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Kasner universes in $f(T, \hat{B})$ gravity

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Abstract In the context of the modified higher-order teleparallel $f\left(T, \hat{B}\right)$ gravity and specifically in the case of $f\left(T, \hat{B}\right) = T + F\left(\hat{B}\right)$ theory, we investigate the existence of anisotropic solutions in a Bianchi I spacetime. Invariant $\hat{B}$ is defined as $\hat{B} = B - 2T$, in which B is term which relates the Ricciscalar R with the torsion scalar T, $R = -T + B$, and $\hat{B}$ is also a boundary term. We investigate the existence of Kasner and Kasner-like solutions. We find the conditions for the limit of General Relativity, while Kasner-like solutions exist as asymptotic solutions for the field equations. Furthermore, anisotropic exponential solutions are not preferred by the theory. As far as the Kasner universes are concerned, we show that they are always unstable in this fourth-order theory of gravity. Finally, we discuss the condition for the general $f\left(T, \hat{B}\right)$ theory where the Kasner universes are recovered.

1 Introduction

One of the most famous cosmological solutions of General Relativity in vacuum is that of the anisotropic Kasner universe [1]. The latter solution describes the anisotropic evolution of the spacetime in a Bianchi background spacetime which is invariant under a three-dimensional Abelian translation group, the spacetime is known as Bianchi I space [2]. Kasner spacetimes are described by three parameters $p_1,\ p_2,\ p_3$, the indices of the three power-law scale factors, which satisfy the so-called Kasner algebraic relations. Specifically, the Kasner indices take the values of the common points of a three-dimensional unitary sphere, $p_1^2 + p_2^2 + p_3^2 = 1$, and of a plane with the sum of the Kasner parameters to be one; that is, $p_1 + p_2 + p_3 = 1$ [1]. Because of the two Kasner algebraic relations, the Kasner universes are one-parameter family of exact solutions. In the case of the Mixmaster universe, the Kasner solution describes the dynamics and the evolution of the field equations when the effects of the Ricci scalar of the three-dimensional spatial hypersurface are negligible, as we reach the initial singularity. Indeed, the Kasner power-law solutions can approximate the general Bianchi models at intermediate stages of their evolutions and at early and late-time asymptotes [3].

Furthermore, there are also many cosmological applications in the literature of the Kasner universes, these applications cover various areas of cosmological studies such is the particle creation, baryosynthesis and many others [4–14]. Because of the importance of the Kasner solution, there are various studies on the generalization. Kasner-like exact solutions have been investigated in details in literature for higher dimensional theories of gravity and for alternative modified theories of gravity. Kasner-like solutions have the property to admit additional Kasner parameters, with Kasner-like relations similar to that of General Relativity [15–22].

A family of modified theories of gravity which have drawn the attention of cosmologists are these which are based on the modification of the teleparallel theory of gravity. Although General Relativity is based on the use of the torsion-less Levi-Civita connection, in teleparallel gravitational theory the curvature-less Weitzenböck connection is used, while the invariant which is used for the definition of the gravitational Action Integral is that of the torsion scalar T. A natural extension of teleparallel theories of gravity is to define as gravitational Lagrangian a function f of the torsion scalar T, or of other invariants defined by T, see for instance [23–30]. In this work we are interested in the higher-order teleparallel theory of gravity known as $f(T, B)$ theory where B is the boundary term related to the Ricciscalar as $T = R - B$ [30]. $f(T)$ theory of gravity is a second-order theory because T includes only first-order derivatives for the vierbein fields [23]. However, because B includes second-order derivatives of the vierbein fields, $f(T, B)$ theory is a fourth-order theory of gravity. A special case for the function $f(T, B)$ is the $f(T, B) = T + F(B)$. In this case it was found that in cosmological studies the field equations can be derived by a point-like Lagrangian with the same dynamical degrees of freedom as the scalar tensor theories. Indeed, by using a Lagrange multiplier the higher-order derivatives can be attributed

In memory of Loulouditsa.

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to a scalar field [31–33]. Recently, by using this property for the $f(T, B)$ theory the quantization of the field equations and the exact solutions of the Wheeler-DeWitt equations were investigated in [34].

In this study we investigate the existence of Kasner and Kasner-like solutions if $f(T, B)$ theory. It is preferable in alternative theories of gravity the limit of General Relativity to be recovered. In the case of $f(T)$ gravity, Kasner and Kasner-like solutions were found for the first time in [35] while the stability of these solutions was the subject of study in [36, 37]. We extend the previous results in the modified teleparallel theory and we study the limit of General Relativity in $f(T, B) = T + F(B)$ theory in a vacuum Bianchi I background space while we investigate the stability of the Kasner and Kasner-like solutions. In addition we study the existence of anisotropic exponential solutions. The plan of the paper is as follows.

In Sect. 2, we define the cosmological model of our study which is that of $f(T, B)$ theory. We focus on the special case of $f(T, B) = T + F(B)$ theory and we present the gravitational field equations in the case of Bianchi I background space. In Sect. 3 we investigate the limit of General Relativity, and specifically we search for the conditions where Kasner and Kasner-like solutions exist. Moreover, the limit of anisotropic Bianchi I solution with cosmological constant term is also studied. The stability properties of the Kasner solutions are investigated in Sect. 4. Finally, in Sect. 5, we summarize our results and present the algebraic constraint in which for general $f(T, B)$ theory the Kasner solution is recovered.

2 $f(T, B)$ theory

The fundamental geometric objects of teleparallelism are the vierbein fields $e_\mu(x^\sigma)$ [?]. The vierbein fields form an orthonormal basis for the tangent space at each point P such that $g(e_\mu, e_\nu) = e_\mu \cdot e_\nu = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the line element of the Minkowski spacetime, $\eta_{\mu\nu} = \mathrm{diag}(-, +, +, +)$. The commutator relations for the vierbein fields are $[e_\mu, e_\nu] = c^\beta_{\nu\mu} e_\beta$ where $c^\beta_{(\nu\mu)} = 0$.

Consider now the in the nonholonomic coordinates the covariant derivative ∇_μ which is defined by the connection

$$\mathring{\Gamma}^\mu_{\nu\beta} = \left\{ {\mu \atop \nu\beta} \right\} + \hat{\Gamma}^\mu_{\nu\beta} \tag{1}$$

where $\{{\mu \atop \nu\beta}\}$ is the symmetric Levi-Civita connection of Riemannian geometry which is used in General Relativity and

$$\hat{\Gamma}^\mu_{\nu\beta} = \frac{1}{2} g^{\mu\sigma} (c_{\nu\sigma,\beta} + c_{\sigma\beta,\nu} - c_{\mu\beta,\sigma}). \tag{2}$$

is the antisymmetric component with the properties $\hat{\Gamma}_{\mu\nu\beta} = -\hat{\Gamma}_{\nu\mu\beta}$, $\hat{\Gamma}_{\mu\nu\beta} = g_{\mu\sigma} \hat{\Gamma}^\mu_{\nu\beta}$.

When $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu}$ is the flat space, $\hat{\Gamma}^\mu_{\nu\beta}$ correspond to the Weitzenböck connection [?] from where the nonnull torsion tensor can be defined

$$T^\beta_{\mu\nu} = \mathring{\Gamma}^\beta_{\nu\mu} - \mathring{\Gamma}^\beta_{\mu\nu}. \tag{3}$$

The modified higher order teleparallel $f(T, B)$ gravitational theory is a fourth-order of theory of gravity in which the gravitational Action Integral is defined as follows [30]

$$S \equiv \frac{1}{16\pi G} \int d^4 x e (f(T, B)) + S_m, \tag{4}$$

in which $e = \det(e^i_\mu)$, $\mathbf{e}_i(x^\mu)$ are the vierbein fields defining the dynamical variables of teleparallel gravity. T is the scalar of the torsion tensor, $T = S_\beta{}^{\mu\nu} T^\beta{}_{\mu\nu}$, and $S_\beta{}^{\mu\nu}$ is defined as

$$S_\beta{}^{\mu\nu} = \frac{1}{2} (\delta^\mu_\beta T^{\theta\nu}{}_\theta - \delta^\nu_\beta T^{\theta\mu}{}_\theta - \frac{1}{2} (T^{\mu\nu}{}_\beta - T^{\nu\mu}{}_\beta - T_\beta{}^{\mu\nu})). \tag{5}$$

Finally, $B = \frac{2}{e} \partial_\mu \left(e T_\rho{}^{\rho\mu} \right)$ is the boundary term which is related with the Ricciscalar as $T = R - B$.

In the case of vacuum the gravitational field equations are [30]

$$\begin{aligned} 4\pi G e \mathcal{T}_a^{(m)\lambda} = {} & \frac{1}{2} e h_a^\lambda \left(f_{,B}\right)^{;\mu\nu} g_{\mu\nu} - \frac{1}{2} e h_a^\sigma \left(f_{,B}\right)_{;\sigma}{}^{;\lambda} + \frac{1}{4} e \left(B f_{,B} - \frac{1}{4} f \right) h_a^\lambda + (e S_a{}^{\mu\lambda})_{,\mu} f_{,T} \\ & + e \left((f_{,B})_{,\mu} + (f_{,T})_{,\mu} \right) S_a{}^{\mu\lambda} - e f_{,T} T^\sigma{}_{\mu a} S_\sigma{}^{\lambda\mu}, \end{aligned} \tag{6}$$

where $\mathcal{T}_a^{(m)\lambda}$ is the energy-momentum tensor of the matter source. In the following we shall consider that the spacetime is vacuum.

We observe that when $f(T, B) = f(T) + f_1 B$ the second-order $f(T)$ teleparallel gravity is recovered, while when $f(T, B) = f(-T + B)$, the fourth-order $f(R)$-theory is recovered. Finally, when $f(T, B) = \alpha T + \beta B$ Eq. (6) reduce to that of Einstein's field equations of General Relativity.

In our consideration we should define the invariant function $\hat{B}$, that is,

$$\hat{B} = B - 2T, \tag{7}$$

thus in the following we consider the Action Integral

$$S=\frac{1}{16\pi G}\int d^4xe\left(f(T,\hat{B})\right), \tag{8}$$

with $f(T,\hat{B})=f(T,B-2T)$, while we work with the $f(T,\hat{B})=T+F\left(\hat{B}\right)$ theory. From (6) with the change of variable to the field equations correspond to the Action Integral (8)

$$4\pi G e\mathcal{T}_a^{(m)\lambda}=\frac{1}{2}eh_a^\lambda\left(f_{,\hat{B}}\right)^{;\mu\nu}g_{\mu\nu}-\frac{1}{2}eh_a^\sigma\left(f_{,\hat{B}}\right)_{;\sigma}^{\ \ ;\lambda}+\frac{1}{4}e\left(\left(\hat{B}+2T\right)f_{,\hat{B}}-\frac{1}{4}f\right)h_a^\lambda+(eS_a{}^{\mu\lambda})_{,\mu}f_{,T}$$
$$+e\left((f_{,\hat{B}})_{,\mu}+(f_{,T})_{,\mu}\right)S_a{}^{\mu\lambda}-ef_{,T}T^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu}. \tag{9}$$

Thus, for the $f(T,\hat{B})=T+F\left(\hat{B}\right)$ theory the field equations read

$$4\pi G e\mathcal{T}_a^{(m)\lambda}=\frac{1}{2}eh_a^\lambda\left(F_{,\hat{B}}\right)^{;\mu\nu}g_{\mu\nu}-\frac{1}{2}eh_a^\sigma\left(F_{,\hat{B}}\right)_{;\sigma}^{\ \ ;\lambda}+\frac{1}{4}e\left(\left(\hat{B}+2T\right)F_{,\hat{B}}-\frac{1}{4}(T+F)\right)h_a^\lambda+(eS_a{}^{\mu\lambda})_{,\mu}$$
$$+e\left((f_{,\hat{B}})_{,\mu}\right)S_a{}^{\mu\lambda}-eT^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu}. \tag{10}$$

or equivalent

$$G_a^\lambda=4\pi G e\mathcal{T}_a^{(m)\lambda}+\mathcal{T}_a^{\left(\hat{B}\right)\lambda}$$

where G_a^λ is the Einstein tensor and

$$\mathcal{T}_a^{\left(\hat{B}\right)\lambda}=\frac{1}{2}eh_a^\lambda\left(F_{,\hat{B}}\right)^{;\mu\nu}g_{\mu\nu}-\frac{1}{2}eh_a^\sigma\left(F_{,\hat{B}}\right)_{;\sigma}^{\ \ ;\lambda}+\frac{1}{4}e\left(\left(\hat{B}+2T\right)F_{,\hat{B}}-\frac{1}{4}F\right)h_a^\lambda+e\left((F_{,\hat{B}})_{,\mu}\right)S_a{}^{\mu\lambda}.$$

2.1 Bianchi I spacetime

We consider the Bianchi I line element

$$ds^2=-dt^2+a^2(t)dx^2+b^2(t)dy^2+c^2(t)dz^2, \tag{11}$$

where without loss of generality we have selected the lapse function to be a constant.

For the vierbein fields we consider the following diagonal frame $h^i_\mu(t)=\mathrm{diag}(1,a(t),a(t),a(t))$, thus we calculate [33]

$$T=2\left(\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}\right),\ \hat{B}=2\left(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}\right) \tag{12}$$

where a dot mark total derivative with respect to the variable t, that is $\dot{a}=\frac{da}{dt}$, $\dot{b}=\frac{db}{dt}$ and $\dot{c}=\frac{dc}{dt}$.

In $f\left(T,\hat{B}\right)=T+F\left(\hat{B}\right)$ the Action Integral (8) reads

$$S=\frac{1}{16\pi G}\int d^4xe\left(T+F\left(\hat{B}\right)\right). \tag{13}$$

Moreover, we introduce the Lagrange multiplier λ such that

$$S=\frac{1}{16\pi G}\int d^4xe\left(T+F\left(\hat{B}\right)-\lambda\left(\hat{B}-2\left(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}\right)\right)\right). \tag{14}$$

Variation with respect to $\hat{B}$ gives that $\lambda=F_{,\hat{B}}$. Thus integration by parts provides the point-like Lagrangian

$$\mathcal{L}\left(a,\dot{a},b,\dot{b},c,\dot{c},\phi,\dot{\phi}\right)=-2\left(a\dot{b}\dot{c}+b\dot{a}\dot{c}+c\dot{a}\dot{b}\right)+2\left(bc\dot{a}\dot{\phi}+ac\dot{b}\dot{\phi}+ab\dot{c}\dot{\phi}\right)-abcV(\phi), \tag{15}$$

where now the scalar field $\phi=f_{\hat{B}}\left(\hat{B}\right)$ and $V(\phi)=\hat{B}F\left(\hat{B}\right)_{,\hat{B}}-F\left(\hat{B}\right)$. The scalar field has been defined in a similar way with that of fourth-order $f(R)$ gravity [38]. We observe that this theory belongs to the family of Hordenski theories [39].

The gravitational field equations are calculated

$$2(H_1H_2+H_1H_3+H_2H_3)+2(H_1+H_3+H_3)\dot{\phi}+V(\phi)=0\,, \tag{16}$$

$$2\left(\dot{H}_2+\dot{H}_3+H_2^2+H_2H_3+H_3^2\right)+2\ddot{\phi}+V(\phi)=0\,, \tag{17}$$

$$2\left(\dot{H}_1+\dot{H}_3+H_1^2+H_1H_3+H_3^2\right)+2\ddot{\phi}+V(\phi)=0\,, \tag{18}$$

$$2\left(\dot{H}_1+\dot{H}_2+H_1^2+H_1H_2+H_2^2\right)+2\ddot{\phi}+V(\phi)=0\,, \tag{19}$$

while for the field ϕ the constraint equation follows

$$2\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + (H_1 + H_2 + H_3)^2\right) + V(\phi)_{\phi} = 0. \tag{20}$$

where $H_1 = \frac{\dot{a}}{a}$, $H_2 = \frac{\dot{b}}{b}$ and $H_3 = \frac{\dot{c}}{c}$.

In the following, we are interested on the investigation of the existence of anisotropic solutions for Bianchi I spacetimes of Kasner and Kasner-like solutions and that of General Relativity solution with the cosmological constant term.

3 Limit of General Relativity

In the case of vacuum from the (6), it follows that the limit of General Relativity is recovered when $F\left(\hat{B}\right) = F_1\hat{B} + F_0$, where F_0 plays the role of the cosmological constant. However, this is not the unique case. Indeed, any solution of General Relativity with or without a cosmological constant is a solution of $T + F\left(\hat{B}\right)$ theory for every $\hat{B} = \hat{B}_0$. Therefore, the contribution of the $F\left(\hat{B}\right)$ in the field equations is summarized by the energy-momentum tensor

$$\mathcal{T}_a^{\left(\hat{B}\right)\lambda} = -\frac{\tilde{\Lambda}}{16\pi G}h_a^{\lambda} , \ \tilde{\Lambda} = \hat{B}_0 F_{,\hat{B}}|_{\hat{B}\to\hat{B}_0}. \tag{21}$$

This indeed is a similar result with that found before by Barrow and Ottewill condition of $f(R)$-gravity [40].

Consider now the field Eqs. (16)–(20) for the Bianchi I background space. They can be written equivalently as $G_a^{\lambda} = \mathcal{T}_a^{\left(\hat{B}\right)\lambda}$,where

$$\mathcal{T}_0^{\left(\hat{B}\right)0} = -2(H_1 + H_3 + H_3)\dot{\phi} - V(\phi), \tag{22}$$

$$\mathcal{T}_1^{\left(\hat{B}\right)1} = \mathcal{T}_2^{\left(\hat{B}\right)2} = \mathcal{T}_3^{\left(\hat{B}\right)3} = 2\ddot{\phi} + V(\phi) , \tag{23}$$

3.1 Kasner universes

The Kasner solution is the vacuum solution of Bianchi I spacetime in General Relativity. The solution describes an anisotropic universe where the scale factors are $a(t) = a_0 t^{p_1}$, $b(t) = b_0 t^{p_2}$ and $c(t) = c_0 t^{p_3}$. As far as the indices p_i are concerned, they satisfy the Kasner relations

$$p_1 + p_2 + p_3 = 1, \tag{24}$$

$$p_1^2 + p_2^2 + p_3^2 = 1. \tag{25}$$

The Kasner relations says that two of the indices p_2, p_3 are positive (or negative) while p_1 is negative (or positive), while the indices take values on the range $|p_i| \leq 1$.

We replace the Kasner solution in the field Eqs. (16)–(20) and we find

$$2\dot{\phi} + tV(\phi) = 0 , \ 2\ddot{\phi} + V(\phi) = 0 \tag{26}$$

which means that $\phi(t) = \phi_0 + \phi_1 t^2$ and $V(\phi) = -2\phi_1$.

However, a constant potential $V(\phi) = V_0$ gives that $F\left(\hat{B}\right) = V_0 + F_1\hat{B}$ which is not an acceptable solution because that is a second-order theory. However, this is not the unique solution. The problem has another special solution which is $\phi(t) = \phi_0$ and $V(\phi) = 0$. The latter reads, $\hat{B}F_{,\hat{B}} - F\left(\hat{B}\right) = 0$, while for the Kasner solution from (12) we calculate $\hat{B} = 0$. The following proposition follows.

Proposition 1 *In $f\left(T, \hat{B}\right) = T + F\left(\hat{B}\right)$ theory of gravity the Kasner universe is recovered for every function $F\left(\hat{B}\right)$ which satisfies the condition $F\left(\hat{B}\right)_{|\hat{B}\to 0} = 0$.*

Let us now investigate if there exist Kasner-like solutions with other relations for the indices. We replace $a(t) = a_0 t^{p_1}$, $b(t) = b_0 t^{p_2}$ and $c(t) = c_0 t^{p_3}$ in the field equations and we find

$$t^2 V(\phi) + 2\left(p_1 p_2 + (p_1 + p_2)p_3 + (p_1 + p_2 + p_3)t\dot{\phi}\right) = 0, \tag{27}$$

$$2\left(p_2^2 + (p_2 + p_3)(p_3 - 1)\right) + t^2\left(V(\phi) + 2\ddot{\phi}\right) = 0, \tag{28}$$

$$2\left(p_1^2 + (p_1 + p_3)(p_3 - 1)\right) + t^2\left(V(\phi) + 2\ddot{\phi}\right) = 0, \tag{29}$$

$$2\left(p_1^2+(p_1+p_2)(p_2-1)\right)+t^2\left(V(\phi)+2\ddot{\phi}\right)=0, \tag{30}$$

Consequently, for $1+p_1+p_2+p_3\neq 0$ from the first two equations we find

$$\phi=\phi_0+\phi_1 t^{1+p_1+p_2+p_3}+\frac{\left(p_2^2+p_3^2-p_2-p_3-p_1(p_2+p_3)\right)}{1+p_1+p_2+p_3}\ln t, \tag{31}$$

$$V(\phi(t))=\frac{2(1-p_1-p_2-p_3)\left(p_2^2+p_2p_3+p_3^2\right)}{(1+p_1+p_2+p_3)t^2}-2(p_1+p_2+p_3)\phi_1 t^{-1+p_1+p_2+p_3}. \tag{32}$$

However, by replacing in the rest of the equations we end with the conditions

$$(p_1-p_2)(1-p_1-p_2-p_3)=0, \tag{33}$$

$$(p_1-p_3)(1-p_1-p_2-p_3)=0, \tag{34}$$

that is, $p_1=p_2$ and $p_1=p_3$ which means that the final solution is the scaling solution of an isotropic Friedmann–Lemaître–Robertson–Walker universe, or the anisotropic Kasner-like solution with the constraint $p_1+p_2+p_3=1$.

However, in the anisotropic solution with $p_1+p_2+p_3=1$. It follows that $\phi(t)=\frac{1}{2}\phi_1 t^2+\phi_0$, $V(\phi(t))=-2\phi_1$ which as before provides that $F\left(\hat{B}\right)=V_0+F_1\hat{B}$. The latter solution it is not accepted for this modified theory because we are interested in nonlinear functional forms of $F\left(\hat{B}\right)$.

Moreover, we calculate $\hat{B}(t)=\frac{2(p_1(p_1-1)+p_2(p_2-1)+p_3(p_3-1))}{t^2}$, which in contrary to the previous does not have the special solution at $\hat{B}=0$ as before. Therefore, there can be asymptotic Kasner-like solutions for the field equation for large values of t, such that $\lim_{t\to\infty}\hat{B}(t)=0$ and $\lim_{t\to\infty}F\left(\hat{B}(t)\right)=0$ with $\phi_1=0$. In the case of $\phi_1\neq 0$ it follows that $F_{,\hat{B}}=\frac{F_0}{\hat{B}}$ which gives $F\left(\hat{B}\right)=F_0\ln\hat{B}$. However, in this case the potential function $V(\phi)$ is derived $V(\phi)=F_0(1+\ln\phi)\simeq\ln t$, which does not admit the constant value as an asymptotic. Hence, necessarily it follows $F_0=0$, i.e., $\phi_1=0$.

3.2 Exponential solution

The Bianchi I exact solution with the cosmological constant terms is that in which the scale factors are $a(t)=a_0e^{p_1t}$, $b(t)=b_0e^{p_2t}$ and $c(t)=c_0e^{p_3t}$, with the relation for the indices $p_1+p_2+p_3=0$. Thus, by replacing the latter scale factors in the field Eqs. (16)–(20) we end

$$\phi(t)=\phi_0+\phi_1 t-\left(p_1^2+p_1p_2+p_2^2\right)t^2,$$
$$V(\phi(t))=2\left(p_1^2+p_1p_2+p_2^2\right).$$

As far as the boundary term $\hat{B}(t)$ is concerned, we calculate $\hat{B}(t)=2\left(p_1^2+p_1p_2+p_2^2\right)$. However, $\phi(t)=F\left(\hat{B}\right)_{,\hat{B}}$, hence $\phi(t)$ cannot be a function of t, that is, $\left(p_1^2+p_1p_2+p_2^2\right)=0$, or $\hat{B}(t)=0$. However, the later algebraic equation does not admit any real solution which means that there is not any anisotropic exponential solution with the cosmological constant term.

In the following, we continue our analysis with the study of the stability properties for the Kasner solution.

4 Stability properties of Kasner universes

In order to study the asymptotic behavior of the field equations in the anisotropic Bianchi I background space as also to investigate the stability conditions for the Kasner solutions, we prefer to work with the Misner variables where the line element (11) is written as follows [2],

$$ds^2=-dt^2+\alpha^2(t)\left(e^{-2\beta_+(t)}dx^2+e^{\beta_+(t)+\sqrt{3}\beta_-(t)}dy^2+e^{\beta_+(t)-\sqrt{3}\beta_-(t)}dz^2\right), \tag{35}$$

where $\beta_+(t)$, $\beta_-(t)$ are the two anisotropic parameters and $\alpha(t)$ is the scale factor of the three-dimensional hypersurface, the point-like Lagrangian (15) takes the following diagonal form

$$\mathcal{L}\left(\alpha,\dot{\alpha},\beta_+,\dot{\beta}_+,\beta_-,\dot{\beta}_-,\phi,\dot{\phi}\right)=-6\alpha\dot{\alpha}^2+\frac{3}{2}\alpha^3\left(\dot{\beta}_+\right)^2+\frac{3}{2}\alpha^3\left(\dot{\beta}_-\right)^2-6\alpha^2\dot{\alpha}\dot{\phi}-\alpha^3V(\phi). \tag{36}$$

Hence, variation with respect to the dependent variables gives that the gravitational field equations

$$6H\left(H+\dot{\phi}\right)-\frac{3}{2}\left(\left(\dot{\beta}_+\right)^2+\left(\dot{\beta}_-\right)^2\right)-V(\phi)=0, \tag{37}$$

$$4\dot{H}+6H^2+\frac{3}{2}\left(\left(\dot{\beta}_+\right)^2+\left(\dot{\beta}_-\right)^2\right)+2\ddot{\phi}-V(\phi)=0, \tag{38}$$

$$\ddot{\beta}_+ + 3H\dot{\beta}_+ = 0, \tag{39}$$

$$\ddot{\beta}_- + 3H\dot{\beta}_- = 0, \tag{40}$$

$$\dot{H}+3H^2-\frac{1}{6}V_{,\phi}=0. \tag{41}$$

4.1 $H-$normalization

Let us define the new variables in the context of the H-normalization,

$$\Sigma_+=\frac{\dot{\beta}_+}{2H}\ ,\ \Sigma_-=\frac{\dot{\beta}_-}{2H}\ ,\ x=-\frac{\dot{\phi}}{H}\ ,\ y=\frac{V(\phi)}{6H^2}\ ,\ \lambda=\frac{V'}{V}\ . \tag{42}$$

In the new variables and for the new independent variable $\tau=\ln a$, the gravitational field equations read

$$\Sigma_+'=\lambda y\Sigma_+\ , \tag{43}$$

$$\Sigma_-^{'}=\lambda y\Sigma_-\ , \tag{44}$$

$$x'=-(3+2\lambda)y+x(3+\lambda y)+3\left(\Sigma_+^2+\Sigma_-^2-1\right)\ , \tag{45}$$

$$y'=y(6+\lambda(x+2y))\ , \tag{46}$$

$$\lambda'=\frac{1}{6}\lambda^2x(\Gamma(\lambda)-1)\ ,\ \Gamma(\lambda)=\frac{V_{,\phi\phi}V}{\left(V_{,\phi}\right)^2}. \tag{47}$$

where $\Sigma_+'=\frac{d\Sigma_+}{d\tau}=\frac{1}{H}\dot{\Sigma}_+$, while the constraint equation becomes

$$1-x-y-\Sigma_+^2-\Sigma_-^2=0. \tag{48}$$

We remark that the variables $\{x, y, \Sigma_+, \Sigma_-\}$ are not bounded and can take values in the whole range of the real numbers. However, we are interested in investigating only the stability of the Kasner universes. In order to perform a complete study of the global dynamics, we should work by using other variables than the H-normalization, that is, because from (37) it is clear that the Hubble function can change sign.

Let us proceed with our analysis by considering some special case for the scalar field potential $V(\phi)$.

4.1.1 Exponential potential

Assume now that $V(\phi)=V_0e^{\lambda\phi}$, then from (47) it follows that $\lambda=cons\not{t}$. As far as the function $F\left(\hat{B}\right)$ is concerned that is calculated to be $F_A\left(\hat{B}\right)=\left(\frac{\hat{B}}{\lambda}\ln\left(\frac{\hat{B}}{\lambda}\right)-1\right)$, we observe that for $\lim_{\hat{B}\to 0}F_A\left(\hat{B}\right)=0$, which means that the Kasner solution can be recovered according to Proposition 1. Hence, for this scalar field potential the stationary points $P=(x(P), y(P), \Sigma_+(P), \Sigma_-(P))$ for the dynamical system (43)–(48) are $P^1=\left(\frac{2}{\lambda}(3+\lambda), -\frac{6+\lambda}{\lambda}, 0, 0\right)$ and $P^2=\left(1-\Sigma_+^2-\Sigma_-^2, 0, \Sigma_+, \Sigma_-\right)$.

Point P^1 describes an isotropic spatially flat FLRW universe, because the anisotropic parameters are zero. The cosmological fluid at this point is found to be that of an ideal gas with constant equation of state parameter $w\left(P^1\right)=-\frac{1}{3}(2\lambda+9)$, from where we conclude that the point describes an accelerated universe when $\lambda<-\frac{9}{2}$. The eigenvalues of the linearized system which follows from the Eqs. (43), (44) and (45) where y has been replaced by the constraint condition (48), are derived $e_1\left(P^1\right)=e_2\left(P^1\right)=e_3\left(P^1\right)=-(6+\lambda)$. Hence, the point is an attractor when $\lambda>-6$.

As far as the family of points P^2 are concerned, they describe anisotropic solutions where the scalar field potential vanishes. As we found before points P^2 describe Kasner universes when $1-\Sigma_+^2-\Sigma_-^2=0$. The eigenvalues of the linearized system around the stationary points are derived to be $e_1\left(P^2\right)=0$, $e_2\left(P^2\right)=0$ and $e_3\left(P^2\right)=6+\lambda\left(1-\Sigma_+^2-\Sigma_-^2\right)$, from where we conclude that the Kasner universes are always unstable solutions of the field equations. For other values of the anisotropic parameters in which $1-\Sigma_+^2-\Sigma_-^2\neq 0$, the final result is that of Kasner-like solution derived in the previous section.

4.1.2 Arbitrary potential

It is straightforward to prove that for every arbitrary potential function $V(\phi)$ in which the corresponding $F\left(\hat{B}\right)$ function satisfies Proposition 1, it always admits the family of stationary points $P_K^2=(0, 0, \Sigma_+, \Sigma_-, \lambda)$, with $1-\Sigma_+^2-\Sigma_-^2=0$, in which λ is an arbitrary point, and these points describe unstable Kasner solutions. Therefore, we conclude that the limit of General Relativity in this specific theory is an unstable point.

As far as the general evolution for the dynamics of the field Eqs. (43)–(48) is concerned, it can be easily recovered by using results of previous studies. Indeed, from Eqs. (39) and (39) the anisotropic functions can be replaced as $\dot{\beta}_+ = \frac{\beta_+^0}{a^3}$, $\dot{\beta}_- = \frac{\beta_-^0}{a^3}$ where by replacing in the rest of the field equations we end with the field equations for the spatially flat FLRW universe with stiff fluid source. Therefore, the detailed analysis presented to [33] is valid.

5 Conclusions

In this study we investigated the existence of anisotropic solutions in a higher-order teleparallel theory of gravity known as $f(T, B)$ theory and specifically in the $f\left(T, \hat{B}\right)$ theory with $\hat{B} = B - 2T$. In particular, we focused on the $f\left(T, \hat{B}\right) = T + F\left(\hat{B}\right)$ theory which is a fourth-order theory defined by the boundary term $\hat{B}$. The higher-order derivatives of the theory can be described by a scalar field in the Einstein frame. That scalar field attributes the higher-order derivatives of the theory and does not belong to the scalar-tensor theories. For this particular theory, we found the conditions where the Kasner universe, which is the vacuum solution of General Relativity, is recovered. In addition we found that anisotropic Kasner-like solutions exist for this specific theory as asymptotic solutions for the field equations. On the other hand, we show that anisotropic exponential solutions which describe the cosmological constant term are not provided in the case of $f\left(T, \hat{B}\right) = T + F\left(\hat{B}\right)$, however, that is not true for the existence of the isotropic de Sitter solution [32].

Moreover, we investigated the stability properties of the theory and we found that the Kasner universes are always unstable solutions for the theory for the arbitrary functional form of $F\left(\hat{B}\right)$. The general asymptotic dynamics and evolution are similar with that studied before for the spatially flat FLRW universe and the stiff matter fluid.

In the general scenario of $f\left(T, \hat{B}\right)$ theory, for the Bianchi I spacetime, from the field Eq. (6 and the definition for the torsion scalar T and for the boundary $\hat{B}$ as given by expressions (12) we find that the Kasner solution of General Relativity is recovered when $f\left(T, \hat{B}\right)$ does not admit as critical point the $\left(T, \hat{B}\right) = (0, 0)$ and the following algebraic condition holds $f\left(T, \hat{B}\right)_{|\left(T,\hat{B}\right)\to(0,0)} = 0$.

This analysis contributes to the subject of the determination of the exact solutions of modified theories of gravity. The derivation of solutions of General relativity is of special interest for the validity of modified theories of gravity. In this work we focused on a higher-order teleparallel theory and we found that the anisotropic Kasner universes are provided by the theory; however, the limit of General Relativity is always an unstable point. In a future work we plan to investigate the existence of spherical symmetric solutions for this higher-order theory.

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