



Lump, periodic and travelling wave solutions to the (2+1)-dimensional pKP-BKP model

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Abstract In this study, the (2+1)-dimensional combined potential Kadomtsev–Petviashvili with B-type Kadomtsev–Petviashvili equation is investigated via two diverse techniques. Firstly, we retrieve the bilinear form of given equation by utilizing Hirota bilinear method. Consequently, the lump waves and collisions among lumps and periodic waves, the collision between lump wave and single, double-kink soliton solutions, the collision among lump, periodic, and single, double-kink soliton solutions as well as periodic wave soliton solutions for the given model are constructed. Lastly, the polynomial-expansion method is implemented to acquire the exact travelling wave solution to the equation. Moreover, 3D, contour and 2D graphs are used to demonstrate the physical nature of many intriguing exact solutions. A wide range of nonlinear partial differential equations can be solved using the considered methods.

1 Introduction

Nonlinear evolution equations (NLEEs) are important in a variety of scientific and technological fields. Nonlinear wave structures have aroused the interest of many academics in recent decades due to their multiple properties seen in various disciplines of modern sciences. In the existence of solitary waves, nonlinear evolution models are employed to simulate the effect of surface for deep water and weakly nonlinear dispersive long waves. As a result, exact solutions of corresponding models are essential for studying dynamical structures and other features of physical phenomena in a variety of fields, including electromagnetism, physical chemistry, geophysics, ionised physics, elastic medium, fluid motion, fluid mechanics, elastic medium, nuclear physics, electrochemistry, optical fibres, energy physics, chemical mechanics, gravity, biostatistics, statistical and natural physics. Recently, bilinear approaches are more frequently used by renowned researcher to deal with many interesting models. Consequently, various forms of solitonic structures are elaborated in the form of lump solution [1], N-soliton solution [2], breather solution [3], bright soliton solution [4], soliton solution [5], dark soliton solution [6], Multiple-soliton solution [7, 8] and interaction solutions [9, 10]. A specific form of rational localized wave solution that decays algebraically to the background wave in space direction is known as the lump wave solution. The lump wave solution was first discovered in the study of Kadomtsev–Petviashvili equation. Consequently, finding lump-type wave solutions to nonlinear evolution equations and examining their dynamical behaviour have gained a lot of attention in the field of nonlinear wave theory. The study of explicit solutions for various soliton equations has played a significant role in modern mathematics with implications in a variety of fields including mathematics, physics and other sciences. There are several well-established approaches for obtaining the exact solutions to various nonlinear models in the literature [11–15].

It is well acknowledged that the construction of exact solutions and the study of integrable properties for NLEEs play a significant role in various nonlinear physical and mathematical events. Such models play a key role to study a wide range of dynamical wave structures efficiently that occur in the real world and can be found in a variety of domains, including physics [16], applied mathematics [17] and engineering sciences [18]. There are numerous methods for obtaining their exact solutions, such as exp-function method [19, 20], Hirota bilinear method [21, 22], Lie group method [23, 24], Bäcklund transformation [25, 26], variable separation method [27, 28], multiple exp-function method [29, 30], Darboux transformation [31, 32], extended tanh function method [33, 34], tanh method [35, 36], homogeneous balance method [37, 38], inverse scattering method [39, 40], multiple exp-function algorithm [41, 42], homoclinic approach [43], ϕ^6 -model expansion method [44], Lie symmetry analysis [45, 46].

This paper is organized as follows: The (2+1)-dimensional pKP-BKP model is described in Sect. 2. In Sect. 3, the bilinear form for Eq. (4) is obtained. While Sect. 4 is devoted to the mathematical analysis of our considered model. The travelling wave solutions are retrieved in Sect. 5. Moreover, Sect. 6 contains discussion and results. Lastly, Sect. 7 contains conclusions.

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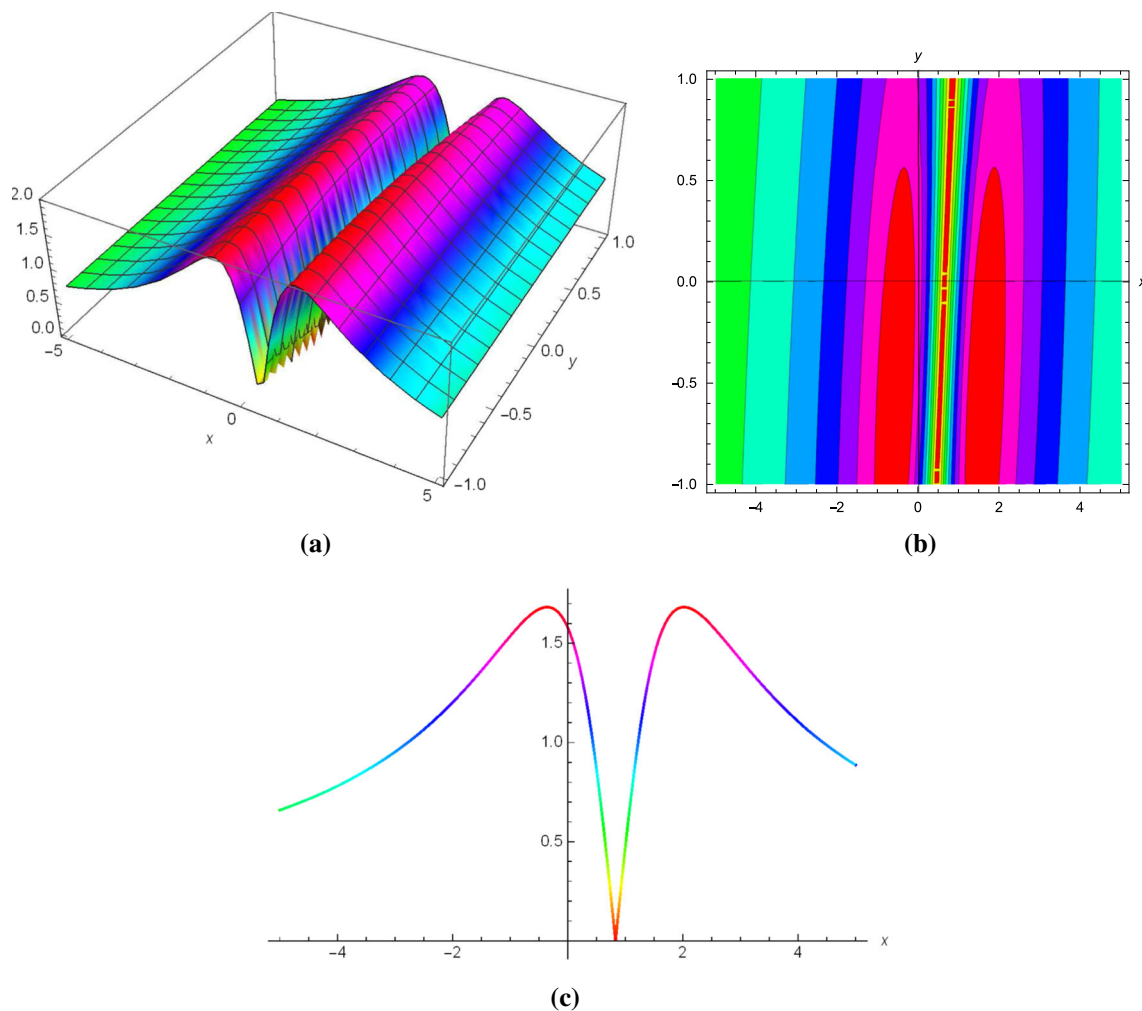


Fig. 1 3D, contour and 2D plots for $|\mu_1(x, y, t)|$ with $\alpha_2 = 2, \alpha_3 = -1.5, \alpha_4 = 0.4, \alpha_5 = 0.5, \alpha_6 = 0.75, g = 2.5, l_1 = 0.4, l_2 = 0.5, m_1 = -1.5, m_2 = 0.45$

2 The (2+1)-dimensional pKP-BKP model

The Kadomtsev-Petviashvili (KP) equation explains waves that are weakly dispersed [47, 48] with modest amplitudes propagating. The KP equation is an dispersive integrable equation with many soliton solutions that can be expressed in the Lax form. The KP hierarchy is a multi-dimensional structure with infinite dimensions with a variety of distinct formulations and symmetries. The B-type KP (B-KP) equation, as part of the KP-category equations, have garnered a significant amount of recent research in fluid dynamics and other areas [49, 50].

The (2 + 1)-dimensional integrable KP model may be expressed as [51]

$$\mu_{xt} + 6(\mu\mu_x)_x + \mu_{xxxx} - \mu_{yy} = 0. \tag{1}$$

Substituting $\mu = \mu_x$ and using integration, the potential KP (pKP) equation

$$\mu_{xt} + 6\mu_x\mu_{xx} + \mu_{xxxx} - \mu_{yy} = 0. \tag{2}$$

is obtained using Eq. (1). Moreover, the (2 + 1)-dimensional integrable B-KP equation may be expressed as

$$(15\mu_x^3 + 15\mu_x\mu_{xxx} + \mu_{xxxxx})_x + 5[\mu_{xxx}y + 3(\mu_x\mu_y)_x] + \mu_{xt} - 5\mu_{yy} = 0. \tag{3}$$

The N-soliton solutions and infinite dimensional Lie algebra structures for Eqs. (1) and (3) are studied recently using various Hirota bilinear forms [52–54]. Consequently, a combined pKP-BKP equation is proposed, that can be expressed as [55]

$$\alpha_1(45\mu_x^2\mu_{xx} + 15\mu_{xx}\mu_{xxx} + 15\mu_x\mu_{xxxx} + \mu_{xxxxx}) + \alpha_2(6\mu_x\mu_{xx} + \mu_{xxx})$$

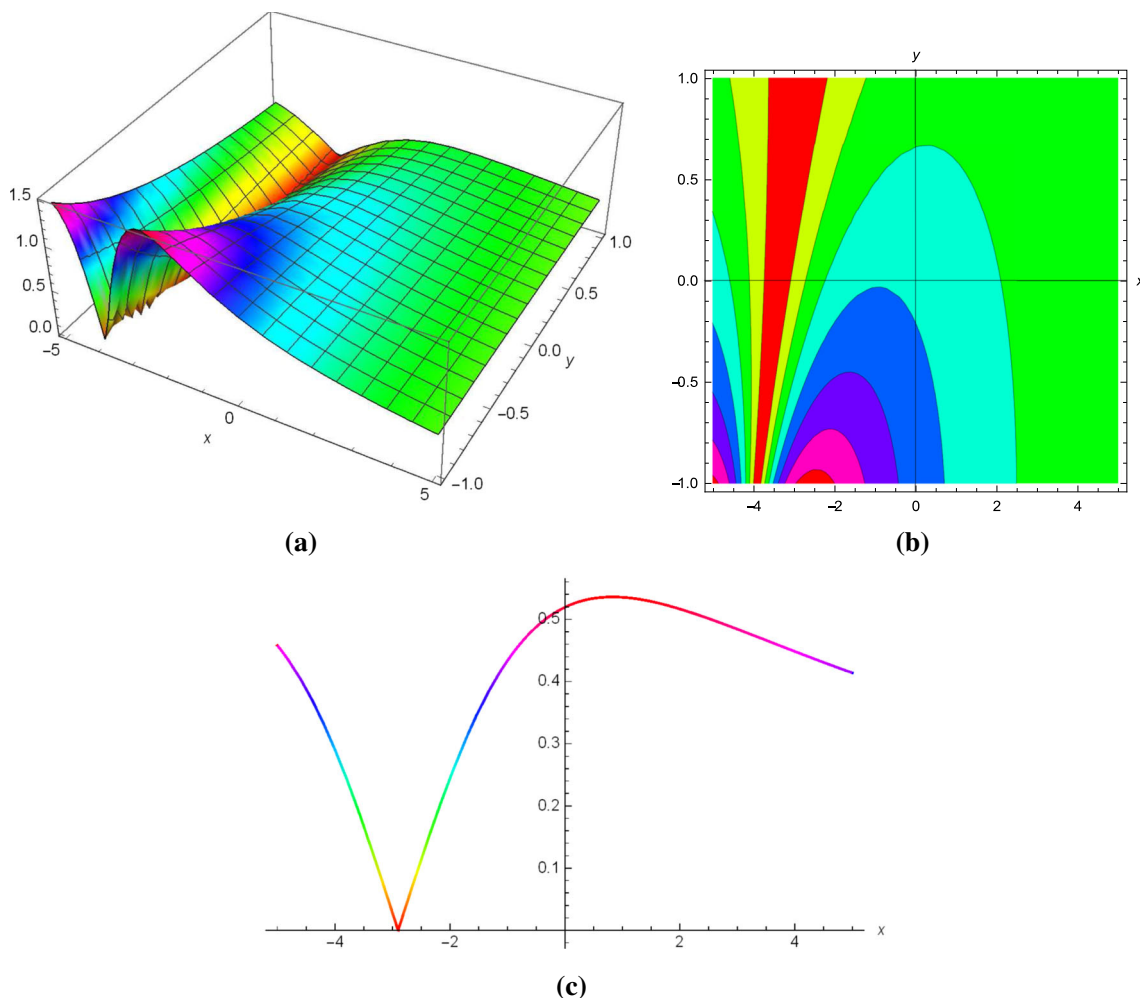


Fig. 2 3D, contour and 2D plots for $|\mu_3(x, y, t)|$ with $\alpha_4 = 0.45, \alpha_5 = 0.55, \alpha_6 = 0.75, \alpha_7 = 2.5, l_1 = -0.4, l_2 = 0.5, l_3 = 0.6$

$$+ \alpha_3(3\mu_x\mu_{xy} + 3\mu_{xx}\mu_y - \mu_{xxx}) + \alpha_4\mu_{xx} + \alpha_5\mu_{xt} + \alpha_6\mu_{yy} = 0, \tag{4}$$

where $\alpha_i (i= 1, 2, \dots, 6)$ are constants. If $\alpha_1 = \alpha_3 = \alpha_4 = 0, \alpha_2 = \alpha_5 = 1, \alpha_6 = -1$, then Eq. (4) represents the pKP Eq. (2), that admits the weakly dispersive waves in a paraxial wave approximation and can alternatively be characterised as the evolution of lengthy ion-acoustic waves of modest amplitude propagating in plasma physics. However, if $\alpha_1 = \alpha_5 = 1, \alpha_2 = \alpha_4 = 0, \alpha_3 = 5, \alpha_6 = -5$, Eq. (4) illustrates the BKP Eq. (3), that is a significant physical model that owns the isospectral soliton and may be characterised as waves in a particular form of nonuniform medium. Recently, a wide range of fresh analysis towards exact solutions to the given model (4) have been published. The N-soliton solution were constructed in [56]. The resonant multi-soliton, M-breather, M-lump and hybrid solutions were obtained in [57]. Lastly, the explicit solution and its soliton molecules were retrieved in [58].

3 Bilinear form of 4

The bilinear approach was first proposed by Hirota [59] in 1971 for constructing multiple solutions to integral nonlinear evolutionary equations. The main idea behind this technique is to employ some dependent variable transformation to change the NLEE in a form where the unknown function appears bilinearly. The bilinear approach is one of the most essential and extensively used methods for solving nonlinear partial differential equations. This strategy is based on using suitable transformations to convert nonlinear models into bilinear forms. Then, on the basis of the bilinear forms, N-soliton solutions, kink solutions, rational solutions, lump solutions, breather solutions, periodic soliton solutions and other exact solutions can be derived. To acquire the bilinear form of Eq. (4), we apply the transformation

$$\mu = 2[\ln F]_x. \tag{5}$$

Consequently, we retrieve the bilinear form as

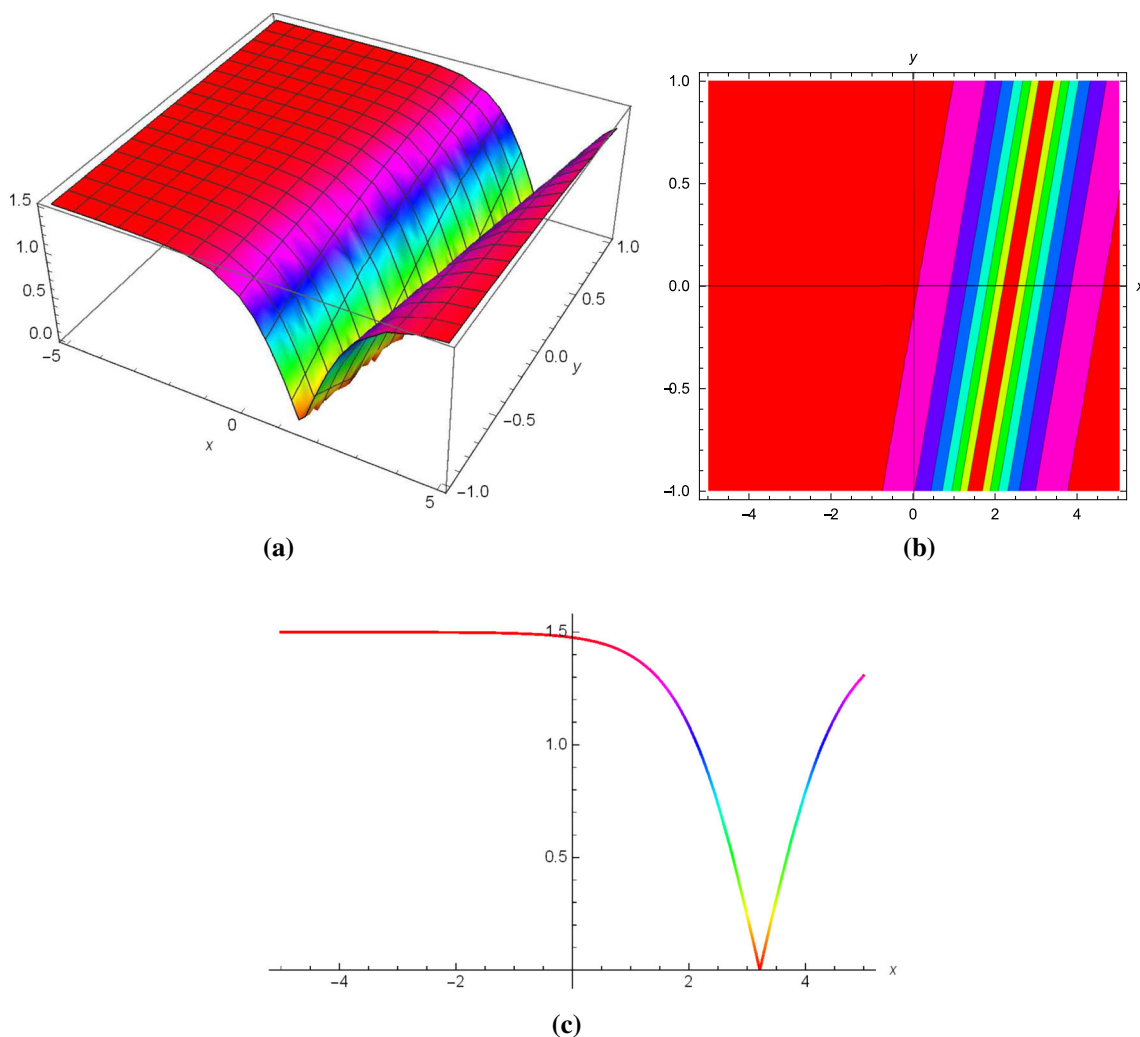


Fig. 3 3D, contour and 2D plots for $|\mu_4(x, y, t)|$ with $\alpha_1 = 0.2, \alpha_2 = 0.25, \alpha_3 = 2.5, \alpha_4 = 0.4, \alpha_5 = 0.5, \alpha_6 = 0.75, n_0 = 0.7, n_1 = -1.5, n_2 = 0.75, n_3 = 0.25, \omega_0 = 1.755, \omega_3 = -0.6$

$$(\alpha_1 D_x^6 + \alpha_2 D_x^4 + \alpha_3 D_x^3 D_y + \alpha_4 D_x^2 + \alpha_5 D_x D_t + \alpha_6 D_y^2) F \cdot F = 0,$$

so that the function D fulfills

$$D_x^m D_t^n (p \cdot q) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n \times p(x, t) \cdot q(x', t') |_{x=x', t=t'}$$

As a result, we get

$$\begin{aligned} &\alpha_1 (F F_{xxxxxx} - 6 F_x F_{xxxxx} + 15 F_{xx} F_{xxxx} - 10 F_{xxx}^2) \\ &+ \alpha_2 (F F_{xxxx} - 4 F_x F_{xxx} + 3 F_{xx}^2) \\ &+ \alpha_3 (F F_{xxy} - 3 F_x F_{xy} + 3 F_{xx} F_y - F_{xxx} F_y) \\ &+ \alpha_4 (F F_{xx} - F_x^2) + \alpha_5 (F F_{xt} - F_x F_t) + \alpha_6 (F F_{yy} - F_y^2) = 0, \end{aligned} \tag{6}$$

obviously if F meets Eq. (4), then $\mu = 2[\ln F]_x$ produces the solution of specified model (4) instantly.

4 Mathematical analysis

In this section, several significant forms are employed to extract the analytical solutions of Eq. (4).

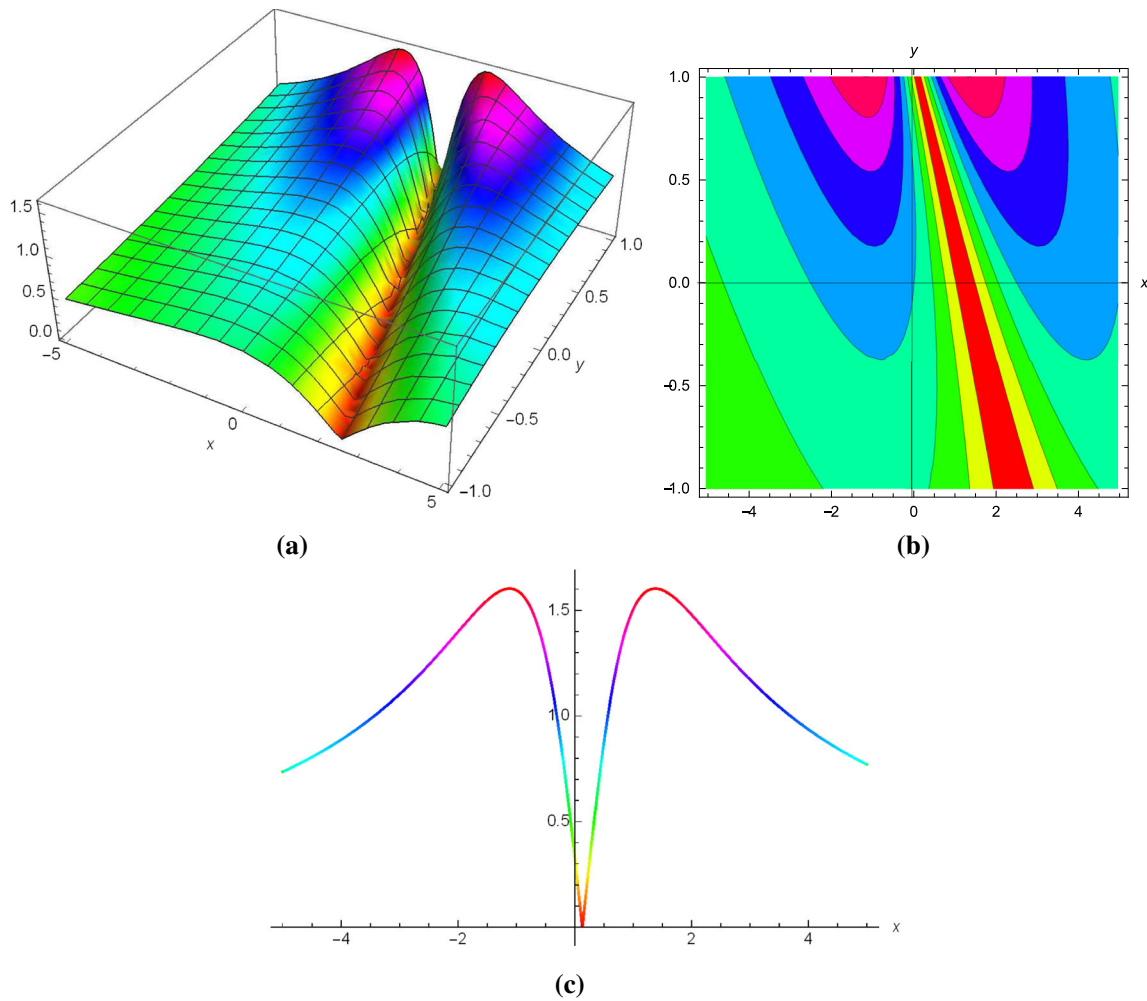


Fig. 4 3D, contour and 2D plots for $|\mu_6(x, y, t)|$ with $\alpha_1 = 0.15, \alpha_2 = 0.2, \alpha_3 = 0.4, \alpha_4 = 0.3, \alpha_5 = 0.5, \alpha_6 = 0.85, l_0 = 0.1, l_3 = 0.4, m_0 = 0.5, m_1 = 0.25, m_3 = 0.35, \omega_1 = 0.5$

4.1 Lump wave solutions

In this portion, we examine the operator F given by

$$F = \xi_1^2 + \xi_2^2 + g, \tag{7}$$

where

$$\xi_1 = l_1x + l_2y + l_3t, \quad \xi_2 = m_1x + m_2y + m_3t,$$

while l_i, m_i 's, g are the constants to be determined. Using Eq. (7) into Eq. (6), we have a system of equations with several parameters. We get the following findings by resolving them utilizing a computational programme such as Mathematica.

Case 1:

$$l_3 = \frac{\alpha_6 l_1 m_2^2 - \alpha_6 l_2^2 l_1 - \alpha_4 l_1^3 - m_1^2 \alpha_4 l_1 - 2\alpha_6 l_2 m_1 m_2}{\alpha_5 (l_1^2 + m_1^2)},$$

$$m_3 = \frac{\alpha_6 l_2^2 m_1 - \alpha_4 l_1^2 m_1 - 2\alpha_6 l_1 l_2 m_2 - \alpha_4 m_1^3 - \alpha_6 m_2^2 m_1}{\alpha_5 (l_1^2 + m_1^2)},$$

here l_1, l_2, m_1, m_2, g are free parameters. Hence utilizing all the above known values, Eq. (7) gives

$$F = g + \left(\frac{(\alpha_6 l_1 m_2^2 - \alpha_6 l_2^2 l_1 - \alpha_4 l_1^3 - m_1^2 \alpha_4 l_1 - 2\alpha_6 l_2 m_1 m_2)t}{\alpha_5 (l_1^2 + m_1^2)} + l_1 x + l_2 y \right)^2$$

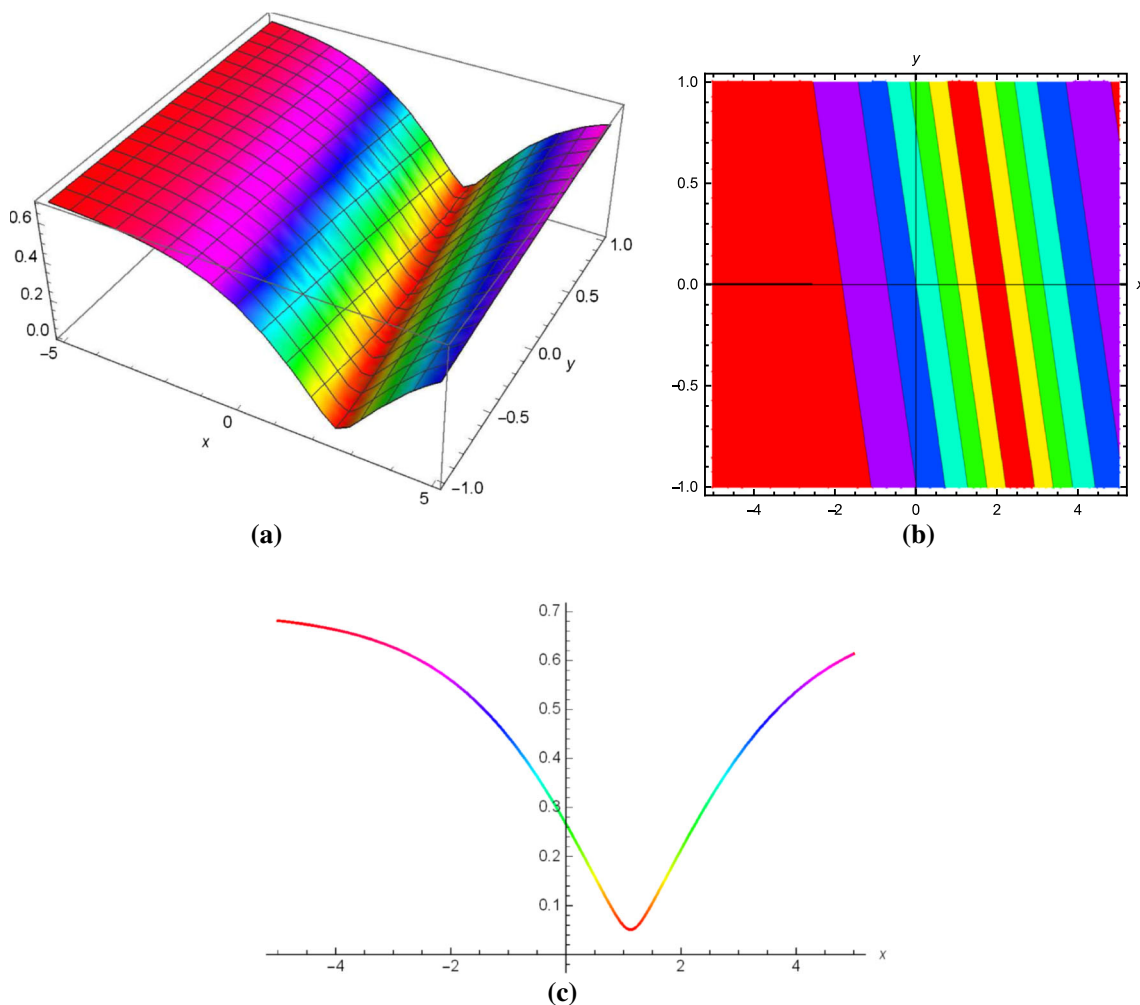


Fig. 5 3D, contour and 2D plots for $|\mu_8(x, y, t)|$ with $\alpha_2 = 0.55, \alpha_3 = 0.5, \alpha_4 = 0.45, \alpha_5 = 0.5, \alpha_6 = 0.75, n_0 = 0.15, n_1 = 0.75, n_2 = -0.65$

$$+ \left(\frac{(\alpha_6 l_2^2 m_1 - \alpha_4 l_1^2 m_1 - 2\alpha_6 l_1 l_2 m_2 - \alpha_4 m_1^3 - \alpha_6 m_2^2 m_1)t}{\alpha_5(l_1^2 + m_1^2)} + m_1 x + m_2 y \right)^2.$$

Utilizing the above result together with Eq. (5), we gain $\mu_1(x, y, t)$.

Case 2:

$$l_1 = 0, l_3 = -\frac{2\alpha_6 l_2 m_2}{\alpha_5 m_1}, m_3 = \frac{\alpha_6 l_2^2 - \alpha_4 m_1^2 - \alpha_6 m_2^2}{\alpha_5 m_1},$$

where l_2, m_1, m_2, g are free parameters. Hence utilizing all the above known values, Eq. (7) gives

$$F = g + \left(l_2 y - \frac{2\alpha_6 l_2 m_2 t}{\alpha_5 m_1} \right)^2 + \left(\frac{t(\alpha_6 l_2^2 - \alpha_4 m_1^2 - \alpha_6 m_2^2)}{\alpha_5 m_1} + m_1 x + m_2 y \right)^2. \tag{8}$$

Utilizing Eq. (8) together with Eq. (5), we obtain the solution

$$\mu_2(x, y, t) = \frac{4m_1 \left(\frac{t(\alpha_6 l_2^2 - \alpha_4 m_1^2 - \alpha_6 m_2^2)}{\alpha_5 m_1} + m_1 x + m_2 y \right)}{g + \left(\frac{t(\alpha_6 l_2^2 - \alpha_4 m_1^2 - \alpha_6 m_2^2)}{\alpha_5 m_1} + m_1 x + m_2 y \right)^2 + \left(l_2 y - \frac{2\alpha_6 l_2 m_2 t}{\alpha_5 m_1} \right)^2}. \tag{9}$$

Case 3:

$$m_1 = i l_1, m_2 = i l_2, m_3 = -\frac{i(2\alpha_4 l_1^2 + \alpha_5 l_3 l_1 + 2\alpha_6 l_2^2)}{\alpha_5 l_1},$$

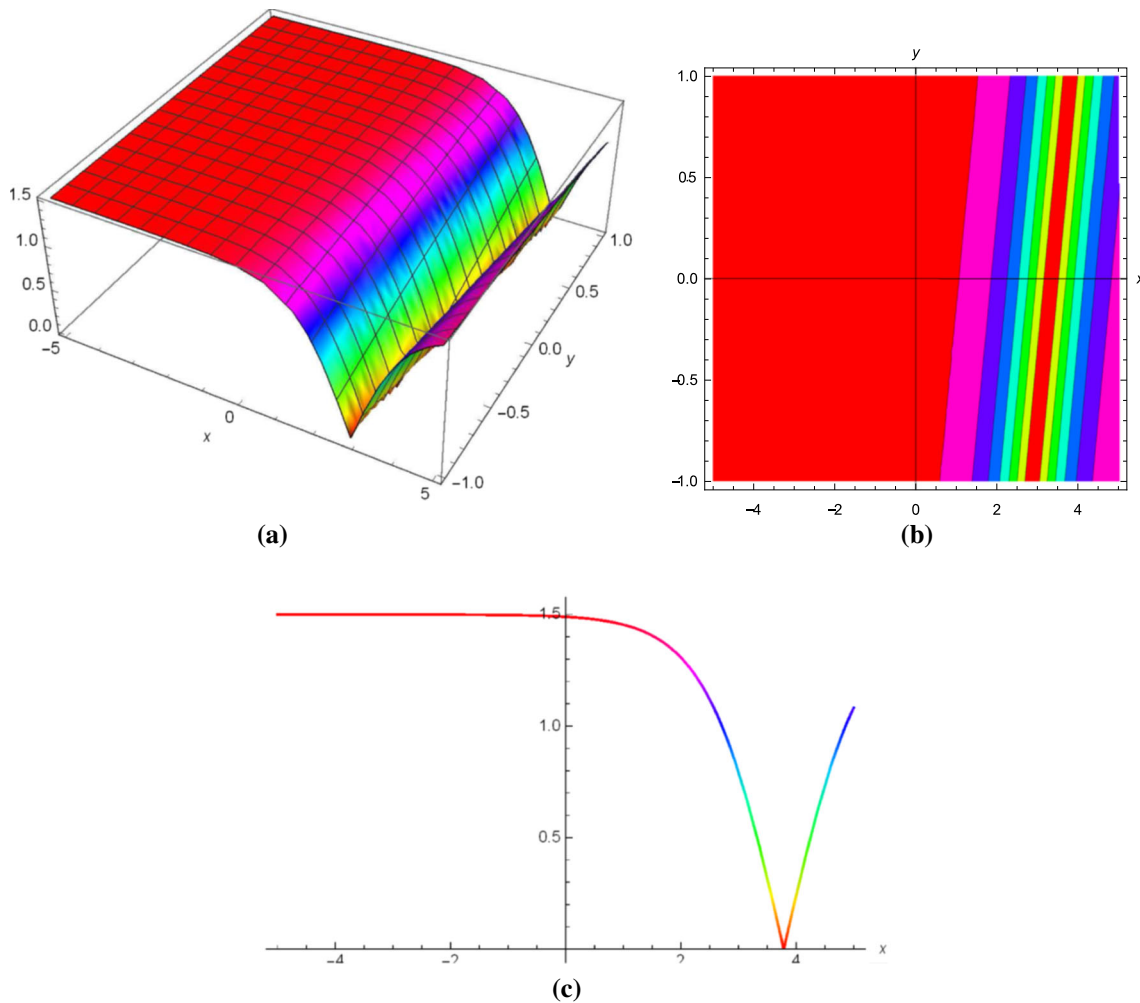


Fig. 6 3D, contour and 2D plots for $|\mu_{10}(x, y, t)|$ with $\alpha_4 = 0.45$, $\alpha_5 = 0.5$, $\alpha_6 = 0.85$, $l_0 = 0.15$, $l_1 = 0.3$, $l_2 = 0.35$, $m_0 = 0.4$, $m_2 = 0.25$, $m_3 = 0.35$, $\omega_1 = 0.5$, $\omega_2 = 0.6$

where l_1, l_2, l_3, g are free parameters. Hence utilizing all the above known values, Eq. (7) gives

$$F = g + (l_3t + l_1x + l_2y)^2 + \left(-\frac{it(2\alpha_4l_1^2 + \alpha_5l_3l_1 + 2\alpha_6l_2^2)}{\alpha_5l_1} + il_1x + il_2y \right)^2. \tag{10}$$

Utilizing Eq. (10) together with Eq. (5), we retrieve the solution

$$\mu_3(x, y, t) = \frac{2l_1(l_3t + l_1x + l_2y) + 2il_1 \left(-\frac{it(2\alpha_4l_1^2 + \alpha_5l_3l_1 + 2\alpha_6l_2^2)}{\alpha_5l_1} + il_1x + il_2y \right)}{g + \left(-\frac{it(2\alpha_4l_1^2 + \alpha_5l_3l_1 + 2\alpha_6l_2^2)}{\alpha_5l_1} + il_1x + il_2y \right)^2 + (l_3t + l_1x + l_2y)^2}. \tag{11}$$

4.2 Collision among lump wave and strip soliton

In this fragment, we use the operator

$$F = \omega_0 + \omega_1\xi_1^2 + \omega_2\xi_2^2 + \omega_3e^{\xi_3}, \tag{12}$$

where

$$\begin{aligned} \xi_1 &= l_0 + l_1x + l_2y + l_3t, & \xi_2 &= m_0 + m_1x + m_2y + m_3t, \\ \xi_3 &= n_0 + n_1x + n_2y + n_3t, \end{aligned}$$

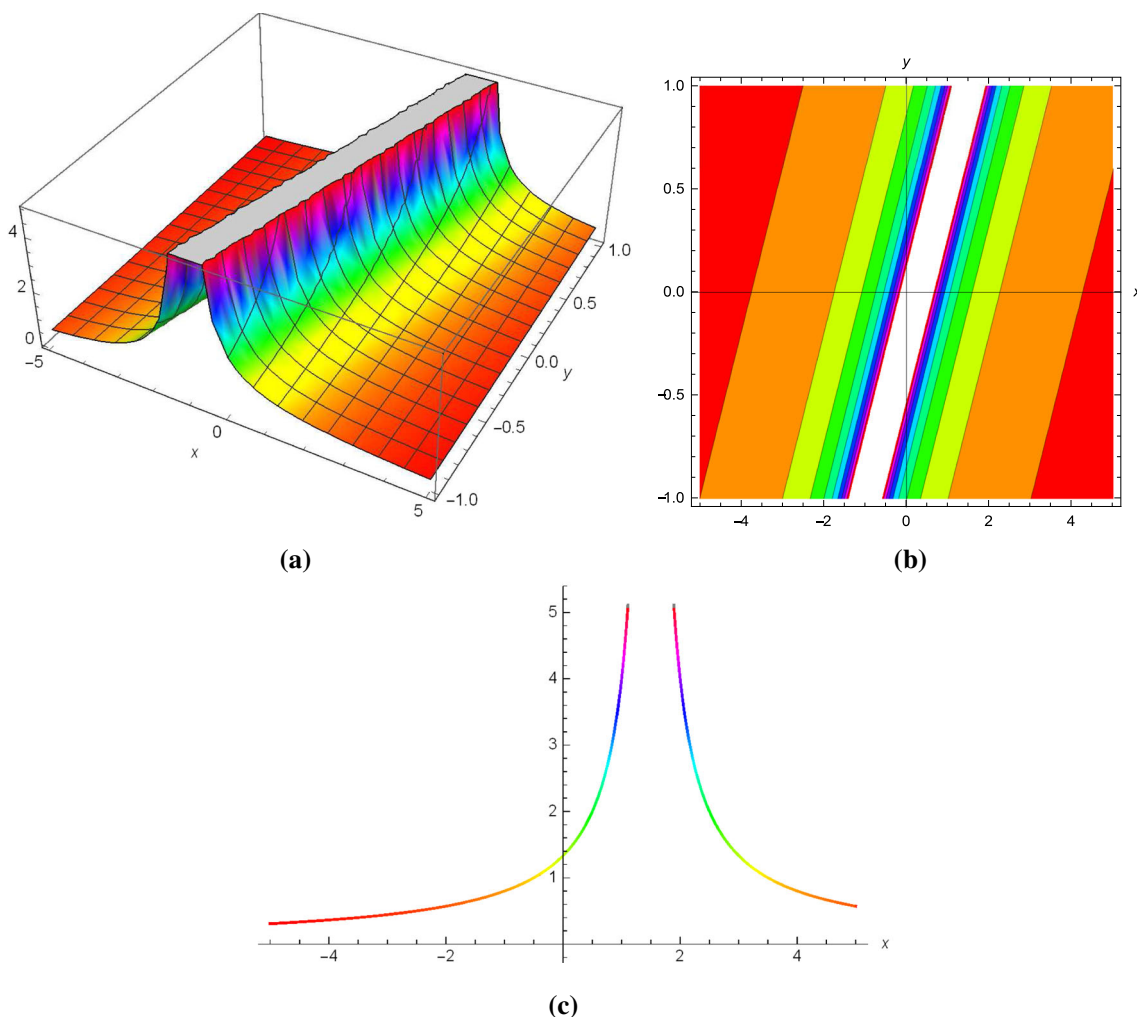


Fig. 7 3D, contour and 2D plots for $|\mu_{12}(x, y, t)|$ with $\alpha_2 = 0.55, \alpha_3 = 0.5, \alpha_4 = 0.45, \alpha_5 = 0.5, \alpha_6 = 0.75, n_0 = 0.15, n_1 = 0.95, n_2 = -0.95$

while l_i, m_i, n_i 's are the constants to be determined. These substitutions for $\xi_1, \xi_2,$ and ξ_3 will be applicable throughout this paper. Using Eq. (12) into Eq. (6), we have a system of equations with several parameters. We get the following findings by resolving them utilizing a computational programme such as Mathematica. Case 1:

$$n_3 = -\frac{\alpha_1 n_1^6 + \alpha_2 n_1^4 + \alpha_3 n_2 n_1^3 + \alpha_4 n_1^2 + \alpha_6 n_2^2}{\alpha_5 n_1}, \omega_1 = 0, \omega_2 = 0,$$

here $n_0, n_1, n_2, \omega_0, \omega_3$ are free parameters. Hence utilizing all the above known values, Eq. (12) gives

$$F = \omega_0 + \omega_3 \exp\left(n_0 - \frac{t(\alpha_1 n_1^6 + \alpha_2 n_1^4 + \alpha_3 n_2 n_1^3 + \alpha_4 n_1^2 + \alpha_6 n_2^2)}{\alpha_5 n_1} + n_1 x + n_2 y\right). \tag{13}$$

Utilizing Eq. (13) together with Eq. (5), we acquire the solution

$$\mu_4(x, y, t) = \frac{2\left(n_1 \omega_3 \exp\left(n_0 - \frac{t(\alpha_1 n_1^6 + \alpha_2 n_1^4 + \alpha_3 n_2 n_1^3 + \alpha_4 n_1^2 + \alpha_6 n_2^2)}{\alpha_5 n_1} + n_1 x + n_2 y\right)\right)}{\omega_3 \exp\left(n_0 - \frac{t(\alpha_1 n_1^6 + \alpha_2 n_1^4 + \alpha_3 n_2 n_1^3 + \alpha_4 n_1^2 + \alpha_6 n_2^2)}{\alpha_5 n_1} + n_1 x + n_2 y\right) + \omega_0}. \tag{14}$$

Case 2:

$$l_3 = \frac{-\alpha_6 l_2^2 l_1 \omega_1 - \alpha_4 l_1^3 \omega_1 - \alpha_4 l_1 m_1^2 \omega_2 + \alpha_6 l_1 m_2^2 \omega_2 - 2\alpha_6 l_2 m_1 m_2 \omega_2}{\alpha_5 (l_1^2 \omega_1 + m_1^2 \omega_2)},$$

$$m_3 = \frac{-\alpha_4 l_1^2 m_1 \omega_1 + \alpha_6 l_2^2 m_1 \omega_1 - 2\alpha_6 l_1 l_2 m_2 \omega_1 - \alpha_4 m_1^3 \omega_2 - \alpha_6 m_2^2 m_1 \omega_2}{\alpha_5 (l_1^2 \omega_1 + m_1^2 \omega_2)},$$

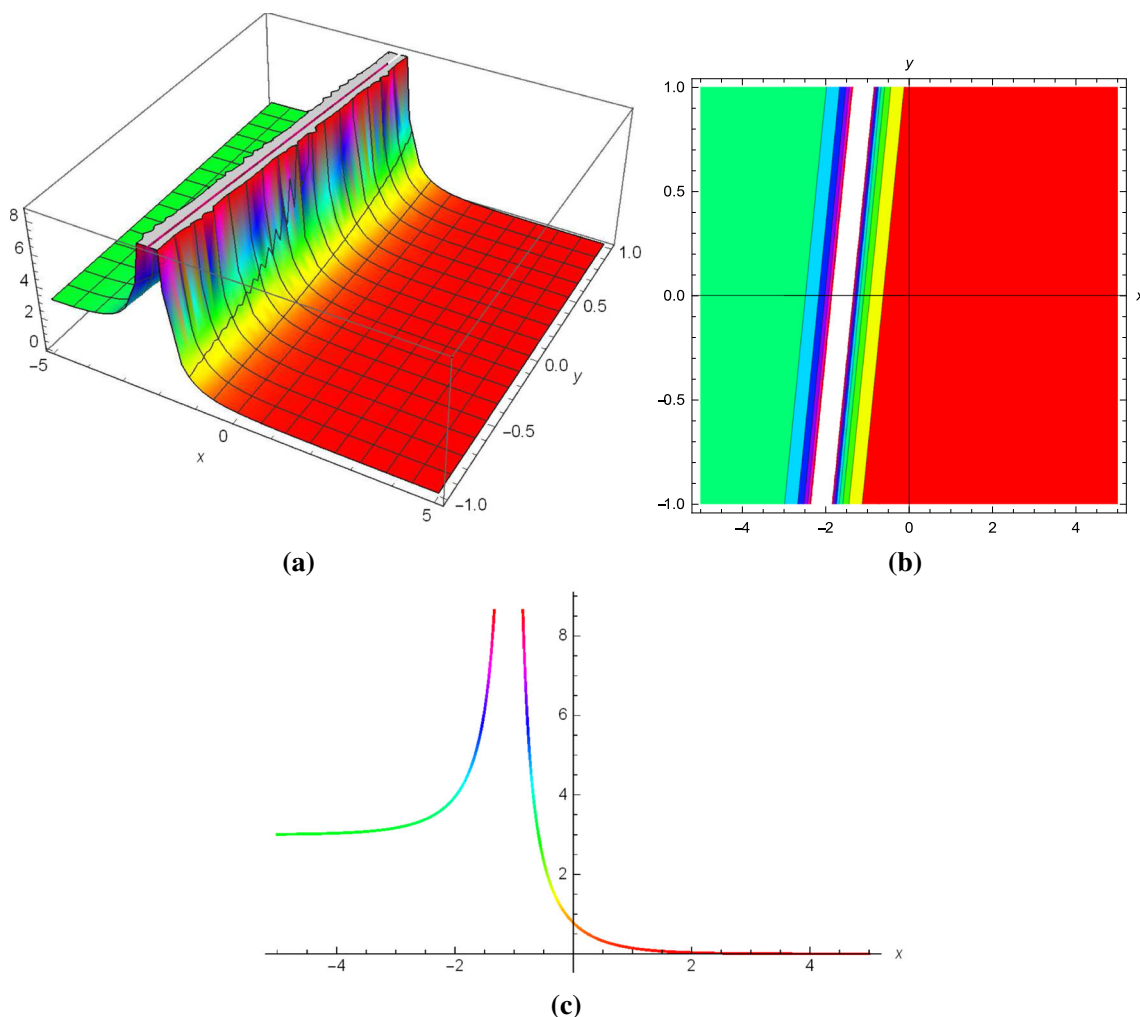


Fig. 8 3D, contour and 2D plots for $|\mu_{13}(x, y, t)|$ with $\alpha_4 = 0.45, \alpha_5 = 0.5, \alpha_6 = 0.75, l_0 = 0.5, l_1 = -2.75, l_2 = 2.55, m_0 = 0.5, m_1 = 1.25, m_3 = 1.55, \omega_1 = 0.5$

$$\omega_0 = -\frac{3(l_1^2\omega_1 + m_1^2\omega_2)^2(\alpha_2l_1^2\omega_1 + \alpha_3l_2l_1\omega_1 + \alpha_2m_1^2\omega_2 + \alpha_3m_1m_2\omega_2)}{\alpha_6\omega_1\omega_2(l_2m_1 - l_1m_2)^2},$$

$$\omega_3 = 0,$$

while $l_0, l_1, l_2, m_0, m_1, m_2, \omega_1, \omega_2$ are free parameters.

Utilizing all the above determined parameters in Eq. (12) and with the aid of Eq. (5), we have $\mu_5(x, y, t)$.

Case 3:

$$l_1 = 0, l_2 = \frac{\alpha_3\alpha_5l_3}{2\alpha_2\alpha_6}, m_2 = -\frac{\alpha_2m_1}{\alpha_3}, \omega_0 = 0,$$

$$\omega_2 = \frac{\alpha_3^4\alpha_5^2l_3^2\omega_1}{4\alpha_2^2\alpha_6m_1(\alpha_6\alpha_2^2m_1 + \alpha_3^2\alpha_4m_1 + \alpha_3^2\alpha_5m_3)}, \omega_3 = 0,$$

where $l_0, l_3, m_0, m_1, m_3, \omega_1$ are free parameters.

Hence utilizing all the above known values, Eq. (12) gives

$$F = \omega_1 \left(l_3t + \frac{\alpha_3\alpha_5l_3y}{2\alpha_2\alpha_6} + l_0 \right)^2 + \frac{\alpha_3^2\alpha_5^4l_3^2\omega_1 \left(m_3t + m_1x - \frac{\alpha_2m_1y}{\alpha_3} + m_0 \right)^2}{4\alpha_2^2\alpha_6m_1(\alpha_6\alpha_2^2m_1 + \alpha_3^2\alpha_4m_1 + \alpha_3^2\alpha_5m_3)}. \tag{15}$$

Utilizing Eq. (15) together with Eq. (5), we gain $\mu_6(x, y, t)$.

Case 4:

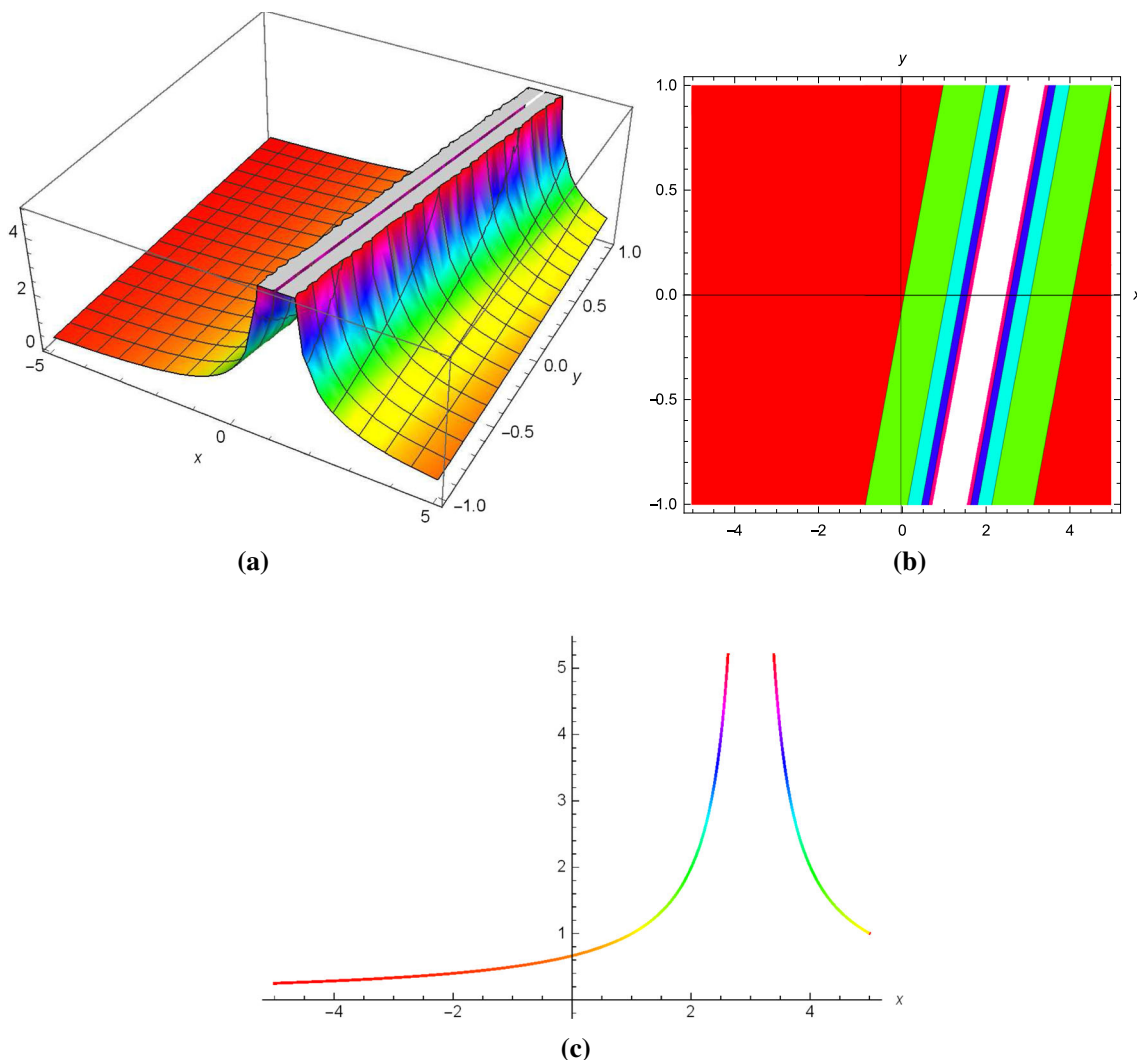


Fig. 9 3D, contour and 2D plots for $|\mu_{15}(x, y, t)|$ with $\alpha_2 = 0.55$, $\alpha_3 = 0.5$, $\alpha_4 = 0.45$, $\alpha_5 = 0.5$, $\alpha_6 = 0.75$, $n_0 = 0.15$, $n_1 = 1.95$, $n_3 = 0.95$

$$l_2 = \frac{i\sqrt{l_1}(-2\alpha_4 l_1 m_1 - \alpha_5 l_3 m_1 - \alpha_5 l_1 m_3)}{\sqrt{2}\sqrt{\alpha_6}\sqrt{m_1}\sqrt{2\alpha_4 l_1 m_1 + \alpha_5 l_3 m_1 + \alpha_5 l_1 m_3}}, \quad \omega_0 = 0,$$

$$\omega_2 = -\frac{l_1^2 \omega_1}{m_1^2}, \quad \omega_3 = 0, \quad m_2 = -\frac{i\sqrt{m_1}\sqrt{2\alpha_4 l_1 m_1 + \alpha_5 l_3 m_1 + \alpha_5 l_1 m_3}}{\sqrt{2}\sqrt{\alpha_6}\sqrt{l_1}},$$

where $l_0, l_1, l_3, m_0, m_1, m_3, \omega_1$ are free parameters.

Hence utilizing all the above known values, Eq. (12) gives

$$F = \omega_1 \left(\frac{i\sqrt{l_1}y(-2\alpha_4 l_1 m_1 - \alpha_5 l_3 m_1 - \alpha_5 l_1 m_3)}{\sqrt{2}\sqrt{\alpha_6}\sqrt{m_1}\sqrt{2\alpha_4 l_1 m_1 + \alpha_5 l_3 m_1 + \alpha_5 l_1 m_3}} + l_3 t + l_1 x + l_0 \right)^2 - \frac{l_1^2 \omega_1 \left(-\frac{i\sqrt{m_1}y\sqrt{2\alpha_4 l_1 m_1 + \alpha_5 l_3 m_1 + \alpha_5 l_1 m_3}}{\sqrt{2}\sqrt{\alpha_6}\sqrt{l_1}} + m_3 t + m_1 x + m_0 \right)^2}{m_1^2}. \tag{16}$$

Utilizing Eq. (16) together with Eq. (5), we gain $\mu_7(x, y, t)$.

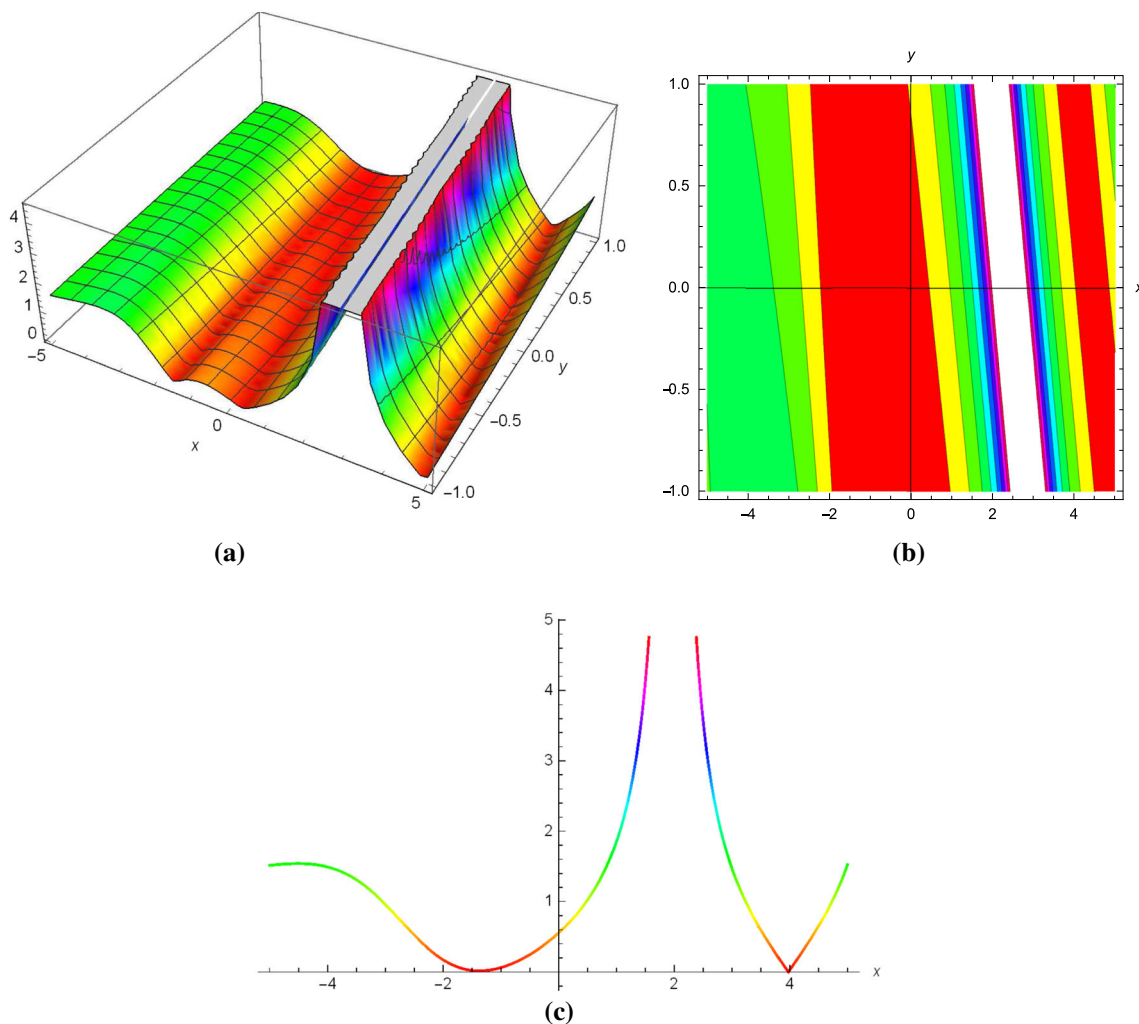


Fig. 10 3D, contour and 2D plots for $|\mu_{16}(x, y, t)|$ with $\alpha_1 = 1.2, \alpha_2 = 2, \alpha_3 = -1.5, \alpha_4 = 0.4, \alpha_5 = 0.5, \alpha_6 = 0.75, m_0 = 0.5, n_0 = -0.85, n_1 = 0.225, n_2 = 0.8, \omega_2 = 0.75, \omega_3 = 0.65$

4.3 Collision between lump wave and double stripes soliton

In this subsection, we review the operator F stated as

$$F = \omega_0 + \omega_1 \xi_1^2 + \omega_2 \xi_2^2 + \omega_3 \cosh(\xi_3). \tag{17}$$

Using Eq. (17) into Eq. (6), we have a system of equations with several parameters. We get the following findings by resolving them utilizing a computational programme such as Mathematica.

Case 1:

$$n_3 = -\frac{4\alpha_2 n_1^4 + 4\alpha_3 n_2 n_1^3 + \alpha_4 n_1^2 + \alpha_6 n_2^2}{\alpha_5 n_1}, \omega_0 = 0, \omega_1 = 0, \omega_2 = 0,$$

where n_0, n_1, n_2, ω_3 are free parameters.

Hence utilizing all the above known values, Eq. (17) gives

$$F = \omega_3 \cosh\left(-\frac{t(4\alpha_2 n_1^4 + 4\alpha_3 n_2 n_1^3 + \alpha_4 n_1^2 + \alpha_6 n_2^2)}{\alpha_5 n_1} + n_1 x + n_2 y + n_0\right). \tag{18}$$

Utilizing Eq. (18) together with Eq. (5), we retrieve the solution

$$\mu_8(x, y, t) = 2n_1 \tanh\left(-\frac{t(4\alpha_2 n_1^4 + 4\alpha_3 n_2 n_1^3 + \alpha_4 n_1^2 + \alpha_6 n_2^2)}{\alpha_5 n_1} + n_1 x + n_2 y + n_0\right).$$

Case 2:

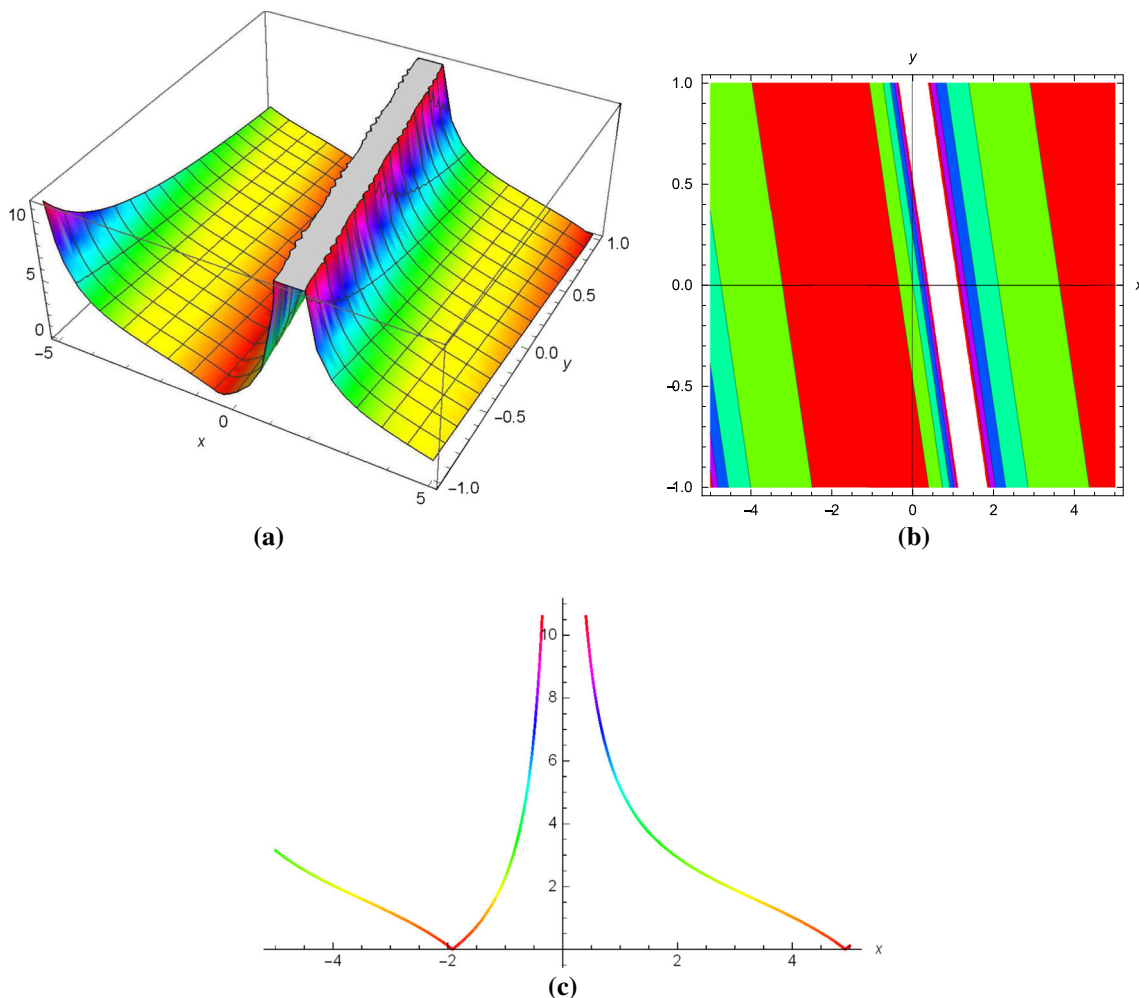


Fig. 11 3D, contour and 2D plots for $|\mu_{17}(x, y, t)|$ with $\alpha_1 = 0.45, \alpha_2 = 0.25, \alpha_3 = 0.25, \alpha_4 = 0.3, \alpha_5 = 0.4, \alpha_6 = 0.35, m_0 = 0.125, m_1 = 0.35, m_2 = 0.25, n_0 = 0.125, \omega_2 = 0.55, \omega_3 = 0.75$

$$l_1 = \frac{l_2 m_1}{m_2}, l_3 = -\frac{l_2(2\alpha_4 m_1^2 + \alpha_5 m_3 m_1 + 2\alpha_6 m_2^2)}{\alpha_5 m_1 m_2}, \omega_2 = -\frac{l_2^2 \omega_1}{m_2^2}, \omega_0 = \omega_3 = 0,$$

where $l_0, l_2, m_0, m_1, m_2, m_3, \omega_1$ are free parameters.

Hence utilizing all the above known values, Eq. (17) gives

$$F = \omega_1 \left(-\frac{l_2 t (2\alpha_4 m_1^2 + \alpha_5 m_3 m_1 + 2\alpha_6 m_2^2)}{\alpha_5 m_1 m_2} + \frac{l_2 m_1 x}{m_2} + l_2 y + l_0 \right)^2 - \frac{l_2^2 \omega_1 (m_3 t + m_1 x + m_2 y + m_0)^2}{m_2^2}. \tag{19}$$

Utilizing Eq. (19) together with Eq. (5), we gain $\mu_9(x, y, t)$.

Case 3:

$$l_3 = \frac{-l_1^2 \alpha_4 \omega_1 - \alpha_6 l_2^2 \omega_1 + \alpha_6 m_2^2 \omega_2}{\alpha_5 l_1 \omega_1}, m_1 = 0, m_3 = -\frac{2\alpha_6 l_2 m_2}{\alpha_5 l_1}, \omega_0 = 0, \omega_3 = 0,$$

where $l_0, l_1, l_2, m_0, m_2, \omega_1, \omega_2$ are free parameters.

Hence utilizing all the above parameters, Eq. (17) gives

$$F = \omega_1 \left(\frac{t(-l_1^2 \alpha_4 \omega_1 - \alpha_6 l_2^2 \omega_1 + \alpha_6 m_2^2 \omega_2)}{\alpha_5 l_1 \omega_1} + l_1 x + l_2 y + l_0 \right)^2$$

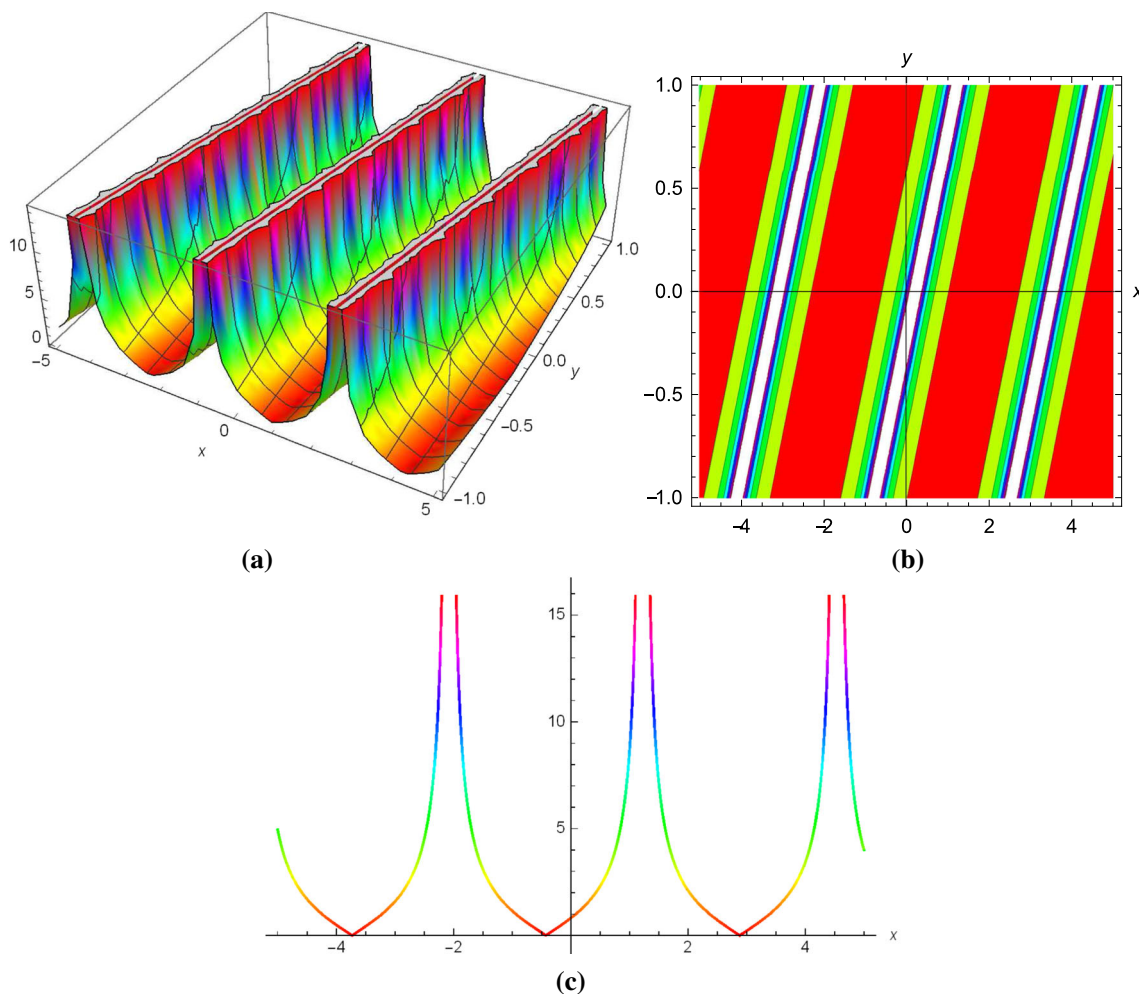


Fig. 12 3D, contour and 2D plots for $|\mu_{18}(x, y, t)|$ with $\alpha_1 = 0.15, \alpha_2 = 0.25, \alpha_3 = 0.5, \alpha_4 = 0.4, \alpha_5 = 0.5, \alpha_6 = 0.55, l_0 = 0.25, l_1 = -0.75, l_2 = 0.35, \omega_1 = 0.5, \omega_2 = 0.85$

$$+ \omega_2 \left(-\frac{2\alpha_6 l_2 m_2 t}{\alpha_5 l_1} + m_2 y + m_0 \right)^2 \tag{20}$$

Utilizing Eq. (20) together with Eq. (5), we have $\mu_{10}(x, y, t)$.

Case 4:

$$n_2 = \frac{\sqrt{16\alpha_3^2 n_1^6 + 4\alpha_6(-4\alpha_2 n_1^4 - \alpha_4 n_1^2 - \alpha_5 n_3 n_1) - 4\alpha_3 n_1^3}}{2\alpha_6},$$

$$\omega_0 = 0, \omega_1 = 0, \omega_2 = 0,$$

where n_0, n_1, n_3, ω_3 are free parameters.

Utilizing all the above determined parameters in Eq. (17) and with the aid of Eq. (5), we gain $\mu_{11}(x, y, t)$.

4.4 Collision between lump and periodic waves

In this portion, we examine the operator F specified by

$$F = \omega_0 + \omega_1 \xi_1^2 + \omega_2 \xi_2^2 + \omega_3 \cos(\xi_3). \tag{21}$$

Using Eq. (21) into Eq. (6), we have a system of equations with several parameters. We get the following findings by resolving them utilizing a computational programme such as Mathematica.

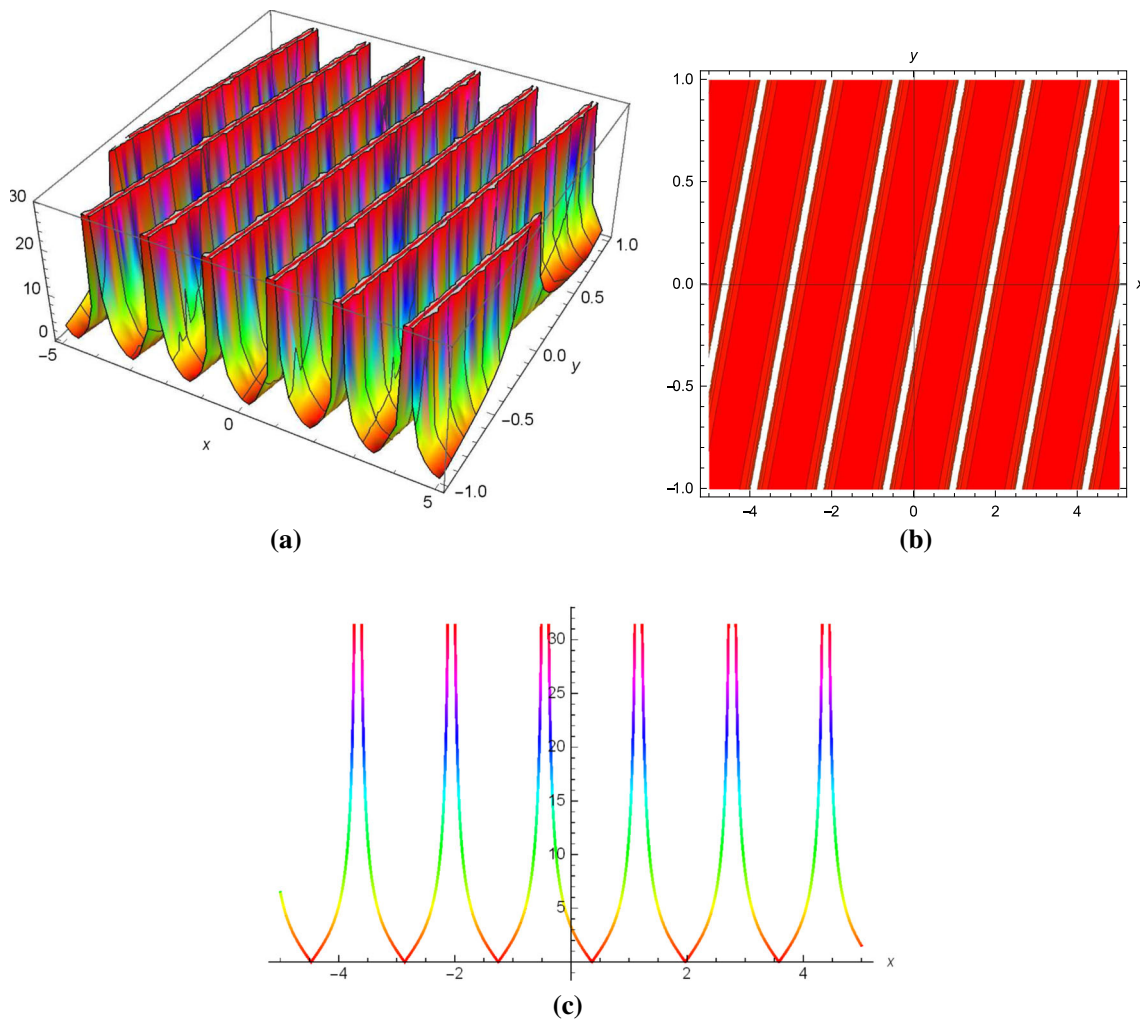


Fig. 13 3D, contour and 2D plots for $|\mu_{19}(x, y, t)|$ with $\alpha_1 = 0.2, \alpha_2 = 0.25, \alpha_3 = 0.5, \alpha_4 = 0.4, \alpha_5 = 0.5, \alpha_6 = 0.55, l_0 = 0.5, l_1 = 0.65, m_0 = 0.5, m_1 = 0.75, m_2 = 0.35, \omega_1 = 1, \omega_3 = 1$

Case 1:

$$n_3 = \frac{4\alpha_2 n_1^4 + 4\alpha_3 n_2 n_1^3 - \alpha_4 n_1^2 - \alpha_6 n_2^2}{\alpha_5 n_1}, \quad \omega_0 = 0, \quad \omega_1 = 0, \quad \omega_2 = 0,$$

where n_0, n_1, n_2, ω_3 are free parameters.

Hence utilizing all the above known values, Eq. (21) gives

$$F = \omega_3 \cos\left(\frac{t(4\alpha_2 n_1^4 + 4\alpha_3 n_2 n_1^3 - \alpha_4 n_1^2 - \alpha_6 n_2^2)}{\alpha_5 n_1} + n_1 x + n_2 y + n_0\right). \tag{22}$$

Utilizing Eq. (22) together with Eq. (5), we obtain the solution

$$\mu_{12}(x, y, t) = -2n_1 \tan\left(\frac{t(4\alpha_2 n_1^4 + 4\alpha_3 n_2 n_1^3 - \alpha_4 n_1^2 - \alpha_6 n_2^2)}{\alpha_5 n_1} + n_1 x + n_2 y + n_0\right).$$

Case 2:

$$l_3 = \frac{-2\alpha_4 l_1^2 m_1 - \alpha_5 l_1^2 m_3 - 2\alpha_6 l_2^2 m_1}{\alpha_5 l_1 m_1}, \quad m_2 = \frac{l_2 m_1}{l_1}, \quad \omega_2 = -\frac{l_1^2 \omega_1}{m_1^2}, \quad \omega_0 = \omega_3 = 0,$$

where $l_0, l_1, l_2, m_0, m_1, m_3, \omega_1$ are free parameters.

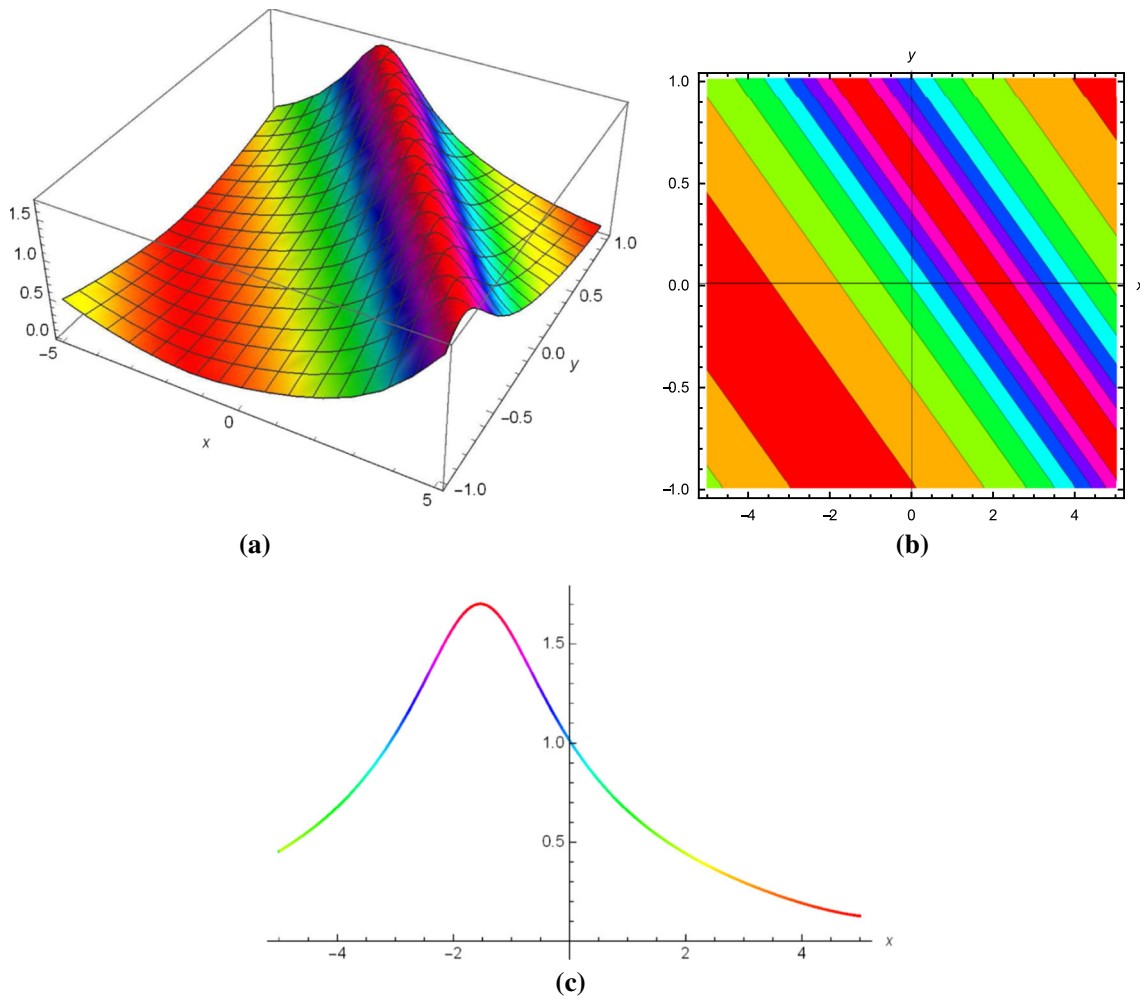


Fig. 14 3D, contour and 2D plots for $|\mu_{21}(x, y, t)|$ with $a = 0.7, b = 0.5, c = 0.5, \alpha_1 = 0.6, \alpha_2 = 1.2, \alpha_3 = 0.5, \alpha_4 = 0.45, \alpha_5 = 0.55, \alpha_6 = 0.75, C_0 = 0.5, \xi_0 = 1.6$

Hence utilizing all the above parameters, Eq. (21) gives

$$\begin{aligned}
 F = & \omega_1 \left(\frac{t(-2\alpha_4 l_1^2 m_1 - \alpha_5 l_1^2 m_3 - 2\alpha_6 l_2^2 m_1)}{\alpha_5 l_1 m_1} + l_1 x + l_2 y + l_0 \right)^2 \\
 & - \frac{l_1^2 \omega_1 \left(\frac{l_2 m_1 y}{l_1} + m_3 t + m_1 x + m_0 \right)^2}{m_1^2}.
 \end{aligned}
 \tag{23}$$

Utilizing Eq. (23) together with Eq. (5), we retrieve the solution $\mu_{13}(x, y, t)$.

Case 3:

$$\begin{aligned}
 l_3 = & \frac{-l_2^2 \alpha_6 l_1 \omega_1 - \alpha_4 l_1^3 \omega_1 - \alpha_4 l_1 m_1^2 \omega_2 + \alpha_6 l_1 m_2^2 \omega_2 - 2\alpha_6 l_2 m_1 m_2 \omega_2}{\alpha_5 (l_1^2 \omega_1 + m_1^2 \omega_2)}, \\
 m_3 = & \frac{-\alpha_4 l_1^2 m_1 \omega_1 + \alpha_6 l_2^2 m_1 \omega_1 - 2\alpha_6 l_1 l_2 m_2 \omega_1 - \alpha_4 m_1^3 \omega_2 - \alpha_6 m_2^2 m_1 \omega_2}{\alpha_5 (l_1^2 \omega_1 + m_1^2 \omega_2)}, \\
 \omega_0 = & 0, \quad \omega_3 = 0,
 \end{aligned}$$

where $l_0, l_1, l_2, m_0, m_1, m_2, \omega_1, \omega_2$ are free parameters.

Utilizing all the above determined parameters in Eq. (21) and with the aid of Eq. (5), we gain $\mu_{14}(x, y, t)$.

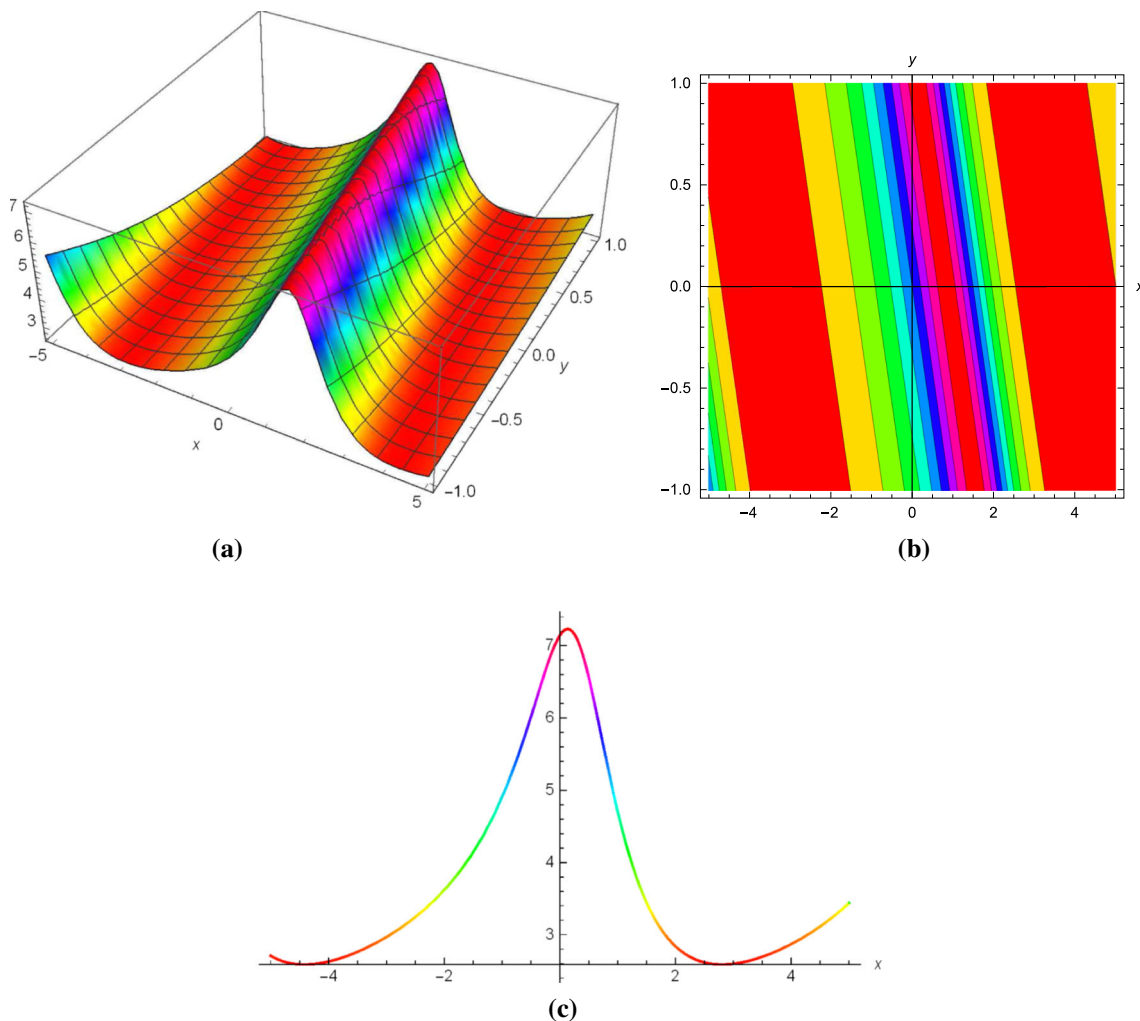


Fig. 15 3D, contour and 2D plots for $|\mu_{23}(x, y, t)|$ with $a = 0.75, b = 0.55, c = 0.65, \alpha_1 = 0.5, \alpha_2 = 1.12, \alpha_3 = 0.45, \alpha_4 = 0.5, \alpha_5 = 0.65, \alpha_6 = 0.5, \xi_0 = 1.56$

Case 4:

$$n_2 = \frac{4\alpha_3 n_1^3 - \sqrt{16\alpha_3^2 n_1^6 + 4\alpha_6(4\alpha_2 n_1^4 - \alpha_4 n_1^2 - \alpha_5 n_3 n_1)}}{2\alpha_6},$$

$$\omega_0 = 0, \omega_1 = 0, \omega_2 = 0,$$

where n_0, n_1, n_3, ω_3 are free parameters.

Utilizing all the above determined parameters in Eq. (21) and with the aid of Eq. (5), we have $\mu_{15}(x, y, t)$.

4.5 Collision among lump wave and double stripes soliton

Here, we get the operator F of the form

$$F = \omega_0 + \omega_1 \xi_1^2 + \omega_2 \cosh(\xi_2) + \omega_3 \cos(\xi_3). \tag{24}$$

Using Eq. (24) into Eq. (6), we have a system of equations with several parameters. We get the following findings by resolving them utilizing a computational programme such as Mathematica.

Case 1:

$$n_3 = \frac{-16\alpha_1 n_1^6 + 4\alpha_2 n_1^4 + 4\alpha_3 n_2 n_1^3 - \alpha_4 n_1^2 - \alpha_6 n_2^2}{\alpha_5 n_1}, \quad m_1 = in_1, \quad m_2 = in_2,$$

$$m_3 = -\frac{i(16\alpha_1 n_1^6 - 4\alpha_2 n_1^4 - 4\alpha_3 n_2 n_1^3 + \alpha_4 n_1^2 + \alpha_6 n_2^2)}{\alpha_5 n_1}, \quad \omega_0 = \omega_1 = 0,$$

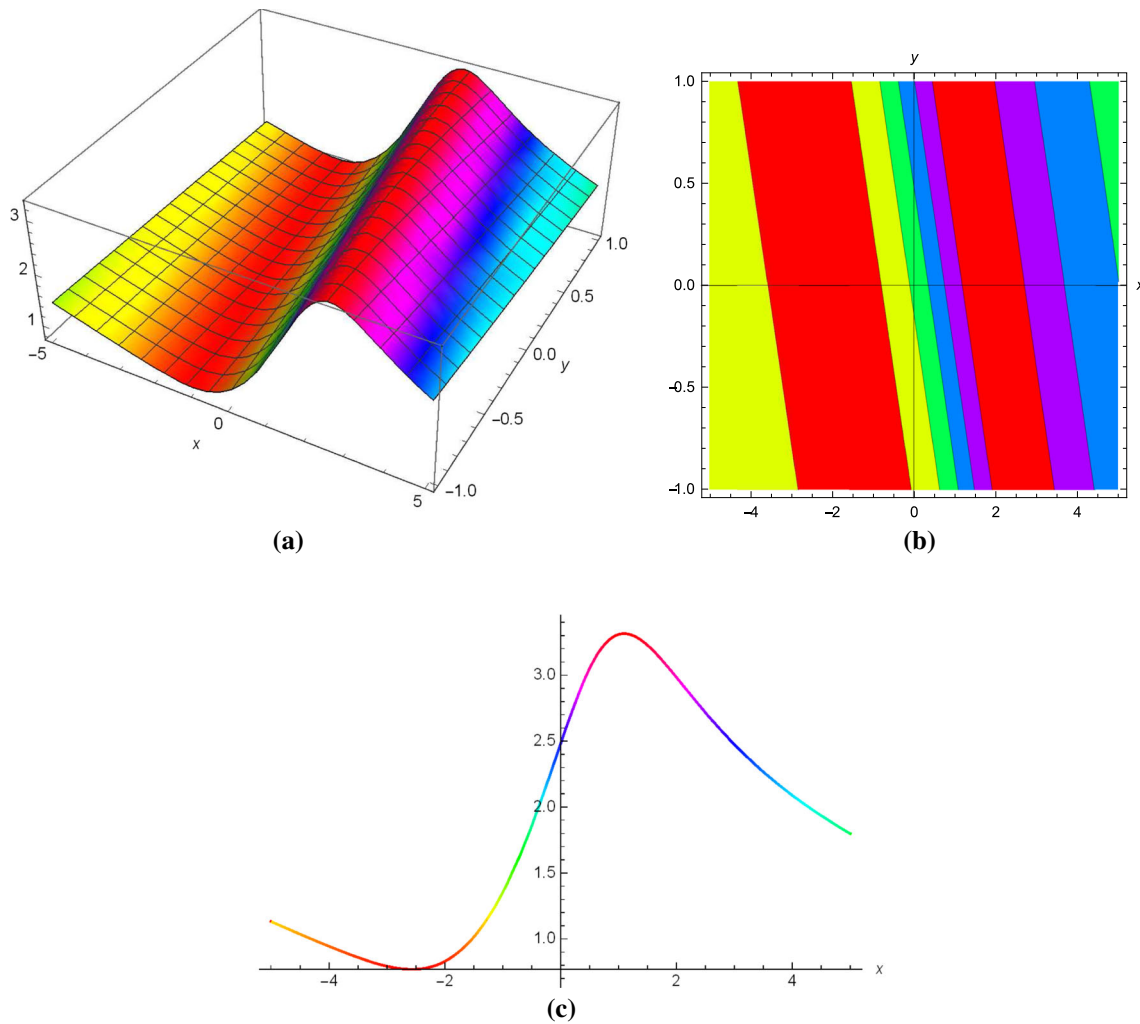


Fig. 16 3D, contour and 2D plots for $|\mu_{23}(x, y, t)|$ with $a = 0.75, b = 0.55, c = 0.55, \alpha_1 = 0.56, \alpha_2 = 1.12, \alpha_3 = 0.65, M = 0.45, \alpha_4 = 0.455, \alpha_5 = 0.55, \alpha_6 = 0.75, \xi_0 = 1.6, C_1 = 0.5$

where $n_0, n_1, n_2, \omega_2, \omega_3$ are free parameters.

Utilizing all the above determined parameters in Eq. (24) and with the aid of Eq. (5), we gain $\mu_{16}(x, y, t)$.

Case 2:

$$m_3 = \frac{-16\alpha_1 m_1^6 - 4\alpha_2 m_1^4 - 4\alpha_3 m_2 m_1^3 - \alpha_4 m_1^2 - \alpha_6 m_2^2}{\alpha_5 m_1}, \quad n_1 = im_1, \quad n_2 = im_2,$$

$$n_3 = -\frac{i(16\alpha_1 m_1^6 + 4\alpha_2 m_1^4 + 4\alpha_3 m_2 m_1^3 + \alpha_4 m_1^2 + \alpha_6 m_2^2)}{\alpha_5 m_1}, \quad \omega_0 = \omega_1 = 0,$$

where $m_0, m_1, m_2, n_0, n_1, n_2, \omega_2, \omega_3$ are free parameters.

Utilizing all the above determined parameters in Eq. (24) and with the aid of Eq. (5), we retrieve the solution $\mu_{17}(x, y, t)$.

4.6 Periodic wave soliton

Lastly, we get the operator F as

$$F = \omega_0 + \omega_1 e^{-\xi_1} + \omega_2 e^{\xi_1} + \omega_3 \cos(\xi_2). \tag{25}$$

Using Eq. (25) into Eq. (6), we have a system of equations with several parameters. We get the following findings by resolving them utilizing a computational programme such as Mathematica.

Case 1:

$$l_3 = \frac{-16\alpha_1 l_1^6 - 4\alpha_2 l_1^4 - 4\alpha_3 l_2 l_1^3 - \alpha_4 l_1^2 - \alpha_6 l_2^2}{\alpha_5 l_1}, \quad \omega_3 = \omega_0 = 0,$$

where $l_0, l_1, l_2, \omega_1, \omega_2$ are free parameters.

Hence utilizing all the above known values, Eq. (25) gives

$$F = \omega_1 \exp\left(\frac{t(16\alpha_1 l_1^6 + 4\alpha_2 l_1^4 + 4\alpha_3 l_2 l_1^3 + \alpha_4 l_1^2 + \alpha_6 l_2^2)}{\alpha_5 l_1} - l_1 x - l_2 y - l_0\right) + \omega_2 \exp\left(\frac{t(-16\alpha_1 l_1^6 - 4\alpha_2 l_1^4 - 4\alpha_3 l_2 l_1^3 - \alpha_4 l_1^2 - \alpha_6 l_2^2)}{\alpha_5 l_1} + l_1 x + l_2 y + l_0\right).$$

Utilizing the above result together with Eq. (5), we gain $\mu_{18}(x, y, t)$.

Case 2:

$$l_2 = \frac{l_1 m_1 (\alpha_3 m_1 (l_1^2 + m_1^2) + \alpha_6 m_2) - \chi}{\alpha_6 m_1^2}, \quad \omega_2 = 0, \quad \omega_0 = 0,$$

$$m_3 = -\frac{m_1^2 (\alpha_4 + 16\alpha_1 m_1^4 - 4\alpha_2 m_1^2 - 4\alpha_3 m_2 m_1) + \alpha_6 m_2^2}{\alpha_5 m_1},$$

$$l_3 = \frac{1}{\alpha_5 \alpha_6 m_1^3} (3\alpha_3 l_1^2 m_1 \chi - 3(\alpha_3^2 + 2\alpha_1 \alpha_6) l_1^5 m_1^3 - 2l_1^3 m_1^2 ((\alpha_3^2 - 10\alpha_1 \alpha_6) m_1^3 + 2\alpha_2 \alpha_6 m_1 + 3\alpha_3 \alpha_6 m_2) + l_1 m_1 (\alpha_3^2 m_1^6 + \alpha_6 m_1^2 (-\alpha_4 + 10\alpha_1 m_1^4 - 2\alpha_3 m_2 m_1) - \alpha_6^2 m_2^2) + \chi (2\alpha_6 m_2 - \alpha_3 m_1^3)),$$

where

$$\chi = \sqrt{m_1^3 (l_1^2 + m_1^2)^2 ((\alpha_3^2 + 5\alpha_1 \alpha_6) l_1^2 m_1 + 3\alpha_6 (-5\alpha_1 m_1^3 + \alpha_2 m_1 + \alpha_3 m_2))},$$

where $l_0, l_1, m_0, m_1, m_2, \omega_1, \omega_3$ are free parameters.

Utilizing all the above determined parameters in Eq. (25) and with the aid of Eq. (5), we get the solution $\mu_{19}(x, y, t)$.

5 Travelling-wave solutions for Eq. (4)

In this section, we employ the polynomial-expansion method to find some new exact travelling wave solutions for Eq. (4). Consequently, we use the transformation stated as

$$\mu(x, y, t) = \Omega(\eta), \quad \eta = ax + by - ct, \tag{26}$$

where a, b are constants and c is the velocity of the wave. Substituting Eq. (26) into Eq. (4), we have

$$\alpha_1 a^6 \Omega^{(6)}(\eta) + 15\alpha_1 a^5 (\Omega^{(4)}(\eta)\Omega'(\eta) + \Omega^{(3)}(\eta)\Omega''(\eta)) + 45\alpha_1 a^4 \Omega'^2(\eta)\Omega''(\eta) + (\alpha_4 a^2 - \alpha_5 ac + \alpha_6 b^2)\Omega''(\eta) + (a^4 \alpha_2 - a^3 \alpha_3 b)\Omega^{(4)}(\eta) + (6\alpha_2 a^3 + 6\alpha_3 a^2 b)\Omega'(\eta)\Omega''(\eta) = 0, \tag{27}$$

through integration, we retrieve the result

$$\alpha_1 a^6 \Omega^{(5)}(\eta) + 15\alpha_1 a^5 (\Omega^{(3)}(\eta)\Omega'(\eta)) + 15\alpha_1 a^4 \Omega'^3(\eta) + (\alpha_4 a^2 - \alpha_5 ac + \alpha_6 b^2)\Omega'(\eta) + (a^4 \alpha_2 - a^3 \alpha_3 b)\Omega^{(3)}(\eta) + (3\alpha_2 a^3 + 3\alpha_3 a^2 b)\Omega'(\eta)^2 = 0, \tag{28}$$

Utilizing the specified method, we suppose the solution for Eq. (28) given by

$$\Omega(\eta) = \xi_0 + \sum_{j=1}^m \xi_j \phi(\eta)^j + \sum_{j=1}^m \zeta_j \phi(\eta)^{-j}, \tag{29}$$

while $\phi(\eta)$ satisfies

$$\phi'(\eta) = \phi(\eta)^2 + \mathcal{R}\phi(\eta) + \mathcal{Q}, \tag{30}$$

here ξ_0, ξ_j 's, ζ_j 's, \mathcal{R}, \mathcal{Q} are constants and m is a positive integer. As a result, we retrieve the following exact solutions:

(1) when $\mathcal{R} = 0, \mathcal{Q} = 0,$

$$\phi(\eta) = -\frac{1}{\eta}; \tag{31}$$

(2) when $\mathcal{R} \neq 0, \mathcal{Q} = 0,$

$$\phi(\eta) = -\frac{\mathcal{R}}{C_0 e^{-\mathcal{R}\eta} - 1}, \tag{32}$$

here C_0 is the constant of integration;

(3) when $\mathcal{R} = 0, \mathcal{Q} \neq 0,$

If $\mathcal{Q} > 0,$ then

$$\begin{aligned} \phi(\eta) &= \sqrt{\mathcal{Q}} \tan(\sqrt{\mathcal{Q}}\eta), \\ \phi(\eta) &= -\sqrt{-\mathcal{Q}} \cot(\sqrt{\mathcal{Q}}\eta), \end{aligned}$$

If $\mathcal{Q} < 0,$ then

$$\begin{aligned} \phi(\eta) &= -\sqrt{-\mathcal{Q}} \tanh(\sqrt{-\mathcal{Q}}\eta), \\ \phi(\eta) &= \sqrt{-\mathcal{Q}} \coth(\sqrt{-\mathcal{Q}}\eta); \end{aligned}$$

(4) when $\mathcal{R} \neq 0, \mathcal{Q} \neq 0,$

$$\phi(\eta) = \frac{\kappa_1 - C_1 \kappa_2 e^{(\kappa_1 - \kappa_2)\eta}}{1 - C_1 e^{(\kappa_1 - \kappa_2)\eta}}, \tag{33}$$

where

$$\kappa_1 = \frac{1}{4}(\sqrt{\mathcal{R}^2 - 4\mathcal{Q}} - \mathcal{R}), \quad \kappa_2 = \frac{1}{4}(-\sqrt{\mathcal{R}^2 - 4\mathcal{Q}} - \mathcal{R}),$$

here C_1 is the constant of integration.

Comparing $\Omega^{(5)}(\eta)$ with $\Omega'(\eta)\Omega^{(3)}(\eta)$ in Eq. (28) we have $m = 1$. As a result, Eq. (29) can be expressed as

$$\Omega(\eta) = \xi_0 + \xi_1 \phi(\eta) + \frac{\zeta_1}{\phi(\eta)}. \tag{34}$$

Utilizing Eq. (34) together with Eq. (30) into Eq. (28) and balancing the coefficients of $\phi^i(\eta)$ to zero, we have a set of equations involving several parameters. We get the following findings by resolving them utilizing a computational programme such as Mathematica. Set 1:

$$\mathcal{R} = \mathcal{Q} = \xi_1 = 0, \quad \zeta_1 = \frac{3\alpha_2 a^3 + 3\alpha_3 a^2 b + Z}{30a^4 \alpha_1},$$

where

$$Z = \sqrt{60\alpha_1 a^4 (-a^2 \alpha_4 + \alpha_5 a c - \alpha_6 b^2) + (3\alpha_2 a^3 + 3\alpha_3 a^2 b)^2},$$

Set 2:

$$\begin{aligned} c &= \frac{-4\alpha_2^2 a^2 + 25\alpha_1 \alpha_4 a^2 - 4\alpha_2 \alpha_3 a b + 24\alpha_3^2 b^2 + 25\alpha_1 \alpha_6 b^2}{25\alpha_1 \alpha_5}, \\ \mathcal{R} &= \frac{\sqrt{-a\alpha_2 - 3\alpha_3 b}}{\sqrt{5a^{3/2} \sqrt{\alpha_1}}}, \quad \zeta_1 = \mathcal{Q} = 0, \quad \xi_1 = -4a \end{aligned}$$

Set 3:

$$\begin{aligned} \mathcal{R} = 0, \quad \mathcal{Q} &= \frac{a\alpha_2 + 3\alpha_3 b}{80a^3 \alpha_1}, \quad \xi_1 = -4a, \quad \zeta_1 = \frac{a\alpha_2 + 3\alpha_3 b}{20a^2 \alpha_1}, \\ c &= \frac{-4\alpha_2^2 a^2 + 25\alpha_1 \alpha_4 a^2 - 4\alpha_2 \alpha_3 a b + 24\alpha_3^2 b^2 + 25\alpha_1 \alpha_6 b^2}{25\alpha_1 \alpha_5} \end{aligned}$$

Set 4:

$$c = \frac{-4\alpha_2^2 a^2 + 25\alpha_1 \alpha_4 a^2 - 4\alpha_2 \alpha_3 a b + 24\alpha_3^2 b^2 + 25\alpha_1 \alpha_6 b^2}{25\alpha_1 \alpha_5},$$

$$\xi_1 = 0, \mathcal{R} = \frac{i\sqrt{-20\alpha_1 a^3 \mathcal{Q} + \alpha_2 a + 3\alpha_3 b}}{\sqrt{5}a^{3/2}\sqrt{\alpha_1}}, \xi_1 = -4a.$$

Hence utilizing all the above determined results, we construct the travelling-wave solutions for Eq. (4) as

Type 1: When $\mathcal{R} = 0$ and $\mathcal{Q} = 0$, we get the solution

$$\mu_{20}(x, y, t) = \xi_0 + \frac{(3\alpha_2 a^3 + 3\alpha_3 a^2 b + Z)(-ax - by + ct)}{30a^4 \alpha_1}, \tag{35}$$

where

$$Z = \sqrt{60\alpha_1 a^4(-a^2\alpha_4 + \alpha_5 ac - \alpha_6 b^2) + (3\alpha_2 a^3 + 3\alpha_3 a^2 b)^2}.$$

Type 2: When $\mathcal{R} \neq 0$ and $\mathcal{Q} = 0$, we obtain the solution

$$\mu_{21}(x, y, t) = \xi_0 + \frac{4\sqrt{-a\alpha_2 - 3\alpha_3 b}}{\sqrt{5}\sqrt{a}\sqrt{\alpha_1} \left(C_0 e^{-\frac{\eta\sqrt{-a\alpha_2 - 3\alpha_3 b}}{\sqrt{5}a^{3/2}\sqrt{\alpha_1}}} - 1 \right)}, \tag{36}$$

where

$$\eta = ax + by - \frac{(-4\alpha_2^2 a^2 + 25\alpha_1 \alpha_4 a^2 - 4\alpha_2 \alpha_3 ab + 24\alpha_3^2 b^2 + 25\alpha_1 \alpha_6 b^2)t}{25a\alpha_1 \alpha_5}.$$

Type 3: When $\mathcal{R} = 0$ and $\mathcal{Q} > 0$, we attain the solution

$$\begin{aligned} \mu_{22}(x, y, t) = \xi_0 + & \frac{(a\alpha_2 + 3\alpha_3 b) \cot\left(\frac{\eta\sqrt{\frac{a\alpha_2 + 3\alpha_3 b}{a^3 \alpha_1}}}{4\sqrt{5}}\right)}{\sqrt{5}a^2 \alpha_1 \sqrt{\frac{a\alpha_2 + 3\alpha_3 b}{a^3 \alpha_1}}} \\ & - \frac{a\sqrt{\frac{a\alpha_2 + 3\alpha_3 b}{a^3 \alpha_1}} \tan\left(\frac{\eta\sqrt{\frac{a\alpha_2 + 3\alpha_3 b}{a^3 \alpha_1}}}{4\sqrt{5}}\right)}{\sqrt{5}}, \end{aligned} \tag{37}$$

where

$$\eta = ax + by - \frac{(-4\alpha_2^2 a^2 + 25\alpha_1 \alpha_4 a^2 - 4\alpha_2 \alpha_3 ab + 24\alpha_3^2 b^2 + 25\alpha_1 \alpha_6 b^2)t}{25a\alpha_1 \alpha_5}.$$

Type 4: When $\mathcal{R} = 0$ and $\mathcal{Q} < 0$, we retrieve the solution

$$\begin{aligned} \mu_{23}(x, y, t) = \xi_0 - & \frac{(a\alpha_2 + 3\alpha_3 b) \coth\left(\frac{\eta\sqrt{\frac{-a\alpha_2 + 3\alpha_3 b}{a^3 \alpha_1}}}{4\sqrt{5}}\right)}{\sqrt{5}a^2 \alpha_1 \sqrt{\frac{-a\alpha_2 + 3\alpha_3 b}{a^3 \alpha_1}}} \\ & + \frac{a\sqrt{\frac{-a\alpha_2 + 3\alpha_3 b}{a^3 \alpha_1}} \tanh\left(\frac{\eta\sqrt{\frac{-a\alpha_2 + 3\alpha_3 b}{a^3 \alpha_1}}}{4\sqrt{5}}\right)}{\sqrt{5}}, \end{aligned} \tag{38}$$

where

$$\eta = ax + by - \frac{(-4\alpha_2^2 a^2 + 25\alpha_1 \alpha_4 a^2 - 4\alpha_2 \alpha_3 ab + 24\alpha_3^2 b^2 + 25\alpha_1 \alpha_6 b^2)t}{25a\alpha_1 \alpha_5}.$$

Type 5: When $\mathcal{R} = 0$ and $\mathcal{Q} < 0$, we have

$$\mu_{24}(x, y, t) = \xi_0 - 4a \left(\frac{\kappa_1 - C_1 \kappa_2 e^{(\kappa_1 - \kappa_2)\eta}}{1 - C_1 e^{(\kappa_1 - \kappa_2)\eta}} \right) \tag{39}$$

where

$$\kappa_1 = \frac{1}{4} \left(\sqrt{-\frac{20\alpha_1 a^3 \mathcal{Q} + \alpha_2 a + 3\alpha_3 b}{5a^3 \alpha_1}} - 4\mathcal{Q} - \frac{i\sqrt{-20\alpha_1 a^3 \mathcal{Q} + \alpha_2 a + 3\alpha_3 b}}{\sqrt{5}a^{3/2}\sqrt{\alpha_1}} \right),$$

$$\kappa_2 = \frac{1}{4} \left(-\sqrt{\frac{-20\alpha_1 a^3 Q + \alpha_2 a + 3\alpha_3 b}{5a^3 \alpha_1}} - 4Q - \frac{i\sqrt{-20\alpha_1 a^3 Q + \alpha_2 a + 3\alpha_3 b}}{\sqrt{5}a^{3/2}\sqrt{\alpha_1}} \right),$$

$$\eta = ax + by - \frac{(-4\alpha_2^2 a^2 + 25\alpha_1 \alpha_4 a^2 - 4\alpha_2 \alpha_3 ab + 24\alpha_3^2 b^2 + 25\alpha_1 \alpha_6 b^2)t}{25a\alpha_1 \alpha_5}.$$

6 Discussion and results

The graphical description of the (2+1)-dimensional pKP-BKP equation are demonstrated in this section. Using a specified range of values, various forms of v-shaped, singular bell-shaped, periodic and bright solitons are displayed. Figure 1 depicts the profiles of $|\mu_1(x, y, t)|$ which exhibits a v-shaped soliton solution using constants $\alpha_2 = 2$, $\alpha_3 = -1.5$, $\alpha_4 = 0.4$, $\alpha_5 = 0.5$, $\alpha_6 = 0.75$, $g = 2.5$, $l_1 = 0.4$, $l_2 = 0.5$, $m_1 = -1.5$, $m_2 = 0.45$. Moreover, Figs. 2,3,4,5,6 represent the v-shaped soliton solutions for various constant values. Figure 7 represents the behaviour of solution $|\mu_3(x, y, t)|$ which illustrates the singular bell shaped solution using parameters $\alpha_4 = 0.45$, $\alpha_5 = 0.55$, $\alpha_6 = 0.75$, $g = 2.5$, $l_1 = -0.4$, $l_2 = 0.5$, $l_3 = 0.6$. Consequently, Figs. 8,9,10,11 demonstrate the singular bell shaped solutions for specified parameters. Singularities may exist almost anywhere and they are surprisingly abundant in the mathematics used by physicists to understand the universe. Figure 12 depicts the behaviour of solution $|\mu_{12}(x, y, t)|$ which describes the periodic solitary waves solution utilizing constants $\alpha_2 = 0.55$, $\alpha_3 = 0.5$, $\alpha_4 = 0.45$, $\alpha_5 = 0.5$, $\alpha_6 = 0.75$, $n_0 = 0.15$, $n_1 = 0.95$, $n_2 = -0.95$. Moreover, Figure 13 display the periodic solitary waves solutions for various constant values. Figure 14 represents the profiles of $|\mu_{16}(x, y, t)|$ which illustrates the bright soliton solution utilizing parameters $\alpha_1 = 1.2$, $\alpha_2 = 2$, $\alpha_3 = -1.5$, $\alpha_4 = 0.4$, $\alpha_5 = 0.5$, $\alpha_6 = 0.75$, $m_0 = 0.5$, $n_0 = -0.85$, $n_1 = 0.225$, $n_2 = 0.8$, $\omega_2 = 0.75$, $\omega_3 = 0.65$. Also, Figs. 15, 16 demonstrate the bright soliton solution for specified parameters. Lastly, we have illustrated 3D, contour and 2D graphs of various retrieved solutions to have a better understanding of the behaviour of solutions.

7 Conclusion

In this manuscript, the (2+1)-dimensional pKP-BKP equation is investigated utilizing the Hirota bilinear method and the polynomial-expansion method. Consequently, we have generated a variety of fascinating exact solutions to the considered equation using various types of functions. Additionally, using Mathematica 11.0, we were able to generate numerous graphical representations of the specified solutions utilizing various parameters. Lastly, we assert that the methodologies under consideration are applicable to a wide range of NLEEs in mathematical physics.

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