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Hybrid relativistic and modified Toda lattice-type system: equivalent form, N-fold Darboux transformation and analytic solutions

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Abstract Studies of the lattice systems are of current interest due to their applications in relativity, optics, condensed matter physics and plasma physics. In this paper, we look into a hybrid relativistic and modified Toda lattice-type system. An equivalent form of that system is provided by virtue of certain transformations. Based on the Lax pair of that equivalent form, we construct an N-fold Darboux matrix and then derive the N-fold Darboux transformation, where N is a positive integer. Some analytic solutions are determined with the help of the associated N-fold Darboux transformation.

1 Introduction

Studies of the lattice systems have been seen of current interest due to their applications in optics, condensed matter physics and plasma physics [1–4]. The theory of relativity has not been separated from the rest of physics, for every physical theory is supposed to conform to the basic relativistic principles, and any concrete physical problem involves a synthesis of relativity and some specific physical theory [5, 6]. The lattice systems and relativistic lattice systems have served as a useful guide in the recent studies on nonlinear waves[1] [7–10].

The Toda lattice system has been introduced to study the vibrations in a chain with nonlinear interactions [21] and waves in an anharmonic lattice [22]. Numerical investigations have indicated that the Toda lattice system behaves remarkably like an integrable nonlinear system, where the integrability means that the system's Hamiltonian can be brought to an integrable form [23]. Subsequently, the complete integrability of the Toda lattice system has been confirmed [24, 25]. With a change of the variables, the Toda lattice system has been written in the Lax form [25–27]. As for the solutions of the Toda lattice system, inverse-scattering solutions have been derived via the inverse-scattering method [26], multi-soliton solutions have been constructed through the Bäcklund transformation method [28], positon solutions have been generated via the Darboux transformation (DT) [29], soliton solutions in the Wronskian form has been obtained [30] and complexiton solutions have been determined through the Casoratian formulation [31].

It has been said that certain relativistic particle systems are not only completely integrable at the classical level, but can also be quantized in such a fashion that the integrability survives [32, 33]. The relativistic Toda lattice system has been derived via the requirement of Poincaré invariance for the Toda lattice system [33]. Lax representation, complete integrability and scattering problem of the relativistic Toda lattice system have been studied [33]. A hereditary recursion operator and an alternative proof of the complete integrability of the relativistic Toda lattice system have been provided via the Lax representation [34]. Three different Lax representations and complete integrability of the relativistic Toda lattice system have been investigated [35]. The direct and inverse spectral problems of the relativistic Toda lattice system have been solved [36]. Conserved quantities, bi-Hamiltonian formulation and recursive structure of the relativistic Toda lattice system have been obtained through the master symmetries [37]. Relation between the discrete time Toda lattice system and relativistic Toda lattice system has been found in the context of the Hamiltonian dynamics [38]. Bilinear forms of the relativistic Toda lattice system have been carried out via the auxiliary dependent variables, based on which the multi-soliton solutions have been constructed in the form of the Casorati determinant [39]. Other relativistic lattice systems such as the discrete-time relativistic Toda lattice system, relativistic Volterra lattice system and relativistic Lotka-Volterra lattice system have also been seen [40, 41]. Bilinear forms and multi-soliton solutions in the form of the Casorati determinant of the discrete-time relativistic Toda lattice system and relativistic Lotka-Volterra lattice system have been derived via the Hirota method

[1] Investigations on certain nonlinear waves of the continuous nonlinear systems have been shown, e.g., in Refs. [11–20].

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and Casoratian technique [42, 43]. N-fold DT and some solutions of the relativistic Volterra lattice system have been obtained through a gauge transformation and Lax pair, where N is a positive integer [44].

Based on the work of Ref. [45], Ref. [46] has presented the following hybrid relativistic and modified Toda lattice-type system:

$$p_{n,t} = \alpha(\Lambda - 1)e^{q_n - q_{n-1}} + \beta^2(\Lambda - 1)e^{q_n - q_{n-1} + p_{n-1}}, \tag{1a}$$

$$q_{n,t} = e^{p_n}\left(1 + \beta^2 e^{q_{n+1} - q_n}\right), \tag{1b}$$

where the integer n stands for a discrete spatial variable, q_n and p_n mean two differentiable functions of the continuous temporal variable t, $q_{n,t} = \frac{dq_n}{dt}$ and $p_{n,t} = \frac{dp_n}{dt}$, Λ is a shift operator defined via $\Lambda p_n = p_{n+1}$, while α and β denote two real constants which are respectively modified-Toda-lattice related and relativistic-Toda-lattice related. Dynamic issues with respect to System (1) have been seen in Refs. [45, 46]. Lax pair, one-fold DT and certain solutions of System (1) have been obtained [46].

However, as far as we know, equivalent form of System (1) has not been reported. In addition, N-fold DT of that equivalent form and some analytic solutions which are different from those in Ref. [46] have not been discussed. In Sect. 2, an equivalent form of System (1) will be presented via certain transformations. In Sect. 3, N-fold Darboux matrix and N-fold DT of that equivalent form will be determined via the equivalent-form Lax pair. In Sect. 4, some analytic solutions of that equivalent form will be obtained based on our N-fold DT. In Sect. 5, we shall give the conclusions.

2 Equivalent form of System (1)

For the convenience of deriving the N-fold DT and some analytic solutions, we take the transformations

$$r_n = e^{p_n}, \qquad s_n = e^{q_n}, \tag{2}$$

and then System (1) is equivalent to the following system:

$$r_{n,t} = r_n\left[\alpha(\Lambda - 1)\frac{s_n}{s_{n-1}} + \beta^2(\Lambda - 1)\frac{r_{n-1}s_n}{s_{n-1}}\right], \tag{3a}$$

$$s_{n,t} = r_n\left(s_n + \beta^2 s_{n+1}\right), \tag{3b}$$

whose Lax pair is expressed as

$$\Lambda\Psi_n = U_n\Psi_n, \qquad U_n = \begin{pmatrix} r_n\lambda - \lambda^{-1} & s_n \\ -\frac{\beta^2 r_n + \alpha}{s_n} & \alpha\lambda \end{pmatrix}, \tag{4a}$$

$$\Psi_{n,t} = V_n\Psi_n, \qquad V_n = \begin{pmatrix} \frac{s_n(\beta^2 r_{n-1} + \alpha)}{s_{n-1}} & -s_n\lambda^{-1} \\ \frac{\beta^2 r_{n-1} + \alpha}{s_{n-1}}\lambda^{-1} & -\lambda^{-2} \end{pmatrix}, \tag{4b}$$

which is consistent with the Lax pair in Ref. [46] under Transformations (2), where $\Psi_n = (\Psi_{1,n}, \Psi_{2,n})^T$, $\Psi_{1,n}$ and $\Psi_{2,n}$ are two differentiable functions of t, λ is a spectral parameter and the superscript "T" refers to the transpose for a vector/matrix.

3 N-fold DT of System (3)

For the purpose of constructing an N-fold DT of System (3), we first introduce a gauge transformation [47, 48]

$$\widetilde{\Psi}_n = M_n\Psi_n, \tag{5}$$

where M_n is a reversible matrix and $\widetilde{\Psi}_n$ is necessary to satisfy

$$\Lambda\widetilde{\Psi}_n = \widetilde{U}_n\widetilde{\Psi}_n, \qquad \widetilde{U}_n = M_{n+1}U_nM_n^{-1}, \tag{6a}$$

$$\widetilde{\Psi}_{n,t} = \widetilde{V}_n\widetilde{\Psi}_n, \qquad \widetilde{V}_n = \left(M_{n,t} + M_nV_n\right)M_n^{-1}, \tag{6b}$$

while $\widetilde{U}_n$ and $\widetilde{V}_n$ have the same structures as U_n and V_n, respectively, with the exception that the previous potentials r_n, s_n have been substituted by new ones $\widetilde{r}_n$, $\widetilde{s}_n$, and the superscript "-1" means the inverse of a matrix. We assume that N-fold Darboux matrix M_n takes the form of a polynomial matrix of λ, as shown below:

$$M_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} = \begin{pmatrix} \lambda^{-N} + \sum_{j=1}^{N} A_n^{(-N+2j)}\lambda^{-N+2j} & \sum_{j=1}^{N} B_n^{(N-2j+1)}\lambda^{N-2j+1} \\ \sum_{j=1}^{N} C_n^{(N-2j+1)}\lambda^{N-2j+1} & \lambda^N + \sum_{j=1}^{N} D_n^{(N-2j)}\lambda^{N-2j} \end{pmatrix}, \tag{7}$$

where A_n, B_n, C_n and D_n are some to-be-determined functions of n and t. Assuming that λ_ι's ($\lambda_\iota \neq 0$, $\iota = 1, 2, \ldots, 2N$) are the $2N$ roots of $\det M_n$, we get

$$\det M_n = A_n^{(N)} \lambda^{-2N} \prod_{\iota=1}^{2N} \left(\lambda^2 - \lambda_\iota^2\right). \tag{8}$$

Consequently, we can use the linear algebraic system shown below to find $A_n^{(-N+2j)}$'s, $B_n^{(N-2j+1)}$'s, $C_n^{(N-2j+1)}$'s and $D_n^{(N-2j)}$'s uniquely:

$$\sum_{j=1}^{N} A_n^{(-N+2j)} \lambda_\iota^{-N+2j} + \delta_{\iota,n} \sum_{j=1}^{N} B_n^{(N-2j+1)} \lambda_\iota^{N-2j+1} = -\lambda_\iota^{-N}, \tag{9a}$$

$$\sum_{j=1}^{N} C_n^{(N-2j+1)} \lambda_\iota^{N-2j+1} + \delta_{\iota,n} \sum_{j=1}^{N} D_n^{(N-2j)} \lambda_\iota^{N-2j} = -\delta_{\iota,n} \lambda_\iota^{N}, \tag{9b}$$

where

$$\delta_{\iota,n} = \frac{\phi_{2,n}(\lambda_\iota) + \theta_\iota \psi_{2,n}(\lambda_\iota)}{\phi_{1,n}(\lambda_\iota) + \theta_\iota \psi_{1,n}(\lambda_\iota)}, \tag{10}$$

$\phi_n(\lambda_\iota) = \left(\phi_{1,n}(\lambda_\iota), \phi_{2,n}(\lambda_\iota)\right)^T$ and $\psi_n(\lambda_\iota) = \left(\psi_{1,n}(\lambda_\iota), \psi_{2,n}(\lambda_\iota)\right)^T$ are the solutions of Lax Pair (4), while the $4N$ parameters λ_ι's, θ_ι's ($\lambda_\iota \neq \lambda_\jmath$, $\iota \neq \jmath$, $\theta_\iota \neq 0$) are set in such a way that the determinant of the coefficients for Eq. (9) is non-zero.

Proposition 1 *The matrix $\widetilde{U}_n$ defined via Eq. (6a) has the same structure as that of U_n, i.e.,*

$$\widetilde{U}_n = \begin{pmatrix} \widetilde{r}_n \lambda - \lambda^{-1} & \widetilde{s}_n \\ -\frac{\beta^2 \widetilde{r}_n + \alpha}{\widetilde{s}_n} & \alpha\lambda \end{pmatrix},$$

where the transformations from the old potentials r_n, s_n into the new ones $\widetilde{r}_n$, $\widetilde{s}_n$ are given by

$$\widetilde{r}_n = \frac{A_{n+1}^{(N)}}{A_n^{(N)}} r_n, \tag{11a}$$

$$\widetilde{s}_n = \frac{s_n + B_n^{(-N+1)}}{D_n^{(-N)}}. \tag{11b}$$

Proof Let $M_n^{-1} = M_n^* / \det M_n$ and

$$M_{n+1} U_n M_n^* = \begin{pmatrix} f_{11,n}(\lambda) & f_{12,n}(\lambda) \\ f_{21,n}(\lambda) & f_{22,n}(\lambda) \end{pmatrix},$$

in which

$$\begin{aligned}
f_{11,n}(\lambda) &= -A_{n+1} D_n \lambda^{-1} + (r_n A_{n+1} D_n - \alpha B_{n+1} C_n)\lambda - \frac{\left(\beta^2 r_n + \alpha\right) B_{n+1} D_n}{s_n} - s_n A_{n+1} C_n, \\
f_{12,n}(\lambda) &= A_{n+1} B_n \lambda^{-1} + (\alpha A_n B_{n+1} - r_n A_{n+1} B_n)\lambda + \frac{\left(\beta^2 r_n + \alpha\right) B_n B_{n+1}}{s_n} + s_n A_n A_{n+1}, \\
f_{21,n}(\lambda) &= -C_{n+1} D_n \lambda^{-1} + (r_n C_{n+1} D_n - \alpha C_n D_{n+1})\lambda - \frac{\left(\beta^2 r_n + \alpha\right) D_n D_{n+1}}{s_n} - s_n C_n C_{n+1}, \\
f_{22,n}(\lambda) &= B_n C_{n+1} \lambda^{-1} + (\alpha A_n D_{n+1} - r_n B_n C_{n+1})\lambda + \frac{\left(\beta^2 r_n + \alpha\right) B_n D_{n+1}}{s_n} + s_n A_n C_{n+1},
\end{aligned} \tag{12}$$

and the superscript "$*$" means the adjoint of a matrix. According to Eq. (4a), (9) and (10), we have

$$A_n(\lambda_\iota) = -\delta_{\iota,n} B_n(\lambda_\iota), \quad C_n(\lambda_\iota) = -\delta_{\iota,n} D_n(\lambda_\iota), \quad \delta_{\iota,n+1} = \frac{-\left(\beta^2 r_n + \alpha\right) + \alpha s_n \lambda_\iota \delta_{\iota,n}}{s_n\left(r_n \lambda_\iota - \lambda_\iota^{-1} + s_n \delta_{\iota,n}\right)}. \tag{13}$$

After that, using Eqs. (12) and (13), we can confirm that λ_ι's are the roots of $f_{11,n}(\lambda)$, $f_{12,n}(\lambda)$, $f_{21,n}(\lambda)$ and $f_{22,n}(\lambda)$.

According to Eq. (12), we realize that $f_{11,n}(\lambda)$ is the polynomial in λ with the highest order $2N + 1$ and lowest order $-2N - 1$, $f_{12,n}(\lambda)$ and $f_{21,n}(\lambda)$ are the polynomials in λ with the highest order $2N$ and lowest order $-2N$, while $f_{22,n}(\lambda)$ is the polynomial in λ with the highest order $2N+1$ and lowest order $-2N+1$. Making use of Eq. (8), it is possible to verify the existence of a matrix H_n such that

$$M_{n+1}U_n M_n^* = \det M_n \cdot H_n, \tag{14}$$

where

$$H_n = \begin{pmatrix} h_{11,n}^{(1)}\lambda + h_{11,n}^{(-1)}\lambda^{-1} & h_{12,n}^{(0)} \\ h_{21,n}^{(0)} & h_{22,n}^{(1)}\lambda \end{pmatrix},$$

while $h_{11,n}^{(1)}$, $h_{11,n}^{(-1)}$, $h_{12,n}^{(0)}$, $h_{21,n}^{(0)}$ and $h_{22,n}^{(1)}$ are some to-be-determined functions of n and t. Following that, rewriting Eq. (14) as

$$M_{n+1}U_n = H_n M_n, \tag{15}$$

and equating the same powers of λ in Eq. (15), we are able to derive

$$h_{11,n}^{(1)} = \frac{A_{n+1}^{(N)}}{A_n^{(N)}} r_n = \widetilde{r}_n, \qquad h_{11,n}^{(-1)} = -1, \qquad h_{12,n}^{(0)} = \frac{s_n + B_n^{(-N+1)}}{D_n^{(-N)}} = \widetilde{s}_n, \qquad h_{22,n}^{(1)} = \alpha, \tag{16}$$

and

$$\begin{aligned} & s_n C_{n+1}^{(-N+1)} + \alpha\left(D_{n+1}^{(-N)} - D_n^{(-N)}\right) - h_{21,n}^{(0)} B_n^{(-N+1)} = 0, \\ & h_{21,n}^{(0)} + C_{n+1}^{(-N+1)} + \frac{\beta^2 r_n + \alpha}{s_n} D_{n+1}^{(-N)} = 0. \end{aligned} \tag{17}$$

Solving Eqs. (9) gives rise to

$$D_n^{(-N)} = \prod_{\iota=1}^{2N} \lambda_\iota^2 A_n^{(N)}, \tag{18}$$

and then Eqs. (17) and (18) bring about

$$h_{21,n}^{(0)} = -\frac{\beta^2 \widetilde{r}_n + \alpha}{\widetilde{s}_n}. \tag{19}$$

From Eqs. (6a), (16) and (19), we obtain the conclusion of Proposition 1. The proof is completed. □

Proposition 2 *The matrix $\widetilde{V}_n$ defined via Eq. (6b) has the same structure as that of V_n, i.e.,*

$$\widetilde{V}_n = \begin{pmatrix} \frac{\widetilde{s}_n(\beta^2\widetilde{r}_{n-1}+\alpha)}{\widetilde{s}_{n-1}} & -\widetilde{s}_n\lambda^{-1} \\ \frac{\beta^2\widetilde{r}_{n-1}+\alpha}{\widetilde{s}_{n-1}}\lambda^{-1} & -\lambda^{-2} \end{pmatrix},$$

under Transformations (11).

Proof Let

$$\left(M_{n,t} + M_n V_n\right) M_n^* = \begin{pmatrix} g_{11,n}(\lambda) & g_{12,n}(\lambda) \\ g_{21,n}(\lambda) & g_{22,n}(\lambda) \end{pmatrix},$$

in which

$$\begin{aligned} g_{11,n}(\lambda) =& B_n C_n \lambda^{-2} + \left[s_n A_n C_n + \frac{B_n D_n\left(\beta^2 r_{n-1} + \alpha\right)}{s_{n-1}} \right]\lambda^{-1} + \frac{s_n A_n D_n\left(\beta^2 r_{n-1}+\alpha\right)}{s_{n-1}} + A_{n,t}D_n - B_{n,t}C_n, \\ g_{12,n}(\lambda) =& - A_n B_n \lambda^{-2} - \left[s_n A_n^2 + \frac{B_n^2\left(\beta^2 r_{n-1}+\alpha\right)}{s_{n-1}} \right]\lambda^{-1} - \frac{s_n A_n B_n\left(\beta^2 r_{n-1}+\alpha\right)}{s_{n-1}} + A_n B_{n,t} - A_{n,t}B_n, \\ g_{21,n}(\lambda) =& C_n D_n \lambda^{-2} + \left[s_n C_n^2 + \frac{D_n^2\left(\beta^2 r_{n-1}+\alpha\right)}{s_{n-1}} \right]\lambda^{-1} + \frac{s_n C_n D_n\left(\beta^2 r_{n-1}+\alpha\right)}{s_{n-1}} + C_{n,t}D_n - C_n D_{n,t}, \\ g_{22,n}(\lambda) =& - A_n D_n \lambda^{-2} - \left[s_n A_n C_n + \frac{B_n D_n\left(\beta^2 r_{n-1}+\alpha\right)}{s_{n-1}} \right]\lambda^{-1} - \frac{s_n B_n C_n\left(\beta^2 r_{n-1}+\alpha\right)}{s_{n-1}} + A_n D_{n,t} - B_n C_{n,t}. \end{aligned} \tag{20}$$

With the aid of Eqs. (4b), (9) and (10), we obtain that

$$
\begin{aligned}
&A_{n,t}(\lambda_\iota) = -\delta_{\iota,n,t} B_n(\lambda_\iota) - \delta_{\iota,n} B_{n,t}(\lambda_\iota),\\
&C_{n,t}(\lambda_\iota) = -\delta_{\iota,n,t} D_n(\lambda_\iota) - \delta_{\iota,n} D_{n,t}(\lambda_\iota),\\
&\delta_{\iota,n,t} = \frac{\beta^2 r_{n-1} + \alpha}{s_{n-1}} \lambda_\iota^{-1} - \left[\lambda_\iota^{-2} + \frac{s_n\left(\beta^2 r_{n-1} + \alpha\right)}{s_{n-1}}\right] \delta_{\iota,n} + s_n \lambda_\iota^{-1} \delta_{\iota,n}^2.
\end{aligned} \tag{21}
$$

In terms of Eqs. (20) and (21), it is possible to check that λ_ι are all the roots of $g_{11,n}(\lambda)$, $g_{12,n}(\lambda)$, $g_{21,n}(\lambda)$ and $g_{22,n}(\lambda)$.

On the basis of Eqs. (20), we know that $g_{11,n}(\lambda)$ is the polynomial in λ with the highest order $2N$ and lowest order $-2N$, $g_{12,n}(\lambda)$ and $g_{21,n}(\lambda)$ are the polynomials in λ with the highest order $2N-1$ and lowest order $-2N-1$, while $g_{22,n}(\lambda)$ is the polynomial in λ with the highest order $2N-2$ and lowest order $-2N-2$. With the help of Eq. (8), it can be demonstrated that there exists a matrix K_n such that

$$
\left(M_{n,t} + M_n V_n\right) M_n^* = \det M_n \cdot K_n, \tag{22}
$$

where

$$
K_n = \begin{pmatrix} k_{11,n}^{(0)} & k_{12,n}^{(-1)} \lambda^{-1} \\ k_{21,n}^{(-1)} \lambda^{-1} & k_{22,n}^{(-2)} \lambda^{-2} \end{pmatrix},
$$

while $k_{11,n}^{(0)}$, $k_{12,n}^{(-1)}$, $k_{21,n}^{(-1)}$ and $k_{22,n}^{(-2)}$ are some to-be-determined functions of n and t. To calculate K_n, we rewrite Eq. (22) as

$$
M_{n,t} + M_n V_n = K_n M_n. \tag{23}
$$

By equating the same powers of λ in Eq. (23), we derive

$$
k_{12,n}^{(-1)} = -\frac{s_n + B_n^{(-N+1)}}{D_n^{(-N)}} = -\widetilde{s}_n, \qquad k_{22,n}^{(-2)} = -1, \tag{24}
$$

and

$$
\begin{aligned}
&k_{11,n}^{(0)} - \frac{s_n + B_n^{(-N+1)}}{D_n^{(-N)}} C_n^{(-N+1)} - \frac{s_n\left(\beta^2 r_{n-1} + \alpha\right)}{s_{n-1}} - \frac{\beta^2 r_{n-1} + \alpha}{s_{n-1}} B_n^{(-N+1)} = 0,\\
&k_{21,n}^{(-1)} - \frac{\beta^2 r_{n-1} + \alpha}{s_{n-1}} D_n^{(-N)} - C_n^{(-N+1)} = 0,\\
&A_{n,t}^{(N)} - k_{11,n}^{(0)} A_n^{(N)} + \frac{s_n\left(\beta^2 r_{n-1} + \alpha\right)}{s_{n-1}} A_n^{(N)} = 0,\\
&D_{n,t}^{(-N)} - k_{21,n}^{(-1)} B_n^{(-N+1)} - s_n C_n^{(-N+1)} = 0.
\end{aligned} \tag{25}
$$

Taking Eqs. (17), (18) and (25) into account, we obtain

$$
k_{11,n}^{(0)} = \frac{\widetilde{s}_n\left(\beta^2 \widetilde{r}_{n-1} + \alpha\right)}{\widetilde{s}_{n-1}}, \qquad k_{21,n}^{(-1)} = \frac{\beta^2 \widetilde{r}_{n-1} + \alpha}{\widetilde{s}_{n-1}}. \tag{26}
$$

From Eqs. (6b), (24) and (26), we arrive at the conclusion of Proposition 2. The proof is completed. □

Further, we provide the following theorem in light of Propositions 1 and 2:

Theorem 1 *Let r_n and s_n be the seed solutions of System (3), $\phi_n(\lambda) = \left(\phi_{1,n}(\lambda), \phi_{2,n}(\lambda)\right)^T$ and $\psi_n(\lambda) = \left(\psi_{1,n}(\lambda), \psi_{2,n}(\lambda)\right)^T$ be two linearly independent solutions of Lax Pair (4), then the N-Fold DT is given by Transformation (5) and*

$$
\widetilde{r}_n = \frac{A_{n+1}^{(N)}}{A_n^{(N)}} r_n, \qquad \widetilde{s}_n = \frac{s_n + B_n^{(-N+1)}}{D_n^{(-N)}},
$$

where

$$
A_n^{(N)} = \frac{\Delta A_n^{(N)}}{\Delta_1}, \qquad B_n^{(-N+1)} = \frac{\Delta B_n^{(-N+1)}}{\Delta_1}, \qquad D_n^{(-N)} = \frac{\Delta D_n^{(-N)}}{\Delta_2},
$$

$$\Delta_1 = \begin{vmatrix} \lambda_1^{-N+2} & \lambda_1^{-N+4} & \cdots & \lambda_1^{N} & \lambda_1^{-N+1}\delta_{1,n} & \lambda_1^{-N+3}\delta_{1,n} & \cdots & \lambda_1^{N-1}\delta_{1,n} \\ \lambda_2^{-N+2} & \lambda_2^{-N+4} & \cdots & \lambda_2^{N} & \lambda_2^{-N+1}\delta_{2,n} & \lambda_2^{-N+3}\delta_{2,n} & \cdots & \lambda_2^{N-1}\delta_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{2N}^{-N+2} & \lambda_{2N}^{-N+4} & \cdots & \lambda_{2N}^{N} & \lambda_{2N}^{-N+1}\delta_{2N,n} & \lambda_{2N}^{-N+3}\delta_{2N,n} & \cdots & \lambda_{2N}^{N-1}\delta_{2N,n} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \lambda_1^{-N+1} & \lambda_1^{-N+3} & \cdots & \lambda_1^{N-1} & \lambda_1^{-N}\delta_{1,n} & \lambda_1^{-N+2}\delta_{1,n} & \cdots & \lambda_1^{N-2}\delta_{1,n} \\ \lambda_2^{-N+1} & \lambda_2^{-N+3} & \cdots & \lambda_2^{N-1} & \lambda_2^{-N}\delta_{2,n} & \lambda_2^{-N+2}\delta_{2,n} & \cdots & \lambda_2^{N-2}\delta_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{2N}^{-N+1} & \lambda_{2N}^{-N+3} & \cdots & \lambda_{2N}^{N-1} & \lambda_{2N}^{-N}\delta_{2N,n} & \lambda_{2N}^{-N+2}\delta_{2N,n} & \cdots & \lambda_{2N}^{N-2}\delta_{2N,n} \end{vmatrix},$$

$\Delta A_n^{(N)}$ and $\Delta B_n^{(-N+1)}$ are respectively produced from Δ_1 by replacing the Nth and $(N+1)$th columns with $(-\lambda_1^{-N}, -\lambda_2^{-N}, \ldots, -\lambda_{2N}^{-N})^T$, while $\Delta D_n^{(-N)}$ is produced from Δ_2 by replacing the $(N+1)$th column with $(-\lambda_1^{N}\delta_{1,n}, -\lambda_2^{N}\delta_{2,n}, \ldots, -\lambda_{2N}^{N}\delta_{2N,n})^T$.

4 Analytic solutions of System (3)

To determine some analytic solutions of System (3), we choose the non-zero seed solutions of System (3) as

$$r_n = 1, \qquad s_n = e^{(1+\beta^2)t}. \tag{27}$$

Utilizing Seed Solutions (27), we obtain the two solutions of Lax Pair (4) as

$$\phi_n = \begin{pmatrix} \phi_{1,n}(\lambda_\iota) \\ \phi_{2,n}(\lambda_\iota) \end{pmatrix} = \begin{pmatrix} \chi_1(\lambda_\iota)[\kappa_1(\lambda_\iota)]^n e^{\mu_1(\lambda_\iota)t} \\ [\kappa_1(\lambda_\iota)]^n e^{[\mu_1(\lambda_\iota)-(1+\beta^2)]t} \end{pmatrix}, \tag{28a}$$

$$\psi_n = \begin{pmatrix} \psi_{1,n}(\lambda_\iota) \\ \psi_{2,n}(\lambda_\iota) \end{pmatrix} = \begin{pmatrix} \chi_2(\lambda_\iota)[\kappa_2(\lambda_\iota)]^n e^{\mu_2(\lambda_\iota)t} \\ [\kappa_2(\lambda_\iota)]^n e^{[\mu_2(\lambda_\iota)-(1+\beta^2)]t} \end{pmatrix}, \tag{28b}$$

where

$$\chi_1(\lambda_\iota) = \frac{1}{2(\beta^2+\alpha)}\left[\lambda_\iota^{-1} + (\alpha-1)\lambda_\iota + \lambda_\iota^{-1}\sqrt{1-2(1+2\beta^2+\alpha)\lambda_\iota^2 + (\alpha-1)^2\lambda_\iota^4}\right],$$
$$\chi_2(\lambda_\iota) = \frac{1}{2(\beta^2+\alpha)}\left[\lambda_\iota^{-1} + (\alpha-1)\lambda_\iota - \lambda_\iota^{-1}\sqrt{1-2(1+2\beta^2+\alpha)\lambda_\iota^2 + (\alpha-1)^2\lambda_\iota^4}\right],$$
$$\kappa_1(\lambda_\iota) = \frac{1}{2}\left[-\lambda_\iota^{-1} + (\alpha+1)\lambda_\iota - \lambda_\iota^{-1}\sqrt{1-2(1+2\beta^2+\alpha)\lambda_\iota^2 + (\alpha-1)^2\lambda_\iota^4}\right],$$
$$\kappa_2(\lambda_\iota) = \frac{1}{2}\left[-\lambda_\iota^{-1} + (\alpha+1)\lambda_\iota + \lambda_\iota^{-1}\sqrt{1-2(1+2\beta^2+\alpha)\lambda_\iota^2 + (\alpha-1)^2\lambda_\iota^4}\right],$$
$$\mu_1(\lambda_\iota) = \frac{1}{2}\left[-\lambda_\iota^{-2} + (2\beta^2+\alpha+1) + \lambda_\iota^{-2}\sqrt{1-2(1+2\beta^2+\alpha)\lambda_\iota^2 + (\alpha-1)^2\lambda_\iota^4}\right],$$
$$\mu_2(\lambda_\iota) = \frac{1}{2}\left[-\lambda_\iota^{-2} + (2\beta^2+\alpha+1) - \lambda_\iota^{-2}\sqrt{1-2(1+2\beta^2+\alpha)\lambda_\iota^2 + (\alpha-1)^2\lambda_\iota^4}\right].$$

According to Eqs. (10), (13) and (28), we have

$$\begin{aligned} \delta_{\iota,n} &= \frac{[\kappa_1(\lambda_\iota)]^n e^{[\mu_1(\lambda_\iota)-(1+\beta^2)]t} + \theta_\iota[\kappa_2(\lambda_\iota)]^n e^{[\mu_2(\lambda_\iota)-(1+\beta^2)]t}}{\chi_1(\lambda_\iota)[\kappa_1(\lambda_\iota)]^n e^{\mu_1(\lambda_\iota)t} + \theta_\iota\chi_2(\lambda_\iota)[\kappa_2(\lambda_\iota)]^n e^{\mu_2(\lambda_\iota)t}}, \\ \delta_{\iota,n+1} &= \frac{-(\beta^2+\alpha) + \alpha e^{(1+\beta^2)t}\lambda_\iota\delta_{\iota,n}}{e^{(1+\beta^2)t}\left[\lambda_\iota - \lambda_\iota^{-1} + e^{(1+\beta^2)t}\delta_{\iota,n}\right]}. \end{aligned} \tag{29}$$

When $N = 2$, making use of Theorem 1, two-fold solutions of System (3) can be expressed as

$$\widetilde{r}_n = \frac{A_{n+1}^{(2)}}{A_n^{(2)}}, \tag{30a}$$
$$\widetilde{s}_n = \frac{e^{(1+\beta^2)t} + B_n^{(-1)}}{D_n^{(-2)}}, \tag{30b}$$

where

$$A_n^{(2)} = \frac{\Delta A_n^{(2)}}{\Delta_1}, \quad B_n^{(-1)} = \frac{\Delta B_n^{(-1)}}{\Delta_1}, \quad D_n^{(-2)} = \frac{\Delta D_n^{(-2)}}{\Delta_2},$$

with

$$\Delta_1 = \begin{vmatrix} 1 & \lambda_1^2 & \lambda_1^{-1}\delta_{1,n} & \lambda_1\delta_{1,n} \\ 1 & \lambda_2^2 & \lambda_2^{-1}\delta_{2,n} & \lambda_2\delta_{2,n} \\ 1 & \lambda_3^2 & \lambda_3^{-1}\delta_{3,n} & \lambda_3\delta_{3,n} \\ 1 & \lambda_4^2 & \lambda_4^{-1}\delta_{4,n} & \lambda_4\delta_{4,n} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \lambda_1^{-1} & \lambda_1 & \lambda_1^{-2}\delta_{1,n} & \delta_{1,n} \\ \lambda_2^{-1} & \lambda_2 & \lambda_2^{-2}\delta_{2,n} & \delta_{2,n} \\ \lambda_3^{-1} & \lambda_3 & \lambda_3^{-2}\delta_{3,n} & \delta_{3,n} \\ \lambda_4^{-1} & \lambda_4 & \lambda_4^{-2}\delta_{4,n} & \delta_{4,n} \end{vmatrix}, \quad \Delta A_n^{(2)} = \begin{vmatrix} 1 & -\lambda_1^{-2} & \lambda_1^{-1}\delta_{1,n} & \lambda_1\delta_{1,n} \\ 1 & -\lambda_2^{-2} & \lambda_2^{-1}\delta_{2,n} & \lambda_2\delta_{2,n} \\ 1 & -\lambda_3^{-2} & \lambda_3^{-1}\delta_{3,n} & \lambda_3\delta_{3,n} \\ 1 & -\lambda_4^{-2} & \lambda_4^{-1}\delta_{4,n} & \lambda_4\delta_{4,n} \end{vmatrix},$$

$$\Delta B_n^{(-1)} = \begin{vmatrix} 1 & \lambda_1^2 & -\lambda_1^{-2} & \lambda_1\delta_{1,n} \\ 1 & \lambda_2^2 & -\lambda_2^{-2} & \lambda_2\delta_{2,n} \\ 1 & \lambda_3^2 & -\lambda_3^{-2} & \lambda_3\delta_{3,n} \\ 1 & \lambda_4^2 & -\lambda_4^{-2} & \lambda_4\delta_{4,n} \end{vmatrix}, \quad \Delta D_n^{(-2)} = \begin{vmatrix} \lambda_1^{-1} & \lambda_1 & -\lambda_1^2\delta_{1,n} & \delta_{1,n} \\ \lambda_2^{-1} & \lambda_2 & -\lambda_2^2\delta_{2,n} & \delta_{2,n} \\ \lambda_3^{-1} & \lambda_3 & -\lambda_3^2\delta_{3,n} & \delta_{3,n} \\ \lambda_4^{-1} & \lambda_4 & -\lambda_4^2\delta_{4,n} & \delta_{4,n} \end{vmatrix}.$$

5 Conclusions

Studies of the lattice systems have been seen of current interest due to their applications in relativity, optics, condensed matter physics and plasma physics. In this paper, we have studied a hybrid relativistic and modified Toda lattice-type system, i.e., System (1). It has been known that System (1) is related to the relativistic Toda lattice and modified Toda lattice. Under Transformations (2), we have given an equivalent form of System (1), i.e., System (3), along with Lax Pair (4). With the aid of Lax Pair (4), we have constructed N-Fold DT (5) and (11) of System (3). Via N-Fold DT (5) and (11) with $N = 2$, we have obtained Two-Fold Solutions (30) of System (3).

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Declarations

Conflict of interest The authors have no conflicts to disclose.

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