beta\rho} - \mathbb{L}^{\rho}_{\phi\rho}\mathbb{L}^{\phi}_{\alpha\beta})\sqrt{-g}d^4x. \tag{8}$$

This action is known as the action of STG which is equivalent to the Einstein–Hilbert action. There are some significant differences between two gravitational paradigms. One of them is that the vanishing of curvature tensor in STG causes the system to appear as flat structure throughout. Furthermore, the gravitational effects occur due to variations in the length of vector itself, rather than rotation of an angle formed by two vectors in parallel transport.

Now, we look at an extension of STG and take an arbitrary function of $\mathcal{Q}$, the above action takes the form

$$S = \int\left[\frac{1}{2k}f(\mathcal{Q}) + L_m\right]\sqrt{-g}d^4x, \tag{9}$$

where

$$\mathcal{Q} = -g^{\alpha\beta}(\mathbb{L}^{\rho}_{\mu\alpha}\mathbb{L}^{\mu}_{\beta\rho} - \mathbb{L}^{\rho}_{\mu\rho}\mathbb{L}^{\mu}_{\alpha\beta}), \tag{10}$$

which leads to $f(\mathcal{Q})$ theory. If we couple this theory with the trace of the energy–momentum tensor, we can obtain $f(\mathcal{Q}, \mathcal{T})$ theory whose action is given as

$$\mathcal{S} = \frac{1}{2k}\int f(\mathcal{Q},\mathcal{T})\sqrt{-g}d^4x + \int L_m\sqrt{-g}d^4x, \tag{11}$$

here L_m represents the matter Lagrangian. The traces of non-metricity tensor are given by

$$\mathcal{Q}_{\rho} = \mathcal{Q}^{\alpha}_{\rho\alpha}, \quad \tilde{\mathcal{Q}}_{\rho} = \mathcal{Q}^{\alpha}_{\rho\alpha}. \tag{12}$$

The superpotential in view of $\mathcal{Q}$ is written as

$$P^{\rho}_{\alpha\beta} = -\frac{1}{2}\mathcal{L}^{\rho}_{\alpha\beta} + \frac{1}{4}(\mathcal{Q}^{\rho} - \tilde{\mathcal{Q}}^{\rho})g_{\alpha\beta} - \frac{1}{4}\delta^{\rho}_{[\alpha}\mathcal{Q}_{\beta]}. \tag{13}$$

Furthermore, the expression of $\mathcal{Q}$ obtained using the superpotential becomes

$$\mathcal{Q} = -\mathcal{Q}_{\rho\alpha\beta}P^{\rho\alpha\beta} = -\frac{1}{4}(-\mathcal{Q}^{\rho\beta\zeta}\mathcal{Q}_{\rho\beta\zeta} + 2\mathcal{Q}^{\rho\beta\zeta}\mathcal{Q}_{\zeta\rho\beta} - 2\mathcal{Q}^{\zeta}\tilde{\mathcal{Q}}_{\zeta} + \mathcal{Q}^{\zeta}\mathcal{Q}_{\zeta}). \tag{14}$$

Taking the variation of S with respect to the metric tensor as zero yields the field equations

$$\begin{aligned}\delta S = 0 &= \int\frac{1}{2}\delta[f(\mathcal{Q},\mathcal{T})\sqrt{-g}] + \delta[L_m\sqrt{-g}]d^4x\\ 0 &= \int\frac{1}{2}\left(\frac{-1}{2}fg_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta} + f_{\mathcal{Q}}\sqrt{-g}\delta\mathcal{Q} + f_{\mathcal{T}}\sqrt{-g}\delta\mathcal{T}\right)\\ &\quad - \frac{1}{2}\mathcal{T}_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta}d^4x.\end{aligned} \tag{15}$$

Furthermore, we define

$$\mathcal{T}_{\alpha\beta} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\alpha\beta}}, \quad \Theta_{\alpha\beta} = g^{\rho\mu}\frac{\delta\mathcal{T}_{\rho\mu}}{\delta g^{\alpha\beta}}, \tag{16}$$

which implies that $\delta\mathcal{T} = \delta(\mathcal{T}_{\alpha\beta}g^{\alpha\beta}) = (\mathcal{T}_{\alpha\beta} + \Theta_{\alpha\beta})\delta g^{\alpha\beta}$. Inserting the aforementioned factors in Eq.(15), we have

$$\begin{aligned}\delta S = 0 = \int\frac{1}{2}\Big\{&\frac{-1}{2}fg_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta} + f_{\mathcal{T}}(\mathcal{T}_{\alpha\beta} + \Theta_{\alpha\beta})\sqrt{-g}\delta g^{\alpha\beta}\\ &- f_{\mathcal{Q}}\sqrt{-g}(P_{\alpha\rho\mu}\mathcal{Q}^{\rho\mu}_{\beta} - 2\mathcal{Q}^{\rho\beta}_{\alpha}P_{\rho\mu\beta})\delta g^{\alpha\beta} + 2f_{\mathcal{Q}}\sqrt{-g}P_{\rho\alpha\beta}\nabla^{\rho}\delta g^{\alpha\beta}\\ &+ 2f_{\mathcal{Q}}\sqrt{-g}P_{\rho\alpha\beta}\nabla^{\rho}\delta g^{\alpha\beta}\Big\} - \frac{1}{2}\mathcal{T}_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta}d^4x.\end{aligned} \tag{17}$$

Integration of the term $2f_{\mathcal{Q}}\sqrt{-g}P_{\rho\alpha\beta}\nabla^{\rho}\delta g^{\alpha\beta}$ along with the boundary conditions takes the form $-2\nabla^{\rho}(f_{\mathcal{Q}}\sqrt{-g}P_{\rho\alpha\beta})\delta g^{\alpha\beta}$. The terms $f_{\mathcal{Q}}$ and $f_{\mathcal{T}}$ represent partial derivatives with respect to $\mathcal{Q}$ and $\mathcal{T}$, respectively. Finally, we obtain the field equations as

$$T_{\alpha\beta} = \frac{-2}{\sqrt{-g}}\nabla_{\rho}(f_{\mathcal{Q}}\sqrt{-g}P^{\rho}_{\alpha\beta}) - \frac{1}{2}fg_{\alpha\beta} + f_{\mathcal{T}}(\mathcal{T}_{\alpha\beta} + \Theta_{\alpha\beta}) - f_{\mathcal{Q}}(P_{\alpha\rho\mu}\mathcal{Q}^{\rho\mu}_{\beta}$$

$$- 2\mathcal{Q}_{\alpha}^{\rho\mu} P_{\rho\mu\beta}). \tag{18}$$

Its covariant derivative yields the nonconservation equation as

$$\nabla_{\alpha} T_{\beta}^{\alpha} = \frac{1}{f_{\mathcal{T}} - 1}\Bigg[\nabla_{\alpha}\Bigg(\frac{1}{\sqrt{-g}}\nabla_{\rho} H_{\beta}^{\rho\alpha}\Bigg) - \nabla_{\alpha}(f_{\mathcal{T}}\Theta_{\beta}^{\alpha}) - \frac{1}{\sqrt{-g}}\nabla_{\rho}\nabla_{\alpha} H_{\beta}^{\rho\alpha} - 2\nabla_{\alpha} A_{\beta}^{\alpha} + \frac{1}{2} f_{\mathcal{T}}\partial_{\beta}\mathcal{T}\Bigg], \tag{19}$$

where hyper-momentum tensor density is defined as

$$H_{\beta}^{\rho\alpha} = \frac{\sqrt{-g}}{2} f_{\mathcal{T}} \frac{\delta\mathcal{T}}{\delta\hat{\Gamma}_{\rho\alpha}^{\beta}} + \frac{\delta\sqrt{-g}L_m}{\delta\hat{\Gamma}_{\rho\alpha}^{\beta}}. \tag{20}$$

3 Reconstruction of GGDE $f(\mathcal{Q}, \mathcal{T})$ model

In this section, we use a correspondence technique to recreate the GGDE $f(\mathcal{Q}, \mathcal{T})$ model. The line element of the isotropic and spatially homogeneous universe model is given by

$$ds^2 = -dt^2 + \mathrm{a}^2(t)[dx^2 + dy^2 + dz^2], \tag{21}$$

where the scale factor is represented by $\mathrm{a}(t)$. The configuration of isotropic matter with four-velocity fluid u_{μ}, pressure (P_M) and normal matter density (ρ_M) is given as

$$\tilde{\mathcal{T}}_{\mu\nu} = (\rho_M + P_M)u_{\mu}u_{\nu} + P_M g_{\mu\nu}. \tag{22}$$

The modified Friedmann equations in the context of $f(\mathcal{Q}, \mathcal{T})$ gravity are expressed as

$$3H^2 = \rho_{eff} = \rho_M + \rho_{GGDE}, \tag{23}$$

$$2\dot{H} + 3H^2 = P_{eff} = P_M + P_{GGDE}, \tag{24}$$

where $H = \frac{\dot{\mathrm{a}}}{\mathrm{a}}$. The dot demonstrates derivative with respect to cosmic time t. The non-metricity $\mathcal{Q}$ in terms of the Hubble parameter is

$$\mathcal{Q} = -\frac{1}{4}\big[-\mathcal{Q}_{\rho\xi\eta}\mathcal{Q}^{\rho\xi\eta} + 2\mathcal{Q}_{\rho\xi\eta}\mathcal{Q}^{\xi\rho\eta} - 2\tilde{\mathcal{Q}}^{\rho}\mathcal{Q}_{\rho} + \mathcal{Q}^{\rho}\mathcal{Q}_{\rho}\big]. \tag{25}$$

Simplifying this equation leads to

$$\mathcal{Q} = 6H^2. \tag{26}$$

Furthermore, ρ_{GGDE} and P_{GGDE} denote the density and pressure of DE, respectively, given as

$$\rho_{GGDE} = \frac{1}{2} f(\mathcal{Q}, \mathcal{T}) - \mathcal{Q} f_{\mathcal{Q}} - f_{\mathcal{T}}(\rho_M + P_M), \tag{27}$$

$$P_{GGDE} = -\frac{1}{2} f(\mathcal{Q}, \mathcal{T}) + 2 f_{\mathcal{Q}\mathcal{Q}} H + 2 f_{\mathcal{Q}}\dot{H} + \mathcal{Q} f_{\mathcal{Q}}. \tag{28}$$

The conservation equation (19) takes the following form for an ideal fluid

$$\dot{\rho}_M + 3H(\rho_M + P_M) = \frac{1}{(f_{\mathcal{T}} - 1)}\Bigg[2\nabla_{\beta}(P_M\mu^{\beta} f_{\mathcal{T}}) + f_{\mathcal{T}}\nabla_{\beta}\mu^{\beta}\mathcal{T} + 2\mu^{\beta}\mathcal{T}_{\alpha\beta}\nabla^{\alpha} f_{\mathcal{T}\mu^{\beta}}\Bigg]. \tag{29}$$

The first field equation (23) leads to

$$\Omega_M + \Omega_{GGDE} = 1, \tag{30}$$

where $\Omega_M = \frac{\rho_M}{3H^2}$ and $\Omega_{GGDE} = \frac{\rho_{GGDE}}{3H^2}$ represent the fractional energy densities of normal matter and dark source, respectively. Dynamic DE models with energy density proportional to Hubble parameter are crucial to explaining the accelerated expansion of the universe.

Next, we will use a correspondence approach in an ideal fluid configuration to create the GGDE $f(\mathcal{Q}, \mathcal{T})$ model with emphasis on the dust case ($P_M = 0$).

3.1 Noninteracting GGDE $f(\mathcal{Q}, \mathcal{T})$ model

Here, we consider the standard $f(\mathcal{Q}, \mathcal{T})$ function in the following form [31]

$$f(\mathcal{Q}, \mathcal{T}) = f_1(\mathcal{Q}) + f_2(\mathcal{T}), \tag{31}$$

where f_1 and f_2 depend upon $\mathcal{Q}$ and $\mathcal{T}$, respectively. In this scenario, it is evident that there is a minimal coupling between curvature and matter constituents. This version of the generic function reveals that the interaction is purely gravitational and hence easy to handle. This can effectively explain the ongoing expansion of the universe. Moreover, the reconstruction methodology demonstrates that such generated models are physically viable [31]. Using dust fluid and Eq.(31), the field equations (23) and (24) yield

$$3H^2 = \rho_{eff} = \rho_M + \rho_{GGDE}, \quad 2\dot{H} + 3H^2 = P_{eff} = P_{GGDE}, \tag{32}$$

where

$$\rho_{GGDE} = \frac{1}{2} f_1(\mathcal{Q}) + \frac{1}{2} f_2(\mathcal{T}) - \mathcal{Q} f_{1\mathcal{Q}} + f_{2\mathcal{T}} \rho_M, \tag{33}$$

$$P_{GGDE} = -\frac{1}{2} f_1(\mathcal{Q}) - \frac{1}{2} f_2(\mathcal{T}) + 2 f_{1\mathcal{Q}} \dot{H} + 2 f_{1\mathcal{Q}\mathcal{Q}} H + \mathcal{Q} f_{1\mathcal{Q}}. \tag{34}$$

The associated conservation equation (29) reduces to

$$\dot{\rho}_M + 3H\rho_M = \frac{1}{f_{2\mathcal{T}} - 1}\left[2\mathcal{T} f_{2\mathcal{T}\mathcal{T}} + f_{2\mathcal{T}} \dot{\mathcal{T}}\right]. \tag{35}$$

This equation is consistent with the standard continuity equation when the right-hand side is assumed to be zero, implying

$$\dot{\rho}_M + 3H\rho_M = 0 \quad \Longrightarrow \quad \rho_M = \rho_0 \mathrm{a}(t)^{-3}, \tag{36}$$

with the constraint

$$f_{2\mathcal{T}} + 2\mathcal{T} f_{2\mathcal{T}\mathcal{T}} = 0, \tag{37}$$

whose solution provides

$$f_2(\mathcal{T}) = a\mathcal{T}^{\frac{1}{2}} + b, \tag{38}$$

where a and b represent the integration constants.

We utilize Eqs.() and (33), in conjunction with the constraint on $f_2(\mathcal{T})$ as given in (38), to formulate a reconstruction framework employing the correspondence approach. The differential equation for $f_1(\mathcal{Q})$ is expressed as

$$\frac{f_1(\mathcal{Q})}{2} - \mathcal{Q} f_{1\mathcal{Q}} + a\mathcal{T}^{\frac{1}{2}} + \frac{1}{2} b = \alpha H + \beta H^2. \tag{39}$$

We use the power-law solution for the scale factor given as follows

$$\mathrm{a}(t) = \mathrm{a}_0 t^m, \quad m > 0. \tag{40}$$

In this context, a_0 denotes the current value of the scale factor. Employing this relation, the expressions for H, its derivative and the non-metricity scalar in terms of cosmic time t are given as follows

$$H = \frac{m}{t}, \quad \dot{H} = -\frac{m}{t^2}, \quad \mathcal{Q} = 6\frac{m^2}{t^2}.$$

Substituting (40) in (36), it follows that

$$\rho_M = d(\mathrm{a}_0 t^m)^{-3}, \tag{41}$$

where $\rho_0 = d$ for the sake of similicity. Using Eq.(40) in (39), we can find the function $f_1(\mathcal{Q})$ as

$$f_1(\mathcal{Q}) = \sqrt{\mathcal{Q}}\Bigg[- \frac{a\sqrt{d}\, 2^{\frac{3m}{8}+1} 3^{\frac{3m}{8}} \left(\frac{\mathcal{Q}}{m^2}\right)^{-\frac{3m}{8}}}{\left(-\frac{3m}{8} - \frac{1}{2}\right)\sqrt{\mathcal{Q}}} + \frac{2b}{\sqrt{\mathcal{Q}}} + \frac{\beta m \mathcal{Q}}{\sqrt{6}} + \frac{2\, 2^{3/4} \alpha \mathcal{Q}}{3\sqrt[4]{3}} \Bigg]$$
$$+ c_1 \sqrt{\mathcal{Q}}, \tag{42}$$

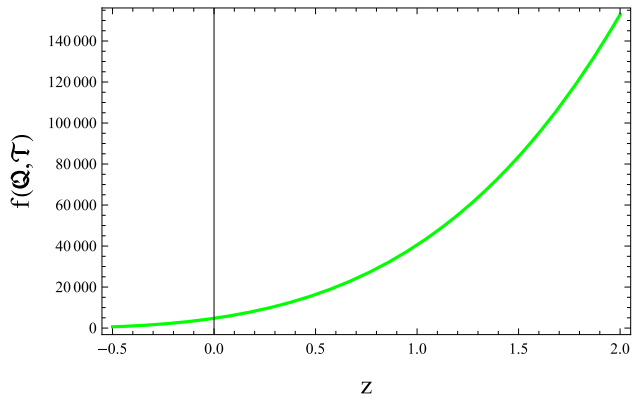

Fig. 1 Plot of GGDE $f(\mathcal{Q}, \mathcal{T})$ model against z

where c_1 is the integration constant. Consequently, the reconstructed $f(\mathcal{Q}, \mathcal{T})$ model is obtained by substituting Eqs.(38) and (42) into (31) as follows

$$f(\mathcal{Q},\mathcal{T}) = \frac{1}{18}\Bigg[-\frac{a\sqrt{d}6^{\frac{3m}{8}+2}\left(\frac{\mathcal{Q}}{m^2}\right)^{-\frac{3m}{8}}}{-\frac{3m}{8}-\frac{1}{2}} + 18\sqrt{\mathrm{ad}} + 54b + 18c_1\sqrt{\mathcal{Q}} + 3\sqrt{6}\beta m\mathcal{Q}^{3/2} + 4\,6^{3/4}\alpha\mathcal{Q}^{3/2}\Bigg]. \tag{43}$$

Now we write down this function in terms of redshift parameter z. The expression for the deceleration parameter is given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 + \frac{1}{m}. \tag{44}$$

In relation to the deceleration parameter, the evolution of the cosmic scale factor can be characterized as

$$\mathrm{a}(t) = t^{\frac{1}{(1+q)}}, \tag{45}$$

Here, we assume a_0 to be equal to unity. It is worth noting that the power-law model corresponds to an expanding universe when $q > -1$. The description of both the expanding phase and the present cosmic evolution is given by

$$H = (1+q)^{-1}t^{-1}, \quad H_0 = (1+q)^{-1}t_0^{-1}. \tag{46}$$

In power-law cosmology, the evolution of the universe expansion is determined by two fundamental parameters, the Hubble constant (H_0) and the deceleration parameter (q). By examining the correlation between the scale factor a and redshift z, we can elucidate

$$H = H_0\Gamma^{1+q}, \tag{47}$$

where $\Gamma = 1 + z$. Using Eq.(47) in (26), $\mathcal{Q}$ appears as

$$\mathcal{Q} = 6H_0^2\Gamma^{2+2q}. \tag{48}$$

The reconstructed model for the GGDE $f(\mathcal{Q}, \mathcal{T})$ in terms of the redshift parameter is derived by substituting this value in Eq.(43), resulting in

$$f(\mathcal{Q},\mathcal{T}) = \frac{16a\sqrt{d}\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{-\frac{3m}{8}}}{3m+4} + \sqrt{\mathrm{ad}} + 3b + \sqrt{6}c_1\sqrt{H_0^2\Gamma^{2q+2}} + (H_0^2\Gamma^{2q+2})^{3/2}6\beta m + 8\sqrt[4]{6}\alpha(H_0^2\Gamma^{2q+2})^{3/2}. \tag{49}$$

For the graphical analysis, we set $a = 1, b = -4$, and $c_1 = -15$. Figure 1 indicates that the reconstructed GGDE model remains positive and increases with respect to z. The GGDE model indicates a rapid expansion, according to this analysis. Figure 2 illustrates the behavior of ρ_{GGDE} and P_{GGDE} with the redshift parameter. The energy density ρ_{GGDE} is positive and increasing, whereas P_{GGDE} is negative and consistent with the DE behavior. We then investigate the properties of ρ_{GGDE} and P_{GGDE} in the context of the reconstructed GGDE $f(\mathcal{Q}, \mathcal{T})$ gravity model. Putting (49) into (33) and (34), we have

$$\rho_{GGDE} = \frac{1}{18}\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{-\frac{3m}{8}}\Bigg[36a\sqrt{d} - 18\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}\Bigg\{\sqrt{\mathrm{ad}}(d-1)$$

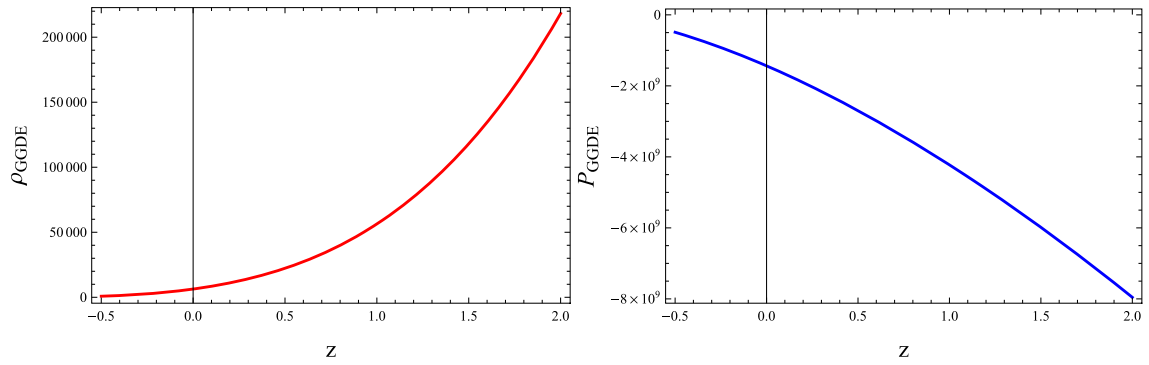

Fig. 2 Plots of ρ_{GGDE} against z (left) and P_{GGDE} against z (right)

$$+ b(d-2) + 2(H_0^2\Gamma^{2q+2})^{3/2}(4\sqrt[4]{6}\alpha + 3\beta m)\Big\}\Big],$$

$$P_{GGDE} = \frac{1}{36}\Bigg[\frac{1}{H_0}\Bigg\{(3H_0-1)\Bigg\{6\sqrt{6}\sqrt{H_0^2\Gamma^{2q+2}}\Bigg\{c_1 + 3\sqrt{6}\beta H_0^2 m\Gamma^{2q+2}$$

$$+ 4\,6^{3/4}\alpha H_0^2\Gamma^{2q+2}\Bigg\} - \frac{72a\sqrt{d}m\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{-\frac{3m}{8}}}{3m+4}\Bigg\}\Bigg\}$$

$$+ \frac{1}{12H_0^3}\Bigg[\Gamma^{-3(q+1)}\Bigg\{\Bigg\{18a\sqrt{d}m(3m+8)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{-\frac{3m}{8}}\Bigg\}(3m+4)$$

$$+ \sqrt{6}\sqrt{H_0^2\Gamma^{2q+2}}\Bigg\{-6c_1 + 18\sqrt{6}\beta H_0^2 m\Gamma^{2q+2} + 24\,6^{3/4}\alpha H_0^2\Gamma^{2q+2}\Bigg\}\Bigg\}\Bigg]$$

$$- \frac{288a\sqrt{d}\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{-\frac{3m}{8}}}{3m+4} - 18\sqrt{\mathrm{ad}} - 54b - 18\sqrt{6}c_1\sqrt{H_0^2\Gamma^{2q+2}}$$

$$- 108\beta m(H_0^2\Gamma^{2q+2})^{3/2} - 144\sqrt[4]{6}\alpha_0 a(H_0^2\Gamma^{2q+2})^{3/2}\Bigg].$$

4 Cosmological analysis

In this section, we explore the evolutionary stages of the universe through different phases. To achieve this, we utilize the reconstructed GGDE $f(\mathcal{Q}, \mathcal{T})$ model under noninteracting conditions, as defined in Eq.(49). Furthermore, we depict the dynamics of various cosmological parameters, including the EoS parameter, $\omega_{GGDE} - \omega'_{GGDE}$ and state finder planes. The stability of this model is also examined.

4.1 Equation of state parameter

The EoS parameter ($\omega = \frac{P}{\rho}$) of DE plays a crucial role in describing both the cosmic inflationary phase and the subsequent expansion of the universe. We analyze the criterion for an accelerating universe, which occurs when the EoS parameter $\omega < -\frac{1}{3}$. When $\omega = -1$, it corresponds to the cosmological constant. However, when $\omega = \frac{1}{3}$ and $\omega = 0$, it signifies the radiation-dominated and matter-dominated universe, respectively. Moreover, the phantom scenario manifests when we assume $\omega < -1$. The expression for the EoS parameter is given by

$$\omega_{GGDE} = \frac{P_{eff}}{\rho_{eff}} = \frac{P_{GGDE}}{\rho_{GGDE} + \rho_M}. \tag{50}$$

Equations (33), (34) and (41) are employed in the aforementioned expression to compute the respective parameter as

$$\omega_{GGDE} = -\Bigg[f^3\Gamma^{-2q-2}\Bigg\{a\sqrt{d}2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}(-432h^4m\Gamma^{4q+4} - 48h^2\Gamma^{2q+2}(12h^2$$

$$\times\Gamma^{2q+2} - 3hm\Gamma^{2q+2}) + 3hm(3m+8)\Gamma^{q+1}) - 6^{\frac{3mx}{8}+\frac{1}{2}}(3m+4)\sqrt{h^2\Gamma^{2q+2}}$$

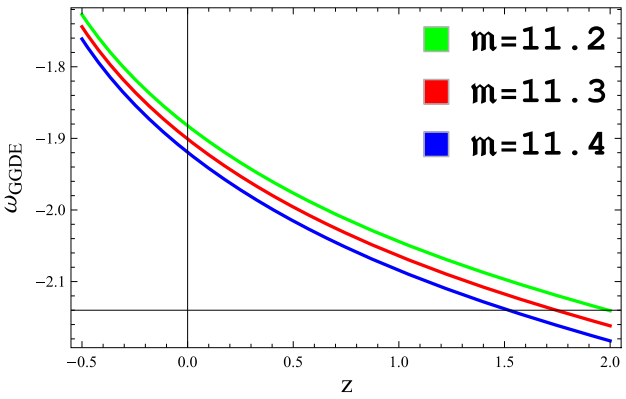

Fig. 3 Plot of EoS parameter against z

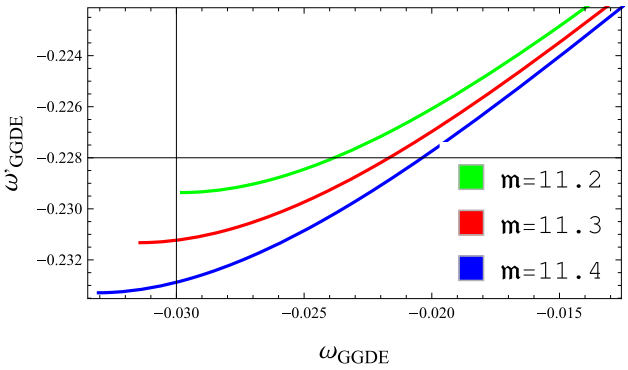

Fig. 4 Plot of $\omega_{GGDE} - \omega'_{GGDE}$ versus z

$$\times \left(\frac{h^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} \Big\{108\sqrt{6}\sqrt{\mathrm{ad}}(H_0^2\Gamma^{2q+2})^{3/2} + 324\sqrt{6}b(H_0^2\Gamma^{2q+2})^{3/2} + 216$$
$$\times c_1 H_0^3\Gamma^{4q+4} + 18c_1 H_0\Gamma^{q+1} - 1296\sqrt{6}\beta H_0^6 m\Gamma^{6q+6} - 1728\, 6^{3/4}\alpha H_0^6\Gamma^{6q+6}$$
$$+ 648\sqrt{6}\beta H_0^5 m\Gamma^{6q+6} + 864\, 6^{3/4}\alpha H_0^5\Gamma^{6q+6} - 54\sqrt{6}\beta H_0^3 m\Gamma^{3q+3} - 72\, 6^{3/4}$$
$$\times \alpha H_0^3\Gamma^{3q+3}\Big\}\Big]\Big[18H_0^2(3m+4)\Big\{a\sqrt{d}\, f^3 2^{\frac{3m}{8}} 3^{\frac{3m}{8}+2} m + H_0^2 6^{\frac{3m}{8}+1}\Gamma^{2q+2}$$
$$\times \left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} (6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha + 3\beta m) - 12d)\Big\}\Big]^{-1}. \tag{51}$$

Figure 3 illustrates the behavior of EoS parameter against z from which one can find that phantom epoch for current as well as late time cosmic evolution.

4.2 $\omega_{GGDE} - \omega'_{GGDE}$ plane

Here, we utilize the $\omega_{GGDE} - \omega'_{GGDE}$ phase plane, where ω'_{GGDE} indicates the evolutionary mode of ω_{GGDE} and prime denotes the derivative with respect to $\mathcal{Q}$. This cosmic plane was established by Caldwell and Linder [32] to investigate the quintessence DE paradigm which can be split into freezing ($\omega_{GGDE} < 0$, $\omega'_{GGDE} < 0$) and thawing ($\omega_{GGDE} < 0$, $\omega'_{GGDE} > 0$) regions. To depict the prevailing cosmic expansion paradigm, the freezing region signifies a more accelerated phase compared to thawing. The cosmic trajectories of $\omega_{GGDE} - \omega'_{GGDE}$ plane for specific choices of m are shown in Fig. 4 which provides the freezing region of the cosmos. The expression of ω'_{GGDE} is given in Appendix **A**.

4.3 State finder analysis

One of the techniques to examine the dynamics of the cosmos using a cosmological perspective is state finder analysis. It is an essential approach for understanding numerous DE models. As a combination of the Hubble parameter and its temporal derivatives, Sahni et al. [33] established two dimensionless parameters (r, s) given as

$$r = \frac{\dddot{a}}{aH^3}, \quad s = \frac{r-1}{3(q-\frac{1}{2})}. \tag{52}$$

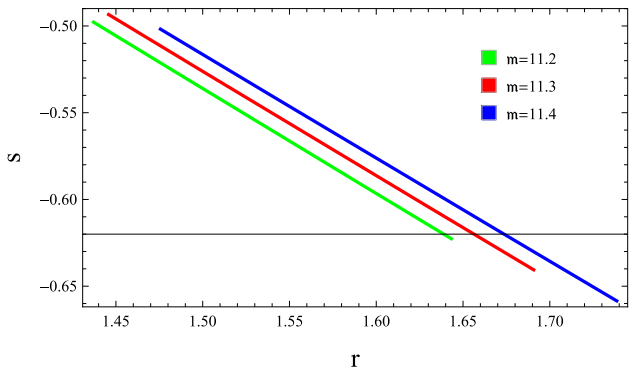

Fig. 5 Plot of $r - s$ plane against z

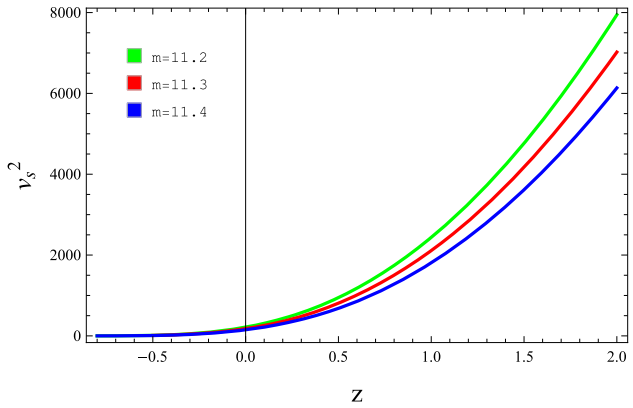

Fig. 6 Plot of v_s^2 against z

The acceleration of cosmic expansion is determined by the parameter s, while the deviation from pure power-law behavior is precisely described by the parameter r. This is a geometric diagnostic that does not favor any particular cosmological paradigm. It is such an approach that does not depend upon any specific model to distinguish between numerous DE scenarios, i.e., CG (Chaplygin gas), HDE (Holographic DE), SCDM (standard CDM) and quintessence.

Several DE scenarios for the appropriate choices of r and s parametric values are given below.

- When $r = 1, s = 0$, it indicates the CDM model.
- If $r = 1, s = 1$, then it denotes SCDM epoch.
- When $r = 1, s = \frac{2}{3}$, this epoch demonstrates the HDE model.
- When we have $r > 1, s < 0$, we get CG scenario.
- Lastly, $r < 1, s > 0$ corresponds to quintessence paradigm.

For our considered setup, the parameters r and s in terms of required factors are given in Appendix **B**. The graphical analysis of $r - s$ phase plane in Fig. 5 gives $r > 1$ and $s < 0$, indicating the CG model.

4.4 Squared speed of sound parameter

Perturbation theory provides a direct analysis for evaluating the stability of the DE model through examination of the sign of (v_s^2). In this context, we investigate the squared speed of sound parameter to analyze the stability of the GGDE $f(\mathcal{Q}, \mathcal{T})$ model, represented by

$$v_s^2 = \frac{P_{GGDE}}{\rho'_{GGDE}}\omega'_{GGDE} + \omega_{GGDE}. \tag{53}$$

A positive value signifies a stable configuration, whereas a negative value indicates unstable behavior for the associated model. Substituting the necessary expressions on the right-hand side of the equation above for the reconstructed model, we derive the squared speed of sound parameter as provided in Appendix **C**. Figure 6 illustrates that the speed component remains positive for all assumed values of m, indicating the stability of the reconstructed GGDE $f(\mathcal{Q}, \mathcal{T})$ model throughout the cosmic evolution.

5 Final remarks

In this study, we have explored various features of the GGDE model within the framework of the recently developed $f(\mathcal{Q}, \mathcal{T})$ theory of gravity. To comprehend the influence of $\mathcal{Q}$ and $\mathcal{T}$ in the GGDE ansatz, we have adopted a specific $f(\mathcal{Q}, \mathcal{T}) = f_1(\mathcal{Q}) + f_2(\mathcal{T})$ model. We have formulated a reconstruction framework involving a power-law scale factor and a corresponding scenario for the flat FRW universe. Subsequently, we have analyzed the EoS parameter, $\omega_{GGDE} - \omega'_{GGDE}$ and state finder planes, and conducted a squared sound speed analysis for the derived model. We summarize the main results as follows.

- In the noninteracting case, the reconstructed GGDE $f(\mathcal{Q}, \mathcal{T})$ model shows an increasing trend for z, indicating the realistic nature of the reconstructed model (Fig. 1).
- The energy density demonstrates positive behavior, showing an increase, while the pressure exhibits negative behavior. These patterns align with the characteristics of DE (Fig. 2).
- The early and late time universe are characterized by the EoS parameter, which involves phantom field and DE. Additionally, we have noted a trend where this parameter assumes increasingly negative values below -1 (Fig. 3). These observations align with the current understanding of accelerated cosmic behavior.
- The evolutionary pattern in the $\omega_{GGDE} - \omega'_{GGDE}$ plane indicates a freezing region for all values of m (Fig. 4). This affirms that the noninteracting GGDE $f(\mathcal{Q}, \mathcal{T})$ gravity model implies a more accelerated expanding universe.
- The $r - s$ plane depicts the CG model as both r and s satisfy the respective model criteria (Fig. 5).
- We have found that the squared speed of sound parameter is positive and hence GGDE $f(\mathcal{Q}, \mathcal{T})$ gravity model is stable for all values of m (Fig. 6).

Finally, we conclude that our results align with the current observational data [34], as provided below

$$\begin{aligned}
\omega_{GGDE} &= -1.023^{+0.091}_{-0.096} \quad \text{(Planck TT+LowP+ext)},\\
\omega_{GGDE} &= -1.006^{+0.085}_{-0.091} \quad \text{(Planck TT+LowP+lensing+ext)},\\
\omega_{GGDE} &= -1.019^{+0.075}_{-0.080} \quad \text{(Planck TT, TE, EE+LowP+ext)}.
\end{aligned} \tag{54}$$

These values have been established using diverse observational methodologies, ensuring a confidence level of 95%. Moreover, we have verified that the cosmic diagnostic state finder parameters for our derived model align with the constraints and limitations on the kinematics of the universe [35]. Sharif and Saba [36] established the correspondence of modified Gauss–Bonnet theory with GGDE paradigm and found phantom phase as well as the stable reconstructed model in noninteracting case. Our findings align with these results. Our results are also consistent with the recent work in $f(\mathcal{Q})$ theory [15].

Data availability No new data were generated or analyzed in support of this research.

Appendix A: Calculation of ω'_{GGDE}

$$\begin{aligned}
\omega'_{GGDE} = &\Bigg[f^3\Gamma^{-4q-4}\Bigg\{2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}(3H_0m(3m+8)\Gamma^{q+1} - 48H_0^2(12H_0^2\Gamma^{2q+2}\\
&- 3H_0m\Gamma^{2q+2})\Gamma^{2q+2} - 432H_0^4m\Gamma^{4q+4}) - 6^{\frac{3m}{8}+\frac{1}{2}}(3m+4)\sqrt{H_0^2\Gamma^{2q+2}}\\
&\times\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}\Bigg\{324\sqrt{6}b(H_0^2\Gamma^{2q+2})^{3/2} + 108\sqrt{6}\sqrt{\mathrm{ad}}(H_0^2\Gamma^{2q+2})^{3/2}\\
&- 726^{3/4}H_0^3\Gamma^{3q+3}\alpha - 1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha + 864\,6^{3/4}H_0^5\Gamma^{6q+6}\alpha\\
&- 54\sqrt{6}H_0^3m\Gamma^{3q+3}\beta - 1296\sqrt{6}H_0^6m\Gamma^{6q+6}\beta + 648\sqrt{6}H_0^5m\Gamma^{6q+6}\beta\\
&+ 18H_0\Gamma^{q+1}c_1 + 216H_0^3\Gamma^{4q+4}c_1\Bigg\}\Bigg\}\Bigg]\Bigg[108H_0^4(3m+4)\Bigg\{6^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}\\
&\times(6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha+3m\beta) - 12d)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} + 2^{\frac{3m}{8}}3^{\frac{3m}{8}+2}\\
&\times a\sqrt{d}f^3m\Bigg\}\Bigg]^{-1} + \Bigg[f^3\Gamma^{-2q-2}\Bigg\{2^{\frac{3m}{8}}3^{\frac{3m}{8}+1}f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha+3m\beta)\\
&\times\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} + 6^{\frac{3m}{8}}(6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha+3m\beta)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}\\
&+ \Bigg[2^{\frac{3m}{8}-3}3^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}(6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha+3m\beta)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}-1}\Bigg]
\end{aligned}$$

$$\times \frac{1}{m}\Bigg\}\Bigg(2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}\{3H_0m(3m+8)\Gamma^{q+1}-48H_0^2(12H_0^2\Gamma^{2q+2}$$
$$-3H_0m\Gamma^{2q+2})\Gamma^{2q+2}-432H_0^4m\Gamma^{4q+4}\}-6^{\frac{3m}{8}+\frac{1}{2}}(3m+4)\Bigg(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Bigg)^{\frac{3m}{8}}$$
$$\times\sqrt{H_0^2\Gamma^{2q+2}}(324\sqrt{6}b\times(H_0^2\Gamma^{2q+2})^{3/2}+108\sqrt{6}\sqrt{\mathrm{ad}}(H_0^2\Gamma^{2q+2})^{3/2}$$
$$-726^{3/4}H_0^3\Gamma^{3q+3}\alpha-1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha+864\,6^{3/4}H_0^5\Gamma^{6q+6}\alpha$$
$$-54\sqrt{6}H_0^3m\Gamma^{3q+3}\beta-1296\sqrt{6}H_0^6m\Gamma^{6q+6}\beta+648\sqrt{6}H_0^5m\Gamma^{6q+6}\beta$$
$$+18H_0\Gamma^{q+1}c_1+216H_0^3\Gamma^{4q+4}c_1\Bigg)\Bigg)\Bigg]\Bigg[18H_0^2(3m+4)\Bigg(6^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}$$
$$\times(6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha+3m\beta)-12d)\Bigg(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Bigg)^{\frac{3m}{8}}+2^{\frac{3m}{8}}3^{\frac{3m}{8}+2}$$
$$\times a\sqrt{d}f^3m\Bigg)^2\Bigg]^{-1}-\Bigg[f^3\Gamma^{-2q-2}(-6^{\frac{3m}{8}+\frac{1}{2}}(3m+4)\sqrt{H_0^2\Gamma^{2q+2}}\Bigg\{-126^{3/4}$$
$$\times H_0\alpha\Gamma^{q+1}-9\sqrt{6}H_0m\beta\Gamma^{q+1}+108H_0^2c_1\Gamma^{2q+2}+36H_0c_1\Gamma^{2q+2}$$
$$-432\,6^{3/4}H_0^4\alpha\Gamma^{4q+4}+2886^{3/4}H_0^3\alpha\Gamma^{4q+4}-324\times\sqrt{6}H_0^4m\beta\Gamma^{4q+4}$$
$$+216\sqrt{6}H_0^3m\beta\Gamma^{4q+4}+81\sqrt{6}b\sqrt{H_0^2\Gamma^{2q+2}}+27\sqrt{6}\sqrt{\mathrm{ad}}\sqrt{H_0^2\Gamma^{2q+2}}\}$$
$$\times\Bigg(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Bigg)^{\frac{3m}{8}}+2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}\Bigg\{-96H_0^2\Gamma^{2q+2}-72H_0^2m\Gamma^{2q+2}$$
$$-8(12H_0^2\Gamma^{2q+2}-3H_0m\Gamma^{2q+2})\Bigg\}-\Bigg[2^{\frac{3m}{8}-\frac{3}{2}}3^{\frac{3m}{8}-\frac{1}{2}}(3m+4)\Bigg(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Bigg)^{\frac{3m}{8}}$$
$$\times\{324\sqrt{6}b(H_0^2\Gamma^{2q+2})^{3/2}+108\sqrt{6}\sqrt{\mathrm{ad}}(H_0^2\Gamma^{2q+2})^{3/2}-6^{3/4}H_0^3\Gamma^{3q+3}\alpha$$
$$-1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha+864\,6^{3/4}H_0^5\Gamma^{6q+6}\alpha-54\sqrt{6}H_0^3m\Gamma^{3q+3}\beta$$
$$-1296\sqrt{6}H_0^6m\Gamma^{6q+6}\beta+\sqrt{6}H_0^5m\Gamma^{6q+6}\beta+18H_0\Gamma^{q+1}c_1+H_0^3\Gamma^{4q+4}c_1\Bigg\}\Bigg]$$
$$\times\Bigg[\sqrt{H_0^2\Gamma^{2q+2}}\Bigg]^{-1}-\Bigg[2^{\frac{3m}{8}-\frac{7}{2}}3^{\frac{3m}{8}+\frac{1}{2}}(3m+4)\sqrt{H_0^2\Gamma^{2q+2}}\Bigg(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Bigg)^{\frac{3m}{8}-1}$$
$$\times(324\sqrt{6}b(H_0^2\Gamma^{2q+2})^{3/2}+108\sqrt{6}(H_0^2\Gamma^{2q+2})^{3/2}-72\,6^{3/4}H_0^3\Gamma^{3q+3}\alpha$$
$$-1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha+864\,6^{3/4}H_0^5\Gamma^{6q+6}\alpha-54\sqrt{6}H_0^3m\Gamma^{3q+3}\beta$$
$$-1296\sqrt{6}H_0^6m\Gamma^{6q+6}\beta+648\sqrt{6}H_0^5m\Gamma^{6q+6}\beta+18H_0\Gamma^{q+1}c_1$$
$$+216H_0^3\Gamma^{4q+4}c_1)]\Bigg[18H_0^2(3m+4)(6^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}(6f^3\sqrt{H_0^2\Gamma^{2q+2}}$$
$$\times(4\sqrt[4]{6}\alpha+3m\beta)-12d)\Bigg(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Bigg)^{\frac{3m}{8}}+2^{\frac{3m}{8}}3^{\frac{3m}{8}+2}a\sqrt{d}f^3m)\Bigg]. \tag{A1}$$

Appendix B: Evaluation of *r* and *s* parameters

$$r=2\Bigg(\frac{3}{2}\Bigg[f^3\Gamma^{-4q-4}(2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}\{3H_0m(3m+8)\Gamma^{q+1}-48H_0^2(12H_0^2\Gamma^{2q+2}$$
$$-\Gamma^{2q+2})\Gamma^{2q+2}-432H_0^4m\Gamma^{4q+4}\}-6^{\frac{3m}{8}+\frac{1}{2}}(3m+4)\sqrt{{}^{H}_{0}2\Gamma^{2q+2}}\Bigg(\frac{h^2\Gamma^{2q+2}}{m^2}\Bigg)^{\frac{3m}{8}}$$
$$\times\Bigg(324\sqrt{6}b(H_0^2\Gamma^{2q+2})^{3/2}+108\sqrt{6}\sqrt{\mathrm{ad}}(H_0^2\Gamma^{2q+2})^{3/2}-72\,6^{3/4}h^3\Gamma^{3q+3}\alpha$$
$$\times\ 6^{3/4}H_0^6\Gamma^{6q+6}\alpha+864\,6^{3/4}h^5\Gamma^{6q+6}\times\alpha-54\sqrt{6}H_0^3m\Gamma^{3q+3}\beta-1296\sqrt{6}H_0^6m$$

$$\times \Gamma^{6q+6}\beta + 648\sqrt{6}H_0^5 m\Gamma^{6q+6}\beta + 18H_0\Gamma^{q+1}c_1 + 216h^3\Gamma^{4q+4}c_1\Big)\Big)\Big]\Big[108h^4$$
$$\times (6^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}(6f^3\sqrt{H_0^2\Gamma^{2q+2}}\Big\{4\sqrt[4]{6}\alpha + 3m\beta\Big\}\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}}$$
$$\times 3^{\frac{3m}{8}+2}a\sqrt{d}\,f^3 m\Big)\Big]^{-1} + \Big[f^3\Gamma^{-2q-2}\Big(2^{\frac{3m}{8}}3^{\frac{3m}{8}+1}f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha$$
$$+ 3m\beta)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}} + 6^{\frac{3m}{8}}\Big(6f^3\sqrt{H_0^2\Gamma^{2q+2}}\Big(4\sqrt[4]{6}\alpha + 3m\beta\Big)$$
$$- 12d\Big)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}} + \frac{1}{m}\Big[2^{\frac{3m}{8}-3}3^{\frac{3m}{8}+1}h^2\Gamma^{2q+2}(6f^3\sqrt{h^2\Gamma^{2q+2}}\Big(4\sqrt[4]{6}\alpha$$
$$+ 3m\beta\Big) - \Big(\frac{h^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}-1}\Big]\Big(2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}\Big(3hm(3m+8)\Gamma^{q+1}$$
$$- 48h^2\Big(12H_0^2\Gamma^{2q+2} - 3H_0 m\Gamma^{2q+2}\Big) - 432h^4 m\Gamma^{4q+4}\Big) - \Big(\frac{h^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}}$$
$$\times \sqrt{h^2\Gamma^{2q+2}}(324\sqrt{6}b(h^2\Gamma^{2q+2}\Big)^{3/2}108\sqrt{6}\sqrt{\mathrm{ad}} + \Big(h^2\Gamma^{2q+2}\Big)^{3/2} - 72\,6^{3/4}$$
$$\times h^3\Gamma^{3q+3}\alpha - 1728\,6^{3/4}h^6\Gamma^{6q+6}\alpha + 864\,6^{3/4}h^5\Gamma^{6q+6}\alpha - 54\sqrt{6}h^3 m\Gamma^{3q+3}\beta$$
$$- 1296\sqrt{6}H_0^6 m\Gamma^{6q+6}\beta + 216H_0^3\Gamma^{4q+4}c_1\Big]\Big[18h^2(3m+4)\Big(6^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}$$
$$\times 6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha 3m\beta) - 12d)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}} + 2^{\frac{3m}{8}}3^{\frac{3m}{8}+2}f^3 m\Big)^2\Big]^{-1}$$
$$- \Big[f^3\Gamma^{-2q-2}\Big(-6^{\frac{3m}{8}+\frac{1}{2}}(3m+4)\sqrt{H_0^2\Gamma^{2q+2}}\Big(-126^{3/4}H_0\alpha\Gamma^{q+1}$$
$$+ 36H_0c_1\Gamma^{2q+2} - 432\,6^{3/4}h^4\alpha\Gamma^{4q+4} + 288\,6^{3/4}h^3\alpha\Gamma^{4q+4} - 324\sqrt{6}H_0^4 m$$
$$\times \beta\Gamma^{4q+4} + 216\sqrt{6}H_0^3 m\beta\Gamma^{4q+4} + \sqrt{H_0^2\Gamma^{2q+2}} + 27\sqrt{6}\sqrt{\mathrm{ad}}\sqrt{H_0^2\Gamma^{2q+2}}\Big)$$
$$\times \Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}} + 2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}\Big\{-H_0^2\Gamma^{2q+2} - H_0^2 m\Gamma^{2q+2}$$
$$- 8(12h^2\Gamma^{2q+2} - 3H_0 m\Gamma^{2q+2})\Big\} - \Big[2^{\frac{3m}{8}-\frac{3}{2}}3^{\frac{3m}{8}-\frac{1}{2}}(3m+4)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}}$$
$$\times \Big(324\sqrt{6}b\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2} + 108\sqrt{6}\sqrt{\mathrm{ad}}\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2} - 72\,6^{3/4}$$
$$\times H_0^3\Gamma^{3q+3}\alpha - 1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha + 864\,6^{3/4}H_0^5\Gamma^{6q+6}\alpha - 54\sqrt{6}$$
$$\times H_0^3 m\Gamma^{3q+3}\beta - \sqrt{6}h^6 m\Gamma^{6q+6}\beta + \sqrt{6}h^5 m\Gamma^{6q+6}\beta + 18H_0\Gamma^{q+1}c_1$$
$$+ 216h^3\Gamma^{4q+4}c_1\Big)\Big] \times \Big[\sqrt{H_0^2\Gamma^{2q+2}}\Big]^{-1} - \frac{1}{m}\Big[2^{\frac{3m}{8}-\frac{7}{2}}3^{\frac{3m}{8}+\frac{1}{2}}\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}-1}$$
$$\times \sqrt{h^2\Gamma^{2q+2}}\Big(324\sqrt{6}b\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2} + \sqrt{6}\sqrt{\mathrm{ad}}\Big(h^2\Gamma^{2q+2}\Big)^{3/2}$$
$$- 72\,6^{3/4}H_0^3\Gamma^{3q+3}\alpha - 1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha + 6^{3/4}H_0^5\Gamma^{6q+6}\alpha - 54\sqrt{6}h^3$$
$$\times m\Gamma^{3q+3}\beta - 1296\sqrt{6}H_0^6 m\Gamma^{6q+6}\beta + 648\sqrt{6}h^5 m\Gamma^{6q+6}\beta + 18h\Gamma^{q+1}c_1$$
$$+ 216h^3\Gamma^{4q+4}c_1\Big)\Big]\Big[18h^2(3m+4)(6^{\frac{3m}{8}+1}h^2\Gamma^{2q+2}(6f^3\sqrt{h^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha$$
$$+ 3m\beta) -)\Big(\frac{h^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}} + 2^{\frac{3m}{8}}3^{\frac{3m}{8}+2}a\sqrt{d}\,f^3 m\Big)\Big]^{-1}\Big) + \frac{1}{2}\Big) + \frac{3}{2}\Big[f^3\Gamma^{-4q-4}$$

$$\times \left(2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}\left(3H_0m(3m+8)\Gamma^{q+1}-48H_0^2\left(12H_0^2\Gamma^{2q+2}-3H_0m\right.\right.\right.$$
$$\times\left.\left.\Gamma^{2q+2}\right)\Gamma^{2q+2}-H_0^4m\Gamma^{4q+4}\right)-6^{\frac{3m}{8}+\frac{1}{2}}(3m+4)\sqrt{H_0^2\Gamma^{2q+2}}\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}$$
$$\times\left(324\sqrt{6}b\left(H_0^2\Gamma^{2q+2}\right)^{3/2}+108\sqrt{6}\sqrt{\mathrm{ad}}\left(H_0^2\Gamma^{2q+2}\right)^{3/2}-72\,6^{3/4}H_0^3\Gamma^{3q+3}\alpha\right.$$
$$-1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha+864\,6^{3/4}H_0^5\Gamma^{6q+6}\alpha-54\sqrt{6}H_0^3m\Gamma^{3q+3}\beta-1296\sqrt{6}H_0^6m$$
$$\left.\times\Gamma^{6q+6}\beta+648\sqrt{6}H_0^5m\Gamma^{6q+6}\beta\Gamma^{q+1}c_1+216H_0^3\Gamma^{4q+4}c_1\right)\Big)\Big]\Big[108H_0^4(3m+4)$$
$$\times(6^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}(\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha+3m\beta)-12d)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}$$
$$\times\left\{2^{\frac{3m}{8}}3^{\frac{3m}{8}+1}f^3\sqrt{H_0^2\Gamma^{2q+2}}\left(4\sqrt[4]{6}\alpha+3m\beta\right)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}+(\sqrt{H_0^2\Gamma^{2q+2}}\right.$$
$$\times(4\sqrt[4]{6}\alpha+3m\beta)-12d)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}+\Big[2^{\frac{3m}{8}-3}3^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}\sqrt{H_0^2\Gamma^{2q+2}}$$
$$\left.\times(4\sqrt[4]{6}\alpha+3m\beta)-12d\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}-1}\Big]\Big[m\Big]^{-1}\right\}\Big(2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}(3H_0$$
$$\times m(3m+8)\Gamma^{q+1}-48H_0^2\left(12H_0^2\Gamma^{2q+2}-3H_0m\Gamma^{2q+2}\right)\Gamma^{2q+2}-432H_0^4m\Gamma^{4q+4})$$
$$-6^{\frac{3m}{8}+\frac{1}{2}}(3m+4)\sqrt{H_0^2\Gamma^{2q+2}}\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}\Big(324\sqrt{6}b\left(H_0^2\Gamma^{2q+2}\right)^{3/2}+108\sqrt{6}$$
$$\times(H_0^2\Gamma^{2q+2})^{3/2}-726^{3/4}H_0^3\Gamma^{3q+3}\alpha-1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha+864\,6^{3/4}H_0^5\Gamma^{6q+6}$$
$$\times\alpha-54\sqrt{6}H_0^3m\Gamma^{3q+3}\beta-\sqrt{6}H_0^6m\Gamma^{6q+6}\beta+\sqrt{6}H_0^5m\Gamma^{6q+6}\beta+H_0\Gamma^{q+1}$$
$$+H_0^3\Gamma^{4q+4}c_1)\Big)\Big]\Big[18H_0^2(3m+4)(6^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}\{6f^3\sqrt{H_0^2\Gamma^{2q+2}}\left(4\sqrt[4]{6}\alpha+\beta\right)$$
$$\times\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}+2^{\frac{3m}{8}}3^{\frac{3m}{8}+2}a\sqrt{d}f^3m\Big)^2\Big]-\Big[f^3\Gamma^{-2q-2}\Big(-6^{\frac{3m}{8}+\frac{1}{2}}(3m+4)$$
$$\times\sqrt{H_0^2\Gamma^{2q+2}}\Big(-126^{3/4}H_0\alpha\Gamma^{q+1}-\sqrt{6}H_0m\beta\Gamma^{q+1}+H_0^2c_1\Gamma^{2q+2}+H_0c_1\Gamma^{2q+2}$$
$$-6^{3/4}H_0^4\alpha\Gamma^{4q+4}+6^{3/4}\times H_0^3\alpha\Gamma^{4q+4}-\sqrt{6}H_0^4m\beta\Gamma^{4q+4}+\sqrt{6}H_0^3m\beta\Gamma^{4q+4}$$
$$+\sqrt{6}b\sqrt{H_0^2\Gamma^{2q+2}}+\sqrt{6}\sqrt{\mathrm{ad}}\sqrt{H_0^2\Gamma^{2q+2}}\Big)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}\Big[\frac{3m}{8}+2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}$$
$$\times\Big(-96H_0^2\Gamma^{2q+2}-72H_0^2m\Gamma^{2q+2}-8\Big(12H_0^2\Gamma^{2q+2}-3H_0m\Gamma^{2q+2}\Big)\Big)-(3m+4)$$
$$\times\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}b(H_0^2\Gamma^{2q+2})^{3/2}\Big(\sqrt{6}+\sqrt{6}\sqrt{\mathrm{ad}}\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2}-72\,6^{3/4}H_0^3$$
$$\times\Gamma^{3q+3}\alpha-6^{3/4}H_0^6\Gamma^{6q+6}\alpha+\Big(324\sqrt{6}\Big(324\sqrt{6}6^{3/4}H_0^5\Gamma^{6q+6}\alpha-54\sqrt{6}H_0^3m$$
$$\times\Gamma^{3q+3}\beta-\sqrt{6}H_0^6m\Gamma^{6q+6}\beta+\sqrt{6}H_0^5m\Gamma^{6q+6}\beta+18H_0\Gamma^{q+1}c_1+H_0^3\Gamma^{4q+4}c_1\Big)\Big]$$
$$\times\Big[\sqrt{H_0^2\Gamma^{2q+2}}\Big]-\Big[2^{\frac{3m}{8}-\frac{7}{2}}3^{\frac{3m}{8}+\frac{1}{2}}\sqrt{H_0^2\Gamma^{2q+2}}\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}-1}\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2}$$
$$+\sqrt{6}\sqrt{\mathrm{ad}}\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2}-6^{3/4}H_0^3\Gamma^{3q+3}\alpha-1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha+864\,6^{3/4}$$
$$\times H_0^5\Gamma^{6q+6}\alpha-54\sqrt{6}H_0^3m\Gamma^{3q+3}\beta-1296\sqrt{6}H_0^6m\Gamma^{6q+6}\beta+648\sqrt{6}H_0^5m\Gamma^{6q+6}\beta$$

$$+ 18H_0\Gamma^{q+1}c_1 + 216H_0^3\Gamma^{4q+4}c_1\Big)\Big]\Big[m\Big]^{-1}\Big]\Big[18H_0^2(3m+4)\Big(6^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}\Big(6f^3$$
$$\times\sqrt{H_0^2\Gamma^{2q+2}}\Big(4\sqrt[4]{6}\alpha+3m\beta\Big)-12d\Big)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}}+2^{\frac{3m}{8}}3^{\frac{3m}{8}+2}a\sqrt{d}\,f^3m\Big)\Big]^{-1}$$
$$-3\Gamma^{-q-1}\Big(f^3\Gamma^{-2q-2}\Big(\Big[2^{\frac{3m}{8}-2}3^{\frac{3m}{8}+1}\alpha+3m\beta\Big)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}}\Big]\Big[\sqrt{H_0^2\Gamma^{2q+2}}\Big]^{-1}$$
$$+\Big[2^{\frac{3m}{8}-4}3^{\frac{3m}{8}}\Big(\frac{3m}{8}-1\Big)H_0^2\Gamma^{2q+2}\Big(6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha+3m\beta)-\Big(\frac{H_0^2}{m^2}\Big)^{\frac{3m}{8}-2}\Big]$$
$$\times\Big[m^3\Big]^{-1}+2^{\frac{3m}{8}-3}3^{\frac{3m}{8}+1}f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha+3m\beta)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}-1}+\Big[2^{\frac{3m}{8}-3}$$
$$\times 3^{\frac{3m}{8}}\Big(6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha+3m\beta)-12d\Big)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}-1}\Big]\Big[m\Big]^{-1}(2^{\frac{3m}{8}+1}$$
$$\times 3^{\frac{3m}{8}+2}a\sqrt{d}\Big(3H_0m\Gamma^{q+1}-48H_0^2\Big(12H_0^2\Gamma^{2q+2}-3H_0m\Gamma^{2q+2}\Big)\Gamma^{2q+2}432H_0^4m$$
$$\times\Gamma^{4q+4}\Big)-6^{\frac{3m}{8}+\frac{1}{2}}-(3m+4)\sqrt{H_0^2\Gamma^{2q+2}}\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}}\Big(324\sqrt{6}b\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2}$$
$$+108\sqrt{6}\sqrt{\mathrm{ad}}\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2}-72\,6^{3/4}H_0^3\Gamma^{3q+3}\alpha-1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha+864\,6^{3/4}$$
$$\times h^5\Gamma^{6q+6}\alpha-54\sqrt{6}h^3m\Gamma^{3q+3}\beta-1296\sqrt{6}h^6m\Gamma^{6q+6}\beta+648\sqrt{6}h^5\times m\Gamma^{6q+6}\beta$$
$$+18h\Gamma^{q+1}c_1+216h^3\Gamma^{4q+4}c_1\Big)\Big)18h^2(3m+4)\Big(6^{\frac{3m}{8}+1}h^2\Gamma^{2q+2}\Big(6f^3\sqrt{h^2\Gamma^{2q+2}}$$
$$\times(4\sqrt[4]{6}\alpha+3m\beta)-12d\Big)\Big(\frac{h^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}}+2^{\frac{3m}{8}}3^{\frac{3m}{8}+2}a\sqrt{d}\,f^3m)^2+f^3\Gamma^{-4q-4}$$
$$\times(-6^{3m}8+\frac{1}{2}(3m+4)\sqrt{h^2\Gamma^{2q+2}}\Big(-6^{3/4}H_0\alpha\Gamma^{q+1}-9\sqrt{6}H_0m\beta\Gamma^{q+1}+108$$
$$\times H_0^2c_1\Gamma^{2q+2}+H_0c_1\Gamma^{2q+2}-6^{3/4}H_0^4\alpha\Gamma^{4q+4}+6^{3/4}H_0^3\alpha\Gamma^{4q+4}-\sqrt{6}H_0^4m\beta\Gamma^{4q+4}$$
$$+\sqrt{6}H_0^3m\beta\Gamma^{4q+4}+81\sqrt{6}b\sqrt{H_0^2\Gamma^{2q+2}}+27\sqrt{6}\sqrt{\mathrm{ad}}\sqrt{H_0^2\Gamma^{2q+2}}\Big)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}}$$
$$\times+2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}\Big(-96H_0^2\Gamma^{2q+2}-72H_0^2m\Gamma^{2q+2}-8\Big(12H_0^2\Gamma^{2q+2}-3H_0m$$
$$\times\Gamma^{2q+2}-2^{\frac{3m}{8}-\frac{3}{2}}3^{\frac{3m}{8}-\frac{1}{2}}(3m+4)\Big(\frac{H_0^2\Gamma^{2q+2}}{m^2}\Big)^{\frac{3m}{8}}\Big(324\sqrt{6}b\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2}+108$$
$$\times\sqrt{6}\sqrt{\mathrm{ad}}\Big(H_0^2\Gamma^{2q+2}\Big)^{3/2}-72\,6^{3/4}H_0^3\Gamma^{3q+3}\alpha-6^{3/4}H_0^6\Gamma^{6q+6}\alpha+6^{3/4}H_0^5$$
$$\times\Gamma^{6q+6}\alpha-54\sqrt{6}H_0^3m\Gamma^{3q+3}\beta-1296\sqrt{6}H_0^6m\Gamma^{6q+6}\beta+648\sqrt{6}H_0^5m\Gamma^{6q+6}\beta$$
$$+18H_0\Gamma^{q+1}c_1\Big].$$

$$s=\frac{1}{2}+\Big[\frac{H_0\alpha\Gamma^{q+1}}{72(H_0^2\Gamma^{2q+2})^{7/2}}-\big[5H_0(3\sqrt{6}\log(6H_0^2\Gamma^{2q+2})\alpha-2\sqrt{6}\alpha+36c_1)\Gamma^{q+1}\big]$$
$$\times\big[\sqrt{6}(H_0^2\Gamma^{2q+2})^{7/2}\big]^{-1}+\Big[(3H_0^2\Gamma^{2q+2}-H_0\Gamma^{2q+2})(\frac{\alpha\Gamma^{-2q-2}}{\sqrt{6}H_0^2}+\frac{2\sqrt{\frac{2}{3}}\beta}{\sqrt{H_0^2\Gamma^{2q+2}}})\Big]$$
$$\times\Big\{\sqrt{H_0^2\Gamma^{2q+2}}\Big\}^{-1}-8\beta-\frac{2\alpha}{\sqrt{H_0^2\Gamma^{2q+2}}}-\frac{2\sqrt{6}c_1}{\sqrt{H_0^2\Gamma^{2q+2}}}-\frac{\alpha\log(6H_0^2\Gamma^{2q+2})}{\sqrt{H_0^2\Gamma^{2q+2}}}$$

$$
\begin{aligned}
&- \left[(3H_0^2\Gamma^{2q+2} - H_0\Gamma^{2q+2})\left(\sqrt{6}\log(6H_0^2\Gamma^{2q+2})\alpha + \alpha + \sqrt{H_0^2\Gamma^{2q+2}}\beta + 12c_1\right)\right] \\
&\times \left[(H_0^2\Gamma^{2q+2})^{3/2}\right]^{-1}\Bigg]\Bigg(-H_0^2\beta\Gamma^{2q+2} + (d+1) + b\left(d+\frac{3}{2}\right) - \sqrt{H_0^2\Gamma^{2q+2}}\alpha \\
&\times \frac{1}{2} + a\sqrt{d} + \frac{d}{f^3}\Bigg)\Bigg]^{-1} - \Bigg[\Bigg(-\frac{\alpha}{24\sqrt{H_0^2\Gamma^{2q+2}}} - \frac{\beta}{6}\Bigg)\Bigg(\Bigg[H_0\Bigg(3\sqrt{6}\log(6H_0^2\Gamma^{2q+2})\alpha \\
&- 2\sqrt{6}\alpha + 36c_1\Bigg)\Gamma^{q+1}\Bigg]\Bigg[36\sqrt{6}\left(H_0^2\Gamma^{2q+2}\right)^{5/2}\Bigg] - 48H_0^2\beta\Gamma^{2q+2} - \sqrt{H_0^2\Gamma^{2q+2}}c_1 \\
&- 12\sqrt{H_0^2\Gamma^{2q+2}}\alpha + 8\sqrt{6}\sqrt{H_0^2\Gamma^{2q+2}}\beta + 12c_1)\Bigg]\Bigg[\sqrt{H_0^2\Gamma^{2q+2}}\Bigg]^{-1} - 48a\sqrt{d})\Bigg] \\
&\times \Bigg[-H_0^2\beta\Gamma^{2q+2} + \sqrt{\mathrm{ad}}(d+1) + b\left(d+\frac{3}{2}\right) - \frac{1}{2}\sqrt{H_0^2\Gamma^{2q+2}}\alpha + a\sqrt{d} + \frac{d}{f^3})^2\Bigg] \\
&+ 36c_1(H_0^2\Gamma^{2q+2})^{9/2} - \frac{H_0\alpha\Gamma^{q+1}}{72(H_0^2\Gamma^{2q+2})^{9/2}} + \sqrt{\frac{2}{3}}(3H_0^2\Gamma^{2q+2} - H_0\Gamma^{2q+2}) \\
&- (\alpha\Gamma^{-4q-4})(6\sqrt{6}H_0^4)^{-1}q + 2\Bigg)^{3/2} + \Bigg[\alpha\log(6H_0^2\Gamma^{2q+2})\Bigg]\Bigg[12(H_0^2\Gamma^{2q+2})^{3/2}\Bigg]^{-1} \\
&+ (3H_0^2\Gamma^{2q+2} - H_0\Gamma^{2q+2})(\sqrt{6}\log\Bigg(6H_0^2\Gamma^{2q+2}])\alpha + 2\sqrt{6}\alpha + \sqrt{h^2\Gamma^{2q+2}}\beta + 12c_1 \\
&\times 12\sqrt{6}(h^2\Gamma^{2q+2})^{5/2} - (3h^2\Gamma^{2q+2} - h\Gamma^{2q+2})\Bigg(\frac{\alpha\Gamma^{-2q-2}}{\sqrt{6}h^2} + \frac{2\sqrt{\frac{2}{3}}\beta}{\sqrt{h^2\Gamma^{2q+2}}}\Bigg)3(h^2\Gamma^{2q+2})^{3/2} \\
&\times 32\Bigg(-h^2\beta\Gamma^{2q+2} + \sqrt{\mathrm{ad}}(d+1) + b\left(d+\frac{3}{2}\right) - \frac{1}{2}\sqrt{h^2\Gamma^{2q+2}}\alpha + a\sqrt{d} + \frac{d}{f^3}\Bigg) \\
&\times -5\Bigg(-\frac{\alpha}{24\sqrt{h^2\Gamma^{2q+2}}} - \frac{\beta}{6}\Bigg)\Bigg(\frac{h\alpha\Gamma^{q+1}}{72(h^2\Gamma^{2q+2})^{7/2}} - (5H_0(3\sqrt{6}\log(6H_0^2\Gamma^{2q+2}) \\
&\times \alpha - 2\sqrt{6}\alpha + 36c_1)\Gamma^{q+1}\Bigg)\Bigg(432\sqrt{6}(H_0^2\Gamma^{2q+2})^{7/2}\Bigg)^{-1} + 2\sqrt{\frac{2}{3}}\Bigg(3H_0^2\Gamma^{2q+2} \\
&- \Bigg[\alpha\log(6H_0^2\Gamma^{2q+2})\Bigg]\Bigg[\sqrt{H_0^2\Gamma^{2q+2}}\Bigg] - \Bigg[(3H_0^2\Gamma^{2q+2} - H_0\Gamma^{2q+2})(6H_0^2\Gamma^{2q+2})\alpha \\
&+ 2\sqrt{6}\alpha + 8\sqrt{6}\sqrt{H_0^2\Gamma^{2q+2}}\beta + 12c_1\Bigg]\Bigg[3\sqrt{6}\Bigg(H_0^2\Gamma^{2q+2}\Bigg)^{3/2}\Bigg]\Bigg]. \qquad \text{(B1)}
\end{aligned}
$$

Appendix C: Determination of v_s^2

$$
\begin{aligned}
v_s^2 = &\Bigg[\left(\frac{h^2\Gamma^{2q+2}}{m^2}\right)^{-\frac{3m}{8}} 36a\sqrt{d} - 18\left(\frac{h^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} 2(4\sqrt[4]{6}\alpha + 3m\beta)(h^2\Gamma^{2q+2})^{3/2} \\
&+ b(d-2) + \sqrt{\mathrm{ad}}(d-1)\Bigg[f^3\Gamma^{-4q-4}2^{\frac{3m}{8}+1}3^{\frac{3m}{8}+2}a\sqrt{d}(3hm(3m+8)\Gamma^{q+1} \\
&- 48h^2(12h^2\Gamma^{2q+2} - 3hm\Gamma^{2q+2})\Gamma^{2q+2} - 432h^4m\Gamma^{4q+4}) - 6^{\frac{3m}{8}+\frac{1}{2}}\sqrt{h^2\Gamma^{2q+2}} \\
&\times \left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} 324\sqrt{6}b(H_0^2\Gamma^{2q+2})^{3/2} + 108\sqrt{6}\sqrt{\mathrm{ad}}(H_0^2\Gamma^{2q+2})^{3/2} \\
&- 72\,6^{3/4}H_0^3\Gamma^{3q+3}\alpha - 1728\,6^{3/4}H_0^6\Gamma^{6q+6}\alpha + 864\,6^{3/4}H_0^5\Gamma^{6q+6}\alpha - 54\sqrt{6}H_0^3 \\
&\times m\Gamma^{3q+3}\beta - 1296\sqrt{6}H_0^6m\Gamma^{6q+6}\beta + 648\sqrt{6}H_0^5m\Gamma^{6q+6}\beta + 18H_0\Gamma^{q+1}c_1 \\
&+ 216H_0^3\Gamma^{4q+4}c_1\Bigg]\Bigg[108H_0^4(3m+4)(6^{\frac{3m}{8}+1}H_0^2\Gamma^{2q+2}(6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha \\
&+ 3m\beta) - 12d)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} + 2^{\frac{3m}{8}}3^{\frac{3m}{8}+2}a\sqrt{d}f^3m)\Bigg]^{-1} + \Bigg[f^3\Gamma^{-2q-2}
\end{aligned}
$$

$$\times 2^{\frac{3m}{8}} 3^{\frac{3m}{8}+1} f^3 \sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha + 3m\beta)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} + (\sqrt{H_0^2\Gamma^{2q+2}}$$

$$\times (4\sqrt[4]{6}\alpha + 3m\beta) - 12d)\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} + \Bigg[2^{\frac{3m}{8}-3} 3^{\frac{3m}{8}+1} H_0^2(z+1)^{2q+2}(6f^3$$

$$\times \sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha - 432H_0^4 m\Gamma^{4q+4}) + 3m\beta)(\frac{H_0^2(z+1)^{2q+2}}{m^2})^{\frac{3m}{8}-1}\Bigg]\Bigg[m\Bigg]^{-1}$$

$$\times 2^{\frac{3m}{8}+1} 3^{\frac{3m}{8}+2} a\sqrt{d} 3H_0 m(3m+8)(z+1)^{q+1} - 48H_0^2(12H_0^2\Gamma^{2q+2} - 3H_0 m$$

$$\times (z+1)^{2q+2})\Gamma^{2q+2} - 6^{\frac{3m}{8}+\frac{1}{2}}(3m+4)\sqrt{H_0^2(z+1)^{2q+2}}\left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}}$$

$$- (H_0^2(z+1)^{2q+2})^{3/2} + 108\sqrt{6}\sqrt{\mathrm{ad}}\left(H_0^2\Gamma^{2q+2}\right)^{3/2} - 72\, 6^{3/4} H_0^3(z+1)^{3q+3}\alpha$$

$$- 1728\, 6^{3/4} H_0^6\Gamma^{6q+6}\alpha + 864\, 6^{3/4} - 54\sqrt{6}H_0^3 m(z+1)^{3q+3}\beta - 1296\sqrt{6}H_0^6$$

$$\times m(z+1)^{6q+6}\beta + 648\sqrt{6}H_0^5 m\Gamma^{6q+6}\beta + H_0\Gamma^{q+1}c_1 + H_0^3(z+1)^{4q+4}c_1\Bigg]$$

$$\times \Bigg[18H_0^2(3m+4)(6^{\frac{3m}{8}+1} H_0^2\Gamma^{2q+2}(6f^3\sqrt{H_0^2\Gamma^{2q+2}}(4\sqrt[4]{6}\alpha + 3m\beta) - 12d)$$

$$\times \left(\frac{H_0^2\Gamma^{2q+2}}{m^2}\right)^{\frac{3m}{8}} + 2^{\frac{3m}{8}} 3^{\frac{3m}{8}+2} a\sqrt{d}\, f^3 m)^2\Bigg]^{-1} - \Bigg[f^3(z+1)^{-2q-2}\Bigg(- 6^{\frac{3m}{8}+\frac{1}{2}}$$

$$\times (3m+4)\sqrt{H_0^2\Gamma^{2q+2}} - 126^{3/4} H_0\alpha(z+1)^{q+1} - 9\sqrt{6}H_0 m\beta\Gamma^{q+1} + 108H_0^2$$

$$\times c_1(z+1)^{2q+2} + 36H_0 c_1\Gamma^{2q+2} - 432\, 6^{3/4} H_0^4\alpha(z+1)^{4q+4} + 6^{3/4} H_0^3\alpha\Gamma^{4q+4}$$

$$- 324\sqrt{6}H_0^4 m\beta(z+1)^{4q+4} + 216\sqrt{6}H_0^3 m\beta\Gamma^{4q+4}\Bigg] + b\sqrt{H_0^2(z+1)^{2q+2}}$$

$$+ 27\sqrt{6}\sqrt{\mathrm{ad}}\sqrt{H_0^2\Gamma^{2q+2}}\left(\frac{H_0^2(z+1)^{2q+2}}{m^2}\right)^{\frac{3m}{8}} + 2^{\frac{3m}{8}+1} 3^{\frac{3m}{8}+2} a\sqrt{d}\Bigg[- 96H_0^2$$

$$\times \Gamma^{2q+2} - 72H_0^2 m(z+1)^{2q+2} - 8(12H_0^2(z+1)^{2q+2} - 3H_0 m\Gamma^{2q+2})\Bigg]. \tag{C1}$$

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